zbMATH Open
THE FIRST RESOURCE FOR MATHEMATICS

Hosseini, S. N.; **Nasab, A. R. Shir Ali**; **Tholen, W.**; **Yeganeh, L.**

Quotients of span categories that are allegories and the representation of regular categories. (English) Zbl 07629352

Appl. Categ. Struct. 30, No. 6, 1177-1201 (2022)

By identifying vertically isomorphic morphisms in *J. Bénabou*'s [Lect. Notes Math. 47, 1–77 (1967; Zbl 1375.18001)] bicategory $\mathcal{S}pan(\mathcal{C})$ of spans of morphisms in a category $\mathcal{C}$ with pullbacks, one obtains the category

$$\mathsf{Span}(\mathcal{C})$$

in which spans are composed horizontally via pullback in $\mathcal{C}$. If $\mathcal{C}$ is regular so that $\mathcal{C}$ has also binary products and a pullback-stable (regular epi, mono)-factorization system, one may similarly form the category

$$\mathsf{Rel}(\mathcal{C})$$

of sets and relations in $\mathcal{C}$, with the horizontal composite of a composable pair of relations obtained as a regular image of their span composite. which is the prototypical example of a unitary and tabular allegory [*P. T. Johnstone*, Sketches of an elephant. A topos theory compendium. I. Oxford: Clarendon Press (2002; Zbl 1071.18001)].

D. Pavlović [J. Pure Appl. Algebra 99, No. 1, 9–34 (1995; Zbl 0829.18002)] considered, without any epi- or mono restrictions, an arbitrary pullback-stable factorization system $(\mathcal{E},\mathcal{M})$ of a category $\mathcal{C}$ with binary products and pullbacks, forming the category

$$\mathsf{Rel}_{\mathcal{M}}(\mathcal{C})$$

whose morphisms are represented by those spans $(A \leftarrow R \rightarrow B)$ whose induced morphism $R \rightarrow A\times B$ lies in $\mathcal{M}$. This paper takes a fresh look at this category by treating it as a quotient category of $\mathsf{Span}(\mathcal{C})$,

The synopsis of the paper goes as follows.

- §2 describes, for any pullback-stable class $\mathcal{E}$ of morphisms in $\mathcal{C}$ containing all isomorphisms and being closed under composition, a compatible equivalence relation $\sim_{\mathcal{E}}$ on $\mathsf{Span}(\mathcal{C})$ which renders its quotient category
$$\mathsf{Span}_{\mathcal{E}}(\mathcal{C})$$
isomorphic to $\mathsf{Rel}_{\mathcal{M}}(\mathcal{C})$ whenever is a factorization partner of $\mathcal{E}$ (Theorem 2.3).
- §3 gives necessary and sufficient conditions for a compatible equivalence condition $\backsim$ on $\mathsf{Span}(\mathcal{C})$ to make its quotient category an allegory (Theorem 3.6).
- §4 shows that the provision $\mathcal{M} \subseteq \mathsf{Mono}(\mathcal{C})$ is necessary for $\mathsf{Rel}_{\mathcal{M}}(\mathcal{C})$ to form an allegory (Theorem 4.6).
- §5 shows that, given any stable factorization system $(\mathcal{E},\mathcal{M})$ in a finitely complete category $\mathcal{C}$, there is a least pullback-stable and composition-closed class $\mathcal{E}_\bullet$ containing $\mathcal{E}$ and making $\mathsf{Span}_{\mathcal{E}_\bullet}(\mathcal{C})$ a unitary tabular allegory (Theorem 5.8).
- §6 sets up the 2-category of unitary tabular allegories on the one hand and that of finitely complete categories rigged out in a pullback-stable factorization system on the other, showing that the construction of Theorem 5.8 gives rise a left ajoint to the 2-functor
$$\mathsf{Map} : \mathfrak{UTabAll} \rightarrow \mathfrak{STabFact}$$
assigning to a unitary tabular allegory its category of Lawverian maps rigged out in its stable factorization system that makes it a regular category. The *Freyd-Scedrov Representation Theorem* [*P. J. Freyd* and *A. Scedrov*, Categories, allegories. Amsterdam etc.: North-Holland (1990; Zbl 0698.18002), 2.154] allows of presenting $\mathfrak{UTabAll}$ as 2-equivalent to the full subcategory $\mathfrak{RegCat}$ of $\mathfrak{STabFact}$ consisting of all regular categories, so that every finitely complete category with a pullback-stable factorization system allows for a reflection into $\mathfrak{RegCat}$.

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

18B10 Categories of spans/cospans, relations, or partial maps
18A32 Factorization systems, substructures, quotient structures, congruences, amalgams
18E08 Regular categories, Barr-exact categories

Keywords:

span category; relation; stable system; regular category; (unitary, tabular) allegory; representation of allegories

Full Text: DOI arXiv

References:

[1] Adámek, J., Herrlich, H., Strecker, G.E.: Abstract and Concrete Categories, John Wiley \& Sons, New York 1990. Online edition: (2004) · Zbl 0695.18001

[2] Bénabou, J.: Introduction to Bicategories. Lecture Notes in Mathematics, vol. 40. Springer, Berlin (1967) · Zbl 1375.18001

[3] Borceux, F.: Handbook of Categorical Algebra 1. Cambridge University Pres, Basic Category Theory (1994) · Zbl 0803.18001

[4] Carboni, A.; Kasangian, S.; Street, R., Bicategories of spans and relations, J. Pure Appl. Algebra, 33, 259-267 (1984)· doi:10.1016/0022-4049(84)90061-6

[5] Freyd, PJ; Scedrov, A., Categories (1990), North Holland, Amsterdam: Allegories, North Holland, Amsterdam

[6] Hosseini, SN; Nasab, AR Shir Ali; Tholen, W., Fraction, restriction and range categories from stable classes of morphisms, J. Pure Appl. Algebra, 224, 9 (2020)· doi:10.1016/j.jpaa.2020.106361

[7] Jayewardene, R.; Wyler, O., Categories of relations and functional relations, Appl. Categ. Struct., 8, 279-305 (1996)· doi:10.1023/A:1008651524610

[8] Johnstone, P., Sketches of an Elephant: A Topos Theory Compendium (2002), Oxford: Oxford University Press, Oxford

[9] Klein, A., Relations in categories, Ill. J. Math., 14, 536-550 (1970)

[10] Meisen, J., On bicategories of relations and pullback spans, Commun. Algebra, 1, 5, 377-401 (1974)· doi:10.1080/00927877408548625

[11] Milius, S.: Relations in Categories, Master's thesis, York University, Toronto, Ontario, June (2000)

[12] Pavlović, D., Maps I: Relative to a factorization system, J. Pure Appl. Algebra, 99, 9-34 (1995)· doi:10.1016/0022-4049(94)00054-M

This reference list is based on information provided by the publisher or from digital mathematics libraries. Its items are heuristically matched to zbMATH identifiers and may contain data conversion errors. It attempts to reflect the references listed in the original paper as accurately as possible without claiming the completeness or perfect precision of the matching.