Cortés-Izurdiaga, Manuel; Crivei, Septimiu; Saorín, Manuel
Reflective and coreflective subcategories. (English) Zbl 07654863
J. Pure Appl. Algebra 227, No. 5, Article ID 107267, 43 p. (2023)

The celebrated Freyd's adjoint functor theorem [*P. Freyd*, Abelian categories. An introduction to the theory of functors. New York-Evanston-London: Harper and Row, Publishers (1964; Zbl 0121.02103); Repr. Theory Appl. Categ. 2003, No. 3, xxiii, 1–164 (2003; Zbl 1041.18001), Chapter 3] claims that a functor $F:\mathcal{B}\rightarrow\mathcal{C}$ with $\mathcal{B}$ complete is of a left adjoint iff it preserves all limits and abides by a so-called solution-set conditions. The conjunction of product-preserving plus the solution-set condition is highly related to the condition that the essential image $\mathrm{Im}(F)$ is a preenveoping categoryf of $\mathcal{C}$. The natural question to aski is whether this condition is also sufficient.

This paper is mainly interested in the case when $\mathcal{C}$ is an arbitrary additive category and F is the inclusion functor. The paper aims to give necessary and sufficient conditions for the subcategory $\mathcal{B}$ to be (co)reflective, asking what are the minimal conditions to add to the preenveloping (resp. precovering) condition on $\mathcal{B}$ for to be a reflective (resp. coreflective) subcategory of $\mathcal{C}$.

The authors' interest in the question comes from a result of *A. Neeman* [Ann. Math. (2) 171, No. 3, 2143–2155 (2010; Zbl 1205.18008), Proposition 1.4] claiming that if $\mathcal{C}$ is a triangulated category with split idempotents, then a full triangulated subcategory $\mathcal{B}$ is coreflective iff it is precovering and closed under direct summands, which was slightly generalized by *M. Saorín* and *J. Šťovíček* [Adv. Math. 228, No. 2, 968–1007 (2011; Zbl 1235.18010), Proposition 3.11] to the mere assumption that $\mathcal{B}$ is a suspended subcategory. The common flavor of these results in triangulated and abelian worlds leads the authors to study the above mentioned question in a general context that includes both worlds.

The synopsis of the paper goes as follows.

- §2 introduces most of the concepts required in this paper.
- §3 gives the basic Theorem 3.1 claiming that if $\mathcal{C}$ is any additive category with split idempotents, pseudokernels and pseudocokernels, then a subcategory $\mathcal{B}$ is coreflective iff it is precovering, closed under direct summands and every morphism in $\mathcal{B}$ has a pseudocokernel in $\mathcal{C}$ that belongs to $\mathcal{B}$.
- §4 applies these theorems to one-sided triangulated categories and pretriangulated categories in the sense of [*A. Beligiannis* and *I. Reiten*, Homological and homotopical aspects of torsion theories. Providence, RI: American Mathematical Society (AMS) (2007; Zbl 1124.18005)].
- §5 considers any abelian category $\mathcal{A}$ and any preenveloping (resp. precovering) subcategory $\mathcal{P}$, in which case the stable category $\underline{\mathcal{A}}=\mathcal{A}/\mathcal{P}$ is a right (resp. left) triangulated category.
- §6 shows that exactly definable additive categories have split idempotents, pseudokernels and pseudocokernels (Proposition 6.3), so that Theorem 3.1 applies to them.
- §7 shows that if $\mathcal{A}$ is a preabelian category, then a subcategory $\mathcal{B}$ is coreflective iff it is precovering and closed under taking cokernels. In the particular case when $\mathcal{A}$ is abelian, a handy criterion for such a coreflective subcategory to be also abelian is given (Theorem 7.6).
- §8 gives necessary and sufficient conditions for the subcategory $\mathrm{Pres}(\mathcal{U})$ of objects presented by some set $\mathcal{U}$ of objects to be coreflective, coreflective abelian or coreflective abelian exact, respectively (Theorems 8.5, 8.6 and 8.9).
- §9 shows several applications of the previous results. It is particularly shown (Theorem 9.14) that fully exact subcategories of module categories over small preadditive categories are always bireflective, being exactly the ones induced by an epimorphism of preadditive categories, which is an extension of the famous theorem of *P. Gabriel* and *J. A. de la Peña* [Commun. Algebra 15, 279–307 (1987; Zbl 0609.16013)] claiming the result for module categories over rings.

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

18A40 Adjoint functors (universal constructions, reflective subcategories, Kan extensions, etc.)
18E05 Preadditive, additive categories
18E10 Abelian categories, Grothendieck categories
18C35 Accessible and locally presentable categories
18G80 Derived categories, triangulated categories

Keywords:

additive category; (co)reflective subcategory; (pre)abelian category; (pre)triangulated category; exactly definable category; finitely accessible category; fully exact subcategory

Full Text: DOI arXiv

References:

[1] Adamek, J.; Rosicky, J., Locally Presentable and Accessible Categories, London Math. Soc. Lecture Note Ser., vol. 189 (1994), Cambridge University Press: Cambridge University Press Cambridge · Zbl 0795.18007

[2] Auslander, M.; Bridger, M., Stable Module Theory, Mem. Amer. Math. Soc., vol. 94 (1969) · Zbl 0204.36402

[3] Balmer, P.; Schlichting, M., Idempotent completions of triangulated categories, J. Algebra, 236, 819-834 (2001) · Zbl 0977.18009

[4] Beligiannis, A.; Marmaridis, N., Left triangulated categories arising from contravariantly finite subcategories, Commun. Algebra, 22, 5021-5036 (1994) · Zbl 0811.18005

[5] Beligiannis, A.; Reiten, I., Homological and homotopical aspects of torsion theories, Mem. Am. Math. Soc., 188 (2007) · Zbl 1124.18005

[6] Bondal, A.; Kapranov, M. M., Representable functors, Serre functors and mutations, Izv. Math., 35 (3):519 (1990) · Zbl 0703.14011

[7] Borceux, F., Handbook of Categorical Algebra, vol. 1 (1994), Cambridge University Press: Cambridge University Press Cambridge

[8] Crawley-Boevey, W., Locally finitely presented additive categories, Commun. Algebra, 22, 1641-1674 (1994) · Zbl 0798.18006

[9] Crivei, S.; Prest, M.; Torrecillas, B., Covers in finitely accessible categories, Proc. Am. Math. Soc., 138, 1213-1221 (2010) · Zbl 1213.18006

[10] El Bashir, R., Covers and directed colimits, Algebr. Represent. Theory, 9, 423-430 (2006) · Zbl 1114.16002

[11] Enochs, E. E.; Iacob, A., Gorenstein injective covers and envelopes over Noetherian rings, Proc. Am. Math. Soc., 143, 1, 5-12 (2015), MR3272726 · Zbl 1307.18013

[12] Freyd, P., Abelian Categories. An Introduction to the Theory of Functors (1964), Harper and Row · Zbl 0121.02103

[13] Gabriel, P.; de la Peña, J. A., Quotients of representation-finite algebras, Commun. Algebra, 15, 279-307 (1987) · Zbl 0609.16013

[14] Göbel, R.; Trlifaj, J., Approximations and Endomorphism Algebras of Modules (2006), Walter de Gruyter: Walter de Gruyter Berlin · Zbl 1121.16002

[15] Happel, D., Triangulated Categories in the Representation Theory of Finite-Dimensional Algebras, London Math. Soc. Lecture Note Ser., vol. 119 (1988), Cambridge University Press: Cambridge University Press Cambridge · Zbl 0635.16017

[16] Hilton, P. J.; Stammbach, U., A Course in Homological Algebra (1997), Springer: Springer New York · Zbl 0863.18001

[17] Holm, H., Approximations by maximal Cohen-Macaulay modules, Pac. J. Math., 277, 355-370 (2015) · Zbl 1328.13014

[18] Jans, J. P., Some aspects of torsion, Pac. J. Math., 15, 1249-1259 (1965) · Zbl 0142.28002

[19] Kasch, F., Modules and Rings (1982), Academic Press: Academic Press London · Zbl 0523.16001

[20] Krause, H., Exactly definable categories, J. Algebra, 201, 456-492 (1998) · Zbl 0917.18005

[21] Krause, H., The stable derived category of a Noetherian scheme, Compos. Math., 141, 5, 1128-1162 (2005), MR2157133 · Zbl 1090.18006

[22] Krause, H., Krull-Schmidt categories and projective covers, Expo. Math., 33, 535-549 (2015) · Zbl 1353.18011

[23] Li, Z. W., The left and right triangulated structures of stable categories, Commun. Algebra, 43, 3725-3753 (2015) · Zbl 1330.18015

[24] MacLane, S., Categories for the Working Mathematician (1998), Springer: Springer New York

[25] Neeman, A., Triangulated Categories (2001), Princeton University Press: Princeton University Press Princeton · Zbl 0974.18008

[26] Neeman, A., Some adjoints in homotopy categories, Ann. Math., 171, 2143-2155 (2010) · Zbl 1205.18008

[27] P. Nicolás, M. Saorín, Pseudo-epimorphisms and semi-orthogonal decompositions, Unpublished notes.

[28] Parra, C. E.; Saorín, M., The HRS tilting process and Grothendieck hearts of t-structures, Contemp. Math., 769, 209-241 (2021)

[29] Parra, C. E.; Saorín, M.; Virili, S., Locally finitely presented and coherent hearts, preprint

[30] Popescu, N., Abelian Categories with Applications to Rings and Modules (1973), Academic Press: Academic Press London · Zbl 0271.18006

[31] Prest, M., Definable additive categories: purity and model theory, Mem. Am. Math. Soc., 210, 987 (2011) · Zbl 1229.03034

[32] Rada, J. P.; Saorín, M., Rings characterized by (pre)envelopes and (pre)covers of their modules, Commun. Algebra, 26, 3, 899-912 (1998) · Zbl 0908.16003

[33] Rada, J. P.; Saorín, M.; del Valle, A., Reflective subcategories, Glasg. Math. J., 42, 97-113 (2000) · Zbl 0943.18001

[34] Roos, J. E., Locally Noetherian categories and generalized strictly linearly compact rings, (Hilton, P., Category Theory, Homology Theory and Their Applications. Category Theory, Homology Theory and Their Applications, Seattle Conference 1968. Category Theory, Homology Theory and Their Applications. Category Theory, Homology Theory and Their Applications, Seattle Conference 1968, Springer LNM, vol. 92 (1969)), 197-277

[35] Saorín, M., Dg algebras with enough idempotents, their dg modules and their derived categories, Algebra Discrete Math., 23, 1, 62-137 (2017) · Zbl 1370.16007

[36] Saorín, M.; Šťovíček, J., On exact categories and applications to triangulated adjoints and model structures, Adv. Math., 228, 968-1007 (2011) · Zbl 1235.18010

[37] Šaroch, J.; Šťovíček, J., Singular compactness and definability for Σ-cotorsion and Gorenstein modules, Sel. Math. New Ser., 26, 2, Article 23 pp. (2020), 40 pp., MR4076700 · Zbl 1444.16010

[38] Stenström, B., Rings of Quotients, Grundlehren der Math., vol. 217 (1975), Springer: Springer Berlin, Heidelberg, New York · Zbl 0296.16001

[39] Swan, R. G., Induced representations and projective modules, Ann. Math., 71, 552-578 (1960) · Zbl 0104.25102

[40] Wisbauer, R., Foundations of Module and Ring Theory (1991), Gordon and Breach: Gordon and Breach Reading

[41] Xu, J., Flat Covers of Modules, Lecture Notes in Math., vol. 1634 (1996), Springer-Verlag: Springer-Verlag Berlin · Zbl 0860.16002