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Presentations and algebraic colimits of enriched monads for a subcategory of arities. (English) Zbl 07646804

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This paper holds twofold purposes,

1. firstly to unfurl a theory of presentations and colimits of enriched monads for subcategories of arities with sufficient generality to accommodate, in case that $\mathcal{V}$ is locally bounded, the Φ-accessible $\mathcal{V}$-monads [*S. Lack* and *J. Rosický*, Appl. Categ. Struct. 19, No. 1, 363–391 (2011; Zbl 1242.18007)] as well as the $\mathcal{J}$-ary $\mathcal{V}$-monads for a small and *eleutheric* system of arities $\mathcal{J} \hookrightarrow \mathcal{C}$ [*R. B. B. Lucyshyn-Wright*, Theory Appl. Categ. 31, 101–137 (2016; Zbl 1337.18002)], and
2. secondly to ensure that the resulting of presentation and algebraic colimits covers in full generality such specific settings as the strongly finitary $\mathcal{V}$-monads of *G. M. Kelly* and *S. Lack* [Appl. Categ. Struct. 1, No. 1, 85–94 (1993; Zbl 0787.18007)], in case that $\mathcal{V}$ is a complete and cocomplete cartesian closed category or, more generally, a π-category in the sense of *F. Borceux* and *B. Day* [J. Pure Appl. Algebra 16, 133–147 (1980; Zbl 0426.18004)], and Wolff's presentations of $\mathcal{V}$-categories by generators and relations for an arbitrary complete and cocomplete $\mathcal{V}$ [*H. Wolff*, J. Pure Appl. Algebra 4, 123–135 (1974; Zbl 0282.18010)].

The authors accomplish these objectives by working with enriched monads for a suitable subcategory of arities $j : \mathcal{J} \hookrightarrow \mathcal{C}$ in a $\mathcal{V}$-category $\mathcal{C}$, where $\mathcal{V}$ is a complete and cocomplete symmetric monoidal closed category that need not be locally presentable. The results apply when $\mathcal{C}$ is a locally bounded $\mathcal{V}$-category over a locally bounded closed category $\mathcal{V}$, and in some cases even without these assumptions.

To get these results, the authors make some modest completeness and cocompleteness assumptions on the $\mathcal{V}$-category $\mathcal{C}$ as well as two main assumptions on the subcategory of arities $j : \mathcal{J} \hookrightarrow \mathcal{C}$.

1. First, the authors generally assume that $j : \mathcal{J} \hookrightarrow \mathcal{C}$ is small and *eleutheric* [*R. B. B. Lucyshyn-Wright*, Theory Appl. Categ. 31, 101–137 (2016; Zbl 1337.18002)], which is a certain exactness condition guaranteeing that the $\mathcal{V}$-endofunctor on $\mathcal{C}$ that are left Kan extensions along j are precisely those preserving left Kan extensions along j. They are called $\mathcal{J}$-ary $\mathcal{V}$-endofunctors.
2. The authors also assume that $j : \mathcal{J} \hookrightarrow \mathcal{C}$ abides by a mild *boundedness* condition, which is defined in terms of certain notions from Kelly's classical paper [*G. M. Kelly*, Seminarber. Fachbereich Math., Fernuniv. 6, 5–82 (1980; Zbl 0437.18003); Bull. Aust. Math. Soc. 22, 1–83 (1980; Zbl 0437.18004)] on transfinite constructions in category theory.

The main results on free $\mathcal{J}$-ary monads, algebraic colimits of $\mathcal{J}$-ary monads, and presentations of $\mathcal{J}$-ary monads then hold for any bounded and eleutheric subcategory of arities $j : \mathcal{J} \hookrightarrow \mathcal{C}$ in a $\mathcal{V}$-category $\mathcal{C}$ abiding by mild assumptions.

The synopsis of the paper goes as follows.

- §2 and §3 defines the notion of an eleuthetic subcategory of arities $j : \mathcal{J} \hookrightarrow \mathcal{C}$ in a $\mathcal{V}$-category $\mathcal{C}$ after reviewing some notation and background
- §4 defines the notions of $\mathcal{J}$-ary $\mathcal{V}$-endofunctor and $\mathcal{J}$-ary $\mathcal{V}$-monad on $\mathcal{C}$.
- §5 unfurls the theory of *algebraically free monads* in the enriched context, generalizing [*G. M. Kelly*, Seminarber. Fachbereich Math., Fernuniv. 6, 5–82 (1980; Zbl 0437.18003); Bull. Aust. Math. Soc. 22, 1–83 (1980; Zbl 0437.18004)].
- §6 defines the notion of a *bounded* subcategory of arities, showing in 6.2.5 and 6.2.6 that if $j : \mathcal{J} \hookrightarrow \mathcal{C}$ is a bounded subcategory of arities in a cocomplete and cotensored $\mathcal{V}$-category $\mathcal{C}$, then the forgetful functor

$$\mathcal{W} : \boldsymbol{Mnd}_{\mathcal{J}}(\mathcal{C}) \to \boldsymbol{End}_{\mathcal{J}}(\mathcal{C})$$

from $\mathcal{J}$-ary $\mathcal{V}$-monads on $\mathcal{C}$ to $\mathcal{J}$-ary $\mathcal{V}$-endfunctors on $\mathcal{C}$ is monadic, and that the free $\mathcal{J}$-ary $\mathcal{V}$-monad on a $\mathcal{J}$-ary $\mathcal{V}$-endfunctor is *algebraically* free. These are the first main results in this paper.

§7 defines the notion of a Σ-algebra for a -signature Σ in $\mathcal{C}$, relative to a subcategory of arities $j : \mathcal{J} \hookrightarrow \mathcal{C}$, showing in 7.7 under certain assumptions that the forgetful functor

$$\boldsymbol{End}_{\mathcal{J}}(\mathcal{C}) \to \boldsymbol{Sig}_{\mathcal{J}}(\mathcal{C})$$

from $\mathcal{J}$-ary $\mathcal{V}$-endfunctors on $\mathcal{C}$ to $\mathcal{J}$-signatures in $\mathcal{C}$ is monadic. It is then shown in 7.9 that the forgetful functor

$$\mathcal{U} : \boldsymbol{Mnd}_{\mathcal{J}}(\mathcal{C}) \to \boldsymbol{Sig}_{\mathcal{J}}(\mathcal{C})$$

has a left adjoint, and that the $\mathcal{V}$-category of algebras for the free $\mathcal{J}$-ary $\mathcal{V}$-monad on a $\mathcal{J}$-signature Σ is isomorphic to the $\mathcal{V}$-category Σ -$\boldsymbol{Alg}$ of Σ-algebras.

§8 establishes by use of *S. Lack* [J. Pure Appl. Algebra 140, No. 1, 65–73 (1999; Zbl 0974.18005)] in 8.2 that the forgetful functor

$$\mathcal{U} : \boldsymbol{Mnd}_{\mathcal{J}}(\mathcal{C}) \to \boldsymbol{Sig}_{\mathcal{J}}(\mathcal{C})$$

is actually *monadic*.

§9 is concerned with algebraic colimits of $\mathcal{J}$-ary $\mathcal{V}$-monads, being divided into three subsections. §9.1 proves some results about limits and colimits in limit $\mathcal{V}$-categories. §9.2 studies the notion of an algebraic colimi of $\mathcal{V}$-monads. §9.3 defines the notion of an algebraic colimit of $\mathcal{J}$-ary $\mathcal{V}$-monads, demonstrating in 9.3.8 that if $j : \mathcal{J} \hookrightarrow \mathcal{C}$ is bounded, then the category $\boldsymbol{Mnd}_{\mathcal{J}}(\mathcal{C})$ of $\mathcal{J}$-ary $\mathcal{V}$-monads on $\mathcal{C}$ has small algebraic colimits.

§10 defines the notion of a *$\mathcal{J}$-presentation* $P = (\Sigma, E)$ for a subcategory of arities $j : \mathcal{J} \hookrightarrow \mathcal{C}$ consisting of $\mathcal{J}$-signature morphisms from a $\mathcal{J}$-signature Γ (the *signature of equations*) to the underlying $\mathcal{J}$-signature of the free $\mathcal{J}$-ary $\mathcal{V}$-monad $\mathbb{T}_\Sigma$ on Σ. It is shown in 10.1.8 that every $\mathcal{J}$-presentation P presents a $\mathcal{J}$-ary $\mathcal{V}$-monad $\mathbb{T}_P$, whose $\mathcal{V}$-category of algebras turns out in 10.1.8 to be isomorphic to the $\mathcal{V}$-category P-$\boldsymbol{Alg}$ of P-algebras for the $\mathcal{J}$-presentations $P = (\Sigma, E)$. It is also shown in 10.1.10 that every $\mathcal{J}$-ary $\mathcal{V}$-monad has a $\mathcal{J}$-presentation.

§11 addresses some specimens of $\mathcal{J}$-presentations, firstly showing that presentations of $\mathcal{V}$-categories by generators and relations are recovered when $\mathcal{J}$ consists of the representables in a power of $\mathcal{V}$, while secondly discussing presentations of strongly finitary $\mathcal{V}$-monads in cartesian closed topological categories over $\boldsymbol{Set}$, dealing with internal modules and affine spaces over internal rings (i.e. semirings).

§12 summarizes the main results in this paper.

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MSC:

18C10	Theories (e.g., algebraic theories), structure, and semantics
18C15	Monads (= standard construction, triple or triad), algebras for monads, homology and derived functors for monads
18C20	Eilenberg-Moore and Kleisli constructions for monads
18C40	Structured objects in a category (group objects, etc.)
18D15	Closed categories (closed monoidal and Cartesian closed categories, etc.)
18D20	Enriched categories (over closed or monoidal categories)
08B20	Free algebras
08C05	Categories of algebras

Keywords:

monad; enriched category; presentations of monads; subcategory of arities; free monad; colimit of monads; locally bounded category

Full Text: arXiv Link

References:

[1] H is J -ary (i.e. H is Φ J -cocontinuous);

[2] H is a left Kan extension along j (equivalently, H ⊠ = Lan j (Hj)).

[3] If C is Ψ J -cocomplete, then (1) and (2) are also equivalent to the following: 3. H preserves small J -flat colimits (i.e. H is Ψ J -cocontinuous).

[4] Moreover, if Ψ is any class of small weights such that j is a free Ψ-cocompletion, then (1) and (2) are equivalent to the following: 4. H is Ψ-cocontinuous.

[5] Proof. Firstly, if Ψ is any class of small weights such that j is a free Ψ-cocompletion, then (2) is equivalent to (4) by [19, 3.6]. In particular, since j is a free Φ J -cocompletion by 3.6, this entails that (2) is equivalent to (1). If C is Ψ J -cocomplete, then j is also a free Ψ J -cocompletion by 3.7, so (2) is equivalent to (3).

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