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A Pirashvili-type theorem for functors on non-empty finite sets. (English) Zbl 07629511

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The principal objective in this paper is to study the category of functors from $\overline{\boldsymbol{Fin}}$ of finite non-empty sets and all morphisms to the category of modules over a fixed commutative ring $\boldsymbol{k}$. Given an essentially small category $\mathcal{C}$, $\mathcal{F}(\mathcal{C};\boldsymbol{k})$ denotes the category of functors from $\mathcal{C}$ to $\boldsymbol{k}$-modules. The authors's interset in $\mathcal{F}\left(\overline{\boldsymbol{Fin}};\boldsymbol{k}\right)$ comes from their functorial approach to higher Hochschild homology to a wedge of circles [*G. Powell* and *C. Vespa*, "Higher Hochschild homology and exponential functors", Preprint, `arXiv:1802.07574`].

T. Pirashvili [Math. Ann. 318, No. 2, 277–298 (2000; Zbl 0963.18006)] gave a Dold-Kan type theorem, called the *Pirashviti's theorem*, claiming an equivalence of categories between the categories $\mathcal{F}(\Gamma;\boldsymbol{k})$ and $\mathcal{F}(\boldsymbol{\Omega};\boldsymbol{k})$, where $\boldsymbol{\Omega}$ is the category of finite sets and surjections. The proof of this theorem is based on a certain family of projective objects $(t^*)^{\otimes n}$ in $\mathcal{F}(\Gamma;\boldsymbol{k})$ for $n\in\mathbb{N}$. The equivalencce follows from the identification

$$\mathrm{Hom}_{\mathcal{F}(\Gamma;\boldsymbol{k})}\left((t^*)^{\otimes a},(t^*)^{\otimes b}\right)\cong\boldsymbol{k}\mathrm{Hom}_{\boldsymbol{\Omega}}(\boldsymbol{b},\boldsymbol{a})$$

for $a,b\in\mathbb{N}$, where

$$\boldsymbol{m}:=\{1,\ldots,m\}$$

for $m\in\mathbb{N}$. This paper aims to give an unpointed analogue of the result, computing the morphisms between the tensor powers of the corresponding functor in the unpointed context.

The synopsis of the paper goes as follows.

§2 is concerned with categories of sets, fixing notation and terminology.

§3 first reviews functors on Γ, comparing the functor categories on Γ and on $\overline{\boldsymbol{Fin}}$ and obtaining the comonad

$$\perp^{\Gamma}\colon\mathcal{F}(\Gamma;\boldsymbol{k})\to\mathcal{F}(\Gamma;\boldsymbol{k})$$

The relationship between contravariant functors on $\boldsymbol{FI}$ and on $\boldsymbol{\Sigma}$ is recalled, obtaining the comonad

$$\perp^{\boldsymbol{\Sigma}}\colon\mathcal{F}(\boldsymbol{\Sigma}^{\mathrm{op}};\boldsymbol{k})\to\mathcal{F}(\boldsymbol{\Sigma}^{\mathrm{op}};\boldsymbol{k})$$

§4 introduces the Koszul complex $\mathsf{Kz}\,F$ in $\boldsymbol{\Sigma}^{\mathrm{op}}$-modules for F an $\boldsymbol{FI}^{\mathrm{op}}$-module. It is shown that the complex is quasi-isomorphic to the normalized cochain complex associated with the opposite of the cosimplicial object $\mathfrak{C}^{\cdot}F$.

§5 provides the technical underpinnings of the paper. The main result is an isomorphism between a cosimplical object arising via the general cosimplicial object constructed from a comonad in Appendix B.2, and another one that is defined in terms of the category $\boldsymbol{\Omega}$ with the aid of a cobar-type cosimplicial construction (Appendix B.1).

§6 is the hear of the paper, computing the cohomology of the Koszul complex of $\boldsymbol{k}\mathrm{Hom}_{\boldsymbol{\Omega}}(-,\boldsymbol{a})$ for $a\in\mathbb{N}^*$. The key result is that its homology is concentrated in the top and bottom degrees (Theorem 6.15), so that the explicit calculation follows with the aid of the Euler-Poincaré characteristic of the complex.

§7 establishes the main theorem by combining the previous results.

Appendix A fixes conventions on the normalized cochain complex associated with a cosimplicial object in an abelian category $\mathcal{A}$.

Appendix B associates to a comonad two natural coaugmented cosimplicial objects of a different nature. The first construction is a cobar-type cosimplicial resolution, extending the canonical augmented simplicial object, while the second one encodes the notion of morphisms of comodules.

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MSC:

18G15 Ext and Tor, generalizations, Künneth formula (category-theoretic aspects)
18A25 Functor categories, comma categories
18G90 Other (co)homology theories (category-theoretic aspects)

Keywords:

category of finite sets and surjections; comonad on Gamma-modules; Koszul complex; $\mathbf{FI}^{\mathrm{op}}$ cohomology

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