Lucyshyn-Wright, Rory B. B.
Bifold algebras and commutants for enriched algebraic theories. (English) Zbl 07629348
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This paper aims to study the interaction of the concepts of bifold algebra and commutant, working in the setting of enriched theories for a system of arities [*R. B. B. Lucyshyn-Wright*, Theory Appl. Categ. 31, 101–137 (2016; Zbl 1337.18002)]. One of the central insights is that the notion of commutant arises functorially via a universal construction in a certain two-sided fibration of bifold algebras over various theories, which allows of studying speical classes of bifold algebras related to commutants, including *commutant bifold algebras* and *balanced bifold algebras*. Several adjunctions and equivalences among these categories of bifold algebras and related categories of algebras over various theories are established with attention to commutative algebras and a new notion of contracommutative algebra that is related by a dual adjunction to commutative algebras. Several examples of commutative bifold algebras and balanced bifold algebas are provided.

The synopsis of the paper goes as follows.

§2 is a review of enriched algebraic theories and commutants for a system of arities

$$\mathcal{J} \to \mathcal{V}$$

where $\mathcal{V}$ is a given symmetric monoidal closed category.

§3 considers the category of algebras over various theories, i.e. pairs $(\mathcal{T}, A)$ of a theory $\mathcal{T}$ and a $\mathcal{T}$-algerba A in $\mathcal{C}$, with *strong morphisms* of algebras, discussing the *full algebra* on an object of $\mathcal{C}$ and its universal property.

§4 arrives at a convenient formalism for commuting pairs of algebra structures on an object and their relationship to the commutant of an algebra $(\mathcal{T}, A)$

$$(\mathcal{T}, A)^{\perp} = (\mathcal{T}_A^{\perp}, A^{\perp})$$

where the theory $\mathcal{T}_A^{\perp}$ is the commutant of $\mathcal{T}$ with respect to A, while $A^{\perp} : \mathcal{T}_A^{\perp} \to \mathcal{C}$ is the associated $\mathcal{T}_A^{\perp}$-algebra on the same carrier as A.

§5 introduces bifold algebras in $\mathcal{C}$, defined as $\mathcal{V}$-functors

$$D : \mathcal{T} \otimes \mathcal{U} \to \mathcal{C}$$

preserving $\mathcal{J}$-cotensors in each variable separately for specified theories $\mathcal{T}$ and $\mathcal{U}$, where $\mathcal{T} \otimes \mathcal{U}$ is the usual monoidal product of -categories.

§6 shows that bifold algebras for a fixed pair of theories $\mathcal{T}$ and $\mathcal{U}$ can be described equivalently as $\mathcal{T} \otimes_{\mathcal{J}} \mathcal{U}$-algebra for a theory called the tensor product, provided $\mathcal{J}$ is small and $\mathcal{V}$ is locally bounded.

§7 defines, by employing a version of the Grothendieck construction for two-sided fibrations [Zbl 0327.18006], a category $\mathsf{BAlg}^{\mathrm{sc}}(\mathcal{C})$ of bifold algebras over various theories, where the morphisms are *strong cross-morphisms of bifold algebras*.

§8 shows that the functors L and R furnishing the left and right faces of a bifold algebra have fully faithful left and right adjoints, respectively, as in the following diagram

$$\begin{tikzcd} \mathsf{Alg}^{\mathrm{s}}(\mathcal{C}) \arrow[r, shift left, "\mathsf{RCom}"] \arrow[r, phantom, "\perp"] & \mathsf{BAlg}^{\mathrm{sc}}(\mathcal{C}) \arrow[l, shift left, "L"] \arrow[r, shift right, "R"'] \arrow[r, phantom, "\top"] & \mathsf{Alg}^{\mathrm{s}}(\mathcal{C})^{\mathrm{op}} \arrow[l, shift right, "\mathsf{LCom}"'] \end{tikzcd}$$

It is shown that we have the *commutant adjunction*

$$\begin{tikzcd} \mathsf{Alg}^{\mathrm{s}}(\mathcal{C}) \arrow[r, shift right, "\mathsf{Com}"'] \arrow[r, phantom, "\top"] & \mathsf{Alg}^{\mathrm{s}}(\mathcal{C})^{\mathrm{op}} \arrow[l, shift right, "\mathsf{Com}^{\mathrm{op}}"'] \end{tikzcd}$$

§9 considers the coreflexive subcategory

$$\mathsf{RComBAlg}^{\mathrm{sc}}(\mathcal{C}) \to \mathsf{BAlg}^{\mathrm{sc}}(\mathcal{C})$$

determined by the coreflective embedding RCom as well as the reflective subcategory

$$\mathsf{LComBAlg}^{\mathrm{sc}}(\mathcal{C}) \to \mathsf{BAlg}^{\mathrm{sc}}(\mathcal{C})$$

determined by the reflective embedding LCom. We obtain equivalences of categories

$$\mathsf{RComBAlg}^{\mathrm{sc}}(\mathcal{C}) \simeq \mathsf{BAlg}^{\mathrm{s}}(\mathcal{C})$$
$$\mathsf{LComBAlg}^{\mathrm{sc}}(\mathcal{C}) \simeq \mathsf{BAlg}^{\mathrm{s}}(\mathcal{C})^{\mathrm{op}}$$

Right-commutant (resp. left-commutant) bifold algebras $D : \mathcal{T} \otimes \mathcal{U} \to \mathcal{C}$ are characterized as those whose transpose $\mathcal{U} \to [\mathcal{T}, \mathcal{C}]$ (resp. $\mathcal{T} \to [\mathcal{U}, \mathcal{C}]$) is fully faithful.

§10 considers *commutant bifold algebras*, which are those bifold algebras that are left-commutant and right-commutant, establishing an equivalence between commutant bifold algebras and *saturated algebras*, which are those algebras such that

$$(\mathcal{T}, A)^{\perp\perp} \cong (\mathcal{T}, A)$$

The commutant adjunction is idempotent, and its fixed points are the saturated algebras forming a reflective subcategory $\mathsf{SatAlg}^{\mathrm{s}}(\mathcal{C})$ of $\mathsf{Alg}^{\mathrm{s}}(\mathcal{C})$. The author establishes equivalences

$$\mathsf{SatAlg}^{\mathrm{s}}(\mathcal{C}) \simeq \mathsf{ComBAlg}^{\mathrm{sc}}(\mathcal{C}) \simeq \mathsf{SatAlg}^{\mathrm{s}}(\mathcal{C})^{\mathrm{op}}$$

under which a commutant bifold algebra corresponds to its left and right faces, respectively, which are commutants of one another. It is also shown that $\mathsf{ComBAlg}^{\mathrm{sc}}(\mathcal{C})$ is reflective in $\mathsf{RComBAlg}^{\mathrm{sc}}(\mathcal{C})$ and coreflective in $\mathsf{LComBAlg}^{\mathrm{sc}}(\mathcal{C})$.

§11 establishes an equivalence of $\mathcal{V}$-categories

$$(\mathcal{T}, \mathcal{U})\text{–}\mathsf{Alg}(\mathcal{C}) \simeq (\mathcal{T}, \mathcal{U})\text{–}\mathsf{CPair}(\mathcal{C})$$

natural in $\mathcal{T}$ and $\mathcal{U}$, between the $\mathcal{V}$-categories of bifold $(\mathcal{T}, \mathcal{U})$-algebras and a $\mathcal{V}$-category $(\mathcal{T}, \mathcal{U})\text{–}\mathsf{CPair}(\mathcal{C})$ of commuting $\mathcal{T}$-$\mathcal{U}$-algebra pairs. The author defines a category of commuting algebra pairs over various theories, with strong cross-morphisms, and it is shown that

$$\mathsf{Alg}^{\mathrm{sc}}(\mathcal{C}) \simeq \mathsf{CPair}^{\mathrm{sc}}(\mathcal{C})$$

On this basis, the author defines the notions of right-commutant, left-commutant, and commutant algebra pairs, for which corollaries to several of the above results are obtained.

§12 establishes several fundamental results on the preservation and reflection of commuting algebra pairs, demonstrating in particular that the hom $\mathcal{V}$-functors

$$\mathcal{C}(G, -) : \mathcal{C} \to \mathcal{V}$$

for the objects G of any enriched generating class $\mathcal{G}$ jointly reflected commutation of algebra pairs.

§13 addresses commutative algebras, which are those algebras commuting with themselves. Two further classes of algebras are investigated: *contracommutaitve algebras*, which are those whose commutant is commutative, and *balanced algebras*, which are those algebras A such that $A \cong A^{\perp}$ in $\mathsf{Alg}(C)$ with $C = |A|$. It is shown that an algebra A is commutative iff $A^{\perp}$ is contracommutative, whileI A is balanced iff A is commutative, contracommutative and saturated. It is also shown that the commutant adjunction restricts to

[1] a dual adjunction between commutative algebras and contracommutative algebras, which further restricts to

[2] a dual equivalence between commutative saturated algebras and contracommutative saturated algebras, which in turn restricts to

[3] an equivalence between the category $\mathsf{BalAlg}^{\mathrm{s}}(\mathcal{C})$ of balanced algebras and its opposite.

- It is shown that $\mathsf{BalAlg}^{\mathrm{s}}(\mathcal{C})$ is a groupoid whose canonical anti-involution is isomorphic to the latter equivalence.

§14 reverts to the study of left- and right-commutant bifold algebras, with special attention to the case in which the left face is commutaitve, or equivalently, the right face is contracommutative. An equivalence between the category of commutative algebras (resp. commutative saturated algebras) and the category of right-commutative (resp. commutant) bifold algebras with commutative left face is obtained. It is shown that any bifold $(\mathcal{T},\mathcal{U})$-algebra of this special kind induces a central morphism from $\mathcal{T}$ to $\mathcal{U}$, leading to the consequence that there is a $\mathcal{V}$-functor

$$\mathcal{U}{-}\mathsf{Alg}(C) \to (\mathcal{T},\mathcal{U}){-}\mathsf{Alg}(C)$$

witnessing that every $\mathcal{U}$-algebra carries the structure of a bifold $(\mathcal{T},\mathcal{U})$-algebra. These results are applicable to the *functional-analytic contexts* [Zbl 1402.46052], which forms the basis for a study of distribution monads via commutants for enriched theories and monads.

§15 defines the concept of a balanced bifold algebra, which is a commutant bifold algebra D whose left and right faces are isomorphic to $\mathsf{Alg}(C)$, where C is the carrier of D. It is established that a commutant bifold algebra is balanced iff its left and right faces are both commutative. An equivalence

$$\mathsf{BalAlg}^{\mathrm{s}}(\mathcal{C}) \simeq \mathsf{BalBAlg}^{\mathrm{s}}(\mathcal{C})$$

between the category of balanced algebras and the category of balanced bifold algebras is established.

§16 gives and develops various examples of commutant bifold algebras and commitants, involving

1. bimodules over pairs of rings;
2. internal rings, rigs (also known as semirings), preordered rings in cartesian closed categories, and internal modules and affine spaces for internal rigs, including convex spaces for preordered rings;
3. semilattices with and without top and/or bottom elements;
4. topological groups, convergence groups, the circle group, and Pontryagin duality, with respect to which it is shown that the circle group $\mathbb{T}$ carries the structure of a commutant bifold algebra in three settings as a consequence of Pontryagin duality
   (a) $\mathbb{T}$ is a balanced bifold algebra in the category of locally compact Hausdorff spaces with respect to the Lawvere theory of abelian groups,
   (b) $\mathbb{T}$ is a balanced bifold algebra with respect to the theory of internal abelian groups in the category of convergence spaces, for the enriched Borceux-Day system of arities,
   (c) $\mathbb{T}$ is a commutant bifold algebra in Set for the infinitary Lawvere-Linton theories of abelian groups and of compact Hausdorff abelian groups;
5. complete lattices with supremum-preserving maps;
6. the abelian group $\mathbb{Z}$ and a theorem of *A. Ehrenfeucht* and *J. Łoś* [Bull. Acad. Pol. Sci., Cl. III 2, 261–263 (1954; Zbl 0055.25304)] on the question of reflexivity of free abelian groups, with respect to which it is shown that is a balanced bifold algebra with respect to the Lawvere-Linton theories of abelian groups and is equivalent to the non-existence of measurable cardinals.

In a subsequent paper [Algebraic dual adjunctions and bifold algebras:enriched algebraic dualization for a system of arities, in preparation], the author will employ the results of this paper to establish biequivalence between certain categories of bifold algebras and certain 2-categories of $\mathcal{J}$-algebraic dual adjunctions, which are adjunctions between pairs of $\mathcal{J}$-algebraic $\mathcal{V}$-categories over $\mathcal{V}$ in the sense of [*R. B. B. Lucyshyn-Wright*, Theory Appl. Categ. 31, 101–137 (2016; Zbl 1337.18002), 12.1], forming the basis for a study of enriched algebraic dualization processes in analysis, order theory and topology.

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MSC:

18C05 Equational categories
18C10 Theories (e.g., algebraic theories), structure, and semantics
18C15 Monads (= standard construction, triple or triad), algebras for monads, homology and derived functors for monads
18C40 Structured objects in a category (group objects, etc.)
18D15 Closed categories (closed monoidal and Cartesian closed categories, etc.)
18D20 Enriched categories (over closed or monoidal categories)
18E99 Categorical algebra
18M05 Monoidal categories, symmetric monoidal categories
08A40 Operations and polynomials in algebraic structures, primal algebras
08A65 Infinitary algebras
08B20 Free algebras
08C05 Categories of algebras

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Lawvere theory; monad; enriched algebraic theory; enriched category; commutant; bifold algebra; tensor product

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