zbMATH Open

THE FIRST RESOURCE FOR MATHEMATICS

Stenzel, Raffael

Univalence and completeness of Segal objects. (English) Zbl 07628595

J. Pure Appl. Algebra 227, No. 4, Article ID 107254, 47 p. (2023)

On the one hand, *univalence*, originally a type-theoretical notion at the heart of Voevodsky's Univalent Foundation Program [*The Univalent Foundations Program*, Homotopy type theory. Univalent foundations of mathematics. Princeton, NJ: Institute for Advanced Study; Raleigh, NC: Lulu Press (2013; Zbl 1298.03002)], has found general importance as a higher categorical property characterizing descent and hence classifying maps in $(\infty, 1)$-categories. On the other hand, *completeness* is a property of Segal spaces introduced by *C. Rezk* [Trans. Am. Math. Soc. 353, No. 3, 973–1007 (2001; Zbl 0961.18008)] characterising those Segal spaces which are $(\infty, 1)$-categories. The principal objective in this paper is first to make rigorous a ostensible analogy between univalence and completeness that has found various informal expressions in the higher categorical research community to date, and second to study its ramifications.

The basic strategy is to understand its quintessence as a translation between internal and external notions, motivated by model categorical considerations of *A. Joyal* and *M. Tierney* [Contemp. Math. 431, 277–326 (2007; Zbl 1138.55016)]. Consequently, the author characterizes the internal notion of univalence in logical model categories by the external notion of completeness defined as the right Quillen conditions of suitably indexed Set-weighted limit functors.

Furthermore, the author extends the analogy, showing that univalent completion in the sense of *B. van den Berg* and *I. Moerdijk* [Math. Ann. 371, No. 3–4, 1337–1350 (2018; Zbl 1400.55007)] translates to Rezk-completion of associated Segal objects as well. Depending on these correspondence, the author exhibits univalence as a homotopical locality condition whenever univalent completion exists.

A connection of Rezk-completeness and univalence via a nerve correspondence $p \mapsto N(p)$ has been investigated by *N. Rasekh* ["Complete Segal objects", Preprint, `arXiv:1805.03561`, §6], where a theory of complete Segal objects in $(\infty, 1)$-categories was developed and univalence of a map p in a locally cartesian closed $(\infty, 1)$-category $\mathcal{C}$ was defined as completeness of its associated Segal object $N(p)$. *B. Ahrens* et al. [Math. Struct. Comput. Sci. 25, No. 5, 1010–1039 (2015; Zbl 1362.18003)] introduced a notion of Rezk-completeness of precategories to categories in the syntax of Homotopy Type Theory, proposing a definition of 'category' for which equality and equivalence of categories agree. They gave a construction corresponding to a truncated version of the Rezk completion for Segal spaces, and also to the stack completion of a prestack.

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

18N45	Categories of fibrations, relations to K-theory, relations to type theory
18N50	Simplicial sets, simplicial objects
18N60	$(\infty, 1)$-categories (quasi-categories, Segal spaces, etc.); ∞-topoi, stable ∞-categories
18C50	Categorical semantics of formal languages
55U35	Abstract and axiomatic homotopy theory in algebraic topology

Keywords:

Segal objects; univalent type theory

Full Text: DOI arXiv

References:

[1] Ahrens, B.; Kapulkin, K.; Shulman, M., Univalent categories and the Rezk completion, Math. Struct. Comput. Sci., 25, 5, 1010-1039 (2015) · Zbl 1362.18003

[2] Arndt, P.; Kapulkin, K., Homotopy-theoretic models of type theory, (Typed Lambda Calculi and Applications. Typed Lambda Calculi and Applications, Lecture Notes in Computer Science, vol. 6690 (2011), Springer: Springer Berlin, Heidelberg), 45-60 · Zbl 1331.03044

[3] Barwick, C., \((\infty, n)\)-cat as a closed model category (2005), University of Pennsylvania: University of Pennsylvania Philadelphia, PA 19104, United States, Ph.D. thesis

[4] Bergner, J. E.; Rezk, C., Comparison of models for \((\infty, n)\)-categories, II, J. Topol., 13, 4, 1554-1581 (2020) · Zbl 1461.18017

[5] van den Berg, B.; Moerdijk, I., Univalent completion, Math. Ann., 371, 3-4, 1337-1350 (2018) · Zbl 1400.55007

[6] Capriotti, P., Models of type theory with strict equality (2016), University of Nottingham: University of Nottingham Nottingham NG7 2RD, United Kingdom, Ph.D. thesis

[7] Cartmell, J., Generalized algebraic theories and contextual categories, Ann. Pure Appl. Log., 32, 209-243 (1986) · Zbl 0634.18003

[8] Cisinski, D. C., Higher Categories and Homotopical Algebra, Cambridge Studies in Advanced Mathematics, vol. 180 (2019), Cambridge University Press · Zbl 1430.18001

[9] Dwyer, W. G.; Kan, D. M., Function complexes in homotopical algebra, Topology, 19, 427-440 (1980) · Zbl 0438.55011

[10] Gambino, N., Weighted limits in simplicial homotopy theory, J. Pure Appl. Algebra, 214, 7, 1193-1199 (2010) · Zbl 1228.18013

[11] Gepner, D.; Kock, J., Univalence in locally Cartesian closed infinity-categories, Forum Math., 29, 3 (2012)

[12] Horel, G., A model structure on internal categories in simplicial sets, Theory Appl. Categ., 30, 20, 704-750 (2015) · Zbl 1317.18018

[13] Hovey, M., Model Categories, Mathematical Surveys and Monographs, vol. 63 (1999), American Mathematical Society · Zbl 0909.55001

[14] Jacobs, B., Comprehension categories and the semantics of type dependency, Theor. Comput. Sci., 107, 169-207 (1993) · Zbl 0804.18007

[15] Jacobs, B., Categorical Logic and Type Theory, Studies in Logic and the Foundations of Mathematics, vol. 141 (1999), Elsevier Science B.V. · Zbl 0911.03001

[16] Joyal, A., Notes on clans and tribes (2017), [Online, v1]

[17] Joyal, A.; Tierney, M., Quasi-categories vs Segal spaces, (Categories in Algebra, Geometry and Mathematical Physics (2006), American Mathematical Society), 277-326 · Zbl 1138.55016

[18] Kapulkin, C., Locally Cartesian closed quasicategories from type theory, J. Topol., 10, 4 (2015)

[19] Kapulkin, C.; Lumsdaine, P. L.; Voevodsky, V., The simplicial model of univalent foundations (2012), [Online]

[20] Kapulkin, C.; Szumiło, K., Quasicategories of frames of cofibration categories, Appl. Categ. Struct., 25, 3, 323-347 (2017) · Zbl 1372.55014

[21] Kapulkin, C.; Szumiło, K., Internal languages of finitely complete \((\infty, 1)\)-categories, Sel. Math. New Ser., 25, 33 (2019) · Zbl 1462.18007

[22] Lumsdaine, P. L.; Warren, M., The local universes model: an overlooked coherence construction for dependent type theories, ACM Trans. Comput. Log., 16, 3, 1-31 (2015) · Zbl 1354.03101

[23] Lurie, J., Higher Topos Theory, Annals of Mathematics Studies, vol. 170 (2009), Princeton University Press · Zbl 1175.18001

[24] The Univalent Foundations Program, Homotopy Type Theory: Univalent Foundations of Mathematics (2013) · Zbl 1298.03002

[25] Rasekh, N., Complete Segal objects (2018), [Online, v1]

[26] Rezk, C., A model for the homotopy theory of homotopy theories, Trans. Am. Math. Soc., 973-1007 (1999) · Zbl 0961.18008

[27] Rezk, C., Toposes and homotopy toposes (version 0.15) (2010)

[28] Shulman, M., Univalence for inverse diagrams and homotopy canonicity, Math. Struct. Comput. Sci., 25, 5, 1203-1277 (2015) · Zbl 1362.03008

[29] Stenzel, R., On univalence, Rezk completeness and presentable quasi-categories (2019), University of Leeds: University of Leeds Leeds LS2 9JT, Ph.D. thesis

[30] Szumiło, K., Homotopy theory of cocomplete quasicategories, Algebraic Geom. Topol., 17, 2, 765-791 (2017) · Zbl 1364.55019

[31] Toën, B., Vers une axiomatisation de la théorie des catégories supérieures, K-Theory, 34, 3, 233-263 (2005) · Zbl 1083.18003