Di Liberti, Ivan; Ramos González, Julia
Exponentiable Grothendieck categories in flat algebraic geometry. (English) Zbl 07526517
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Grothendieck abelian categories play the role of models of possibly noncommutative schemes [*M. Artin* and *J. J. Zhang*, Adv. Math. 109, No. 2, 228–287 (1994; Zbl 0833.14002); *J. T. Stafford* and *M. Van den Bergh*, Bull. Am. Math. Soc., New Ser. 38, No. 2, 171–216 (2001; Zbl 1042.16016; *M. Kontsevich* and *A. L. Rosenberg*, in: The Gelfand Mathematical Seminars, 1996–1999. Dedicated to the memory of Chih-Han Sah. Boston, MA: Birkhäuser. 85–108 (2000; Zbl 1003.14001)], which is motivated by the Gabriel-Rosenberg reconstruction theorem claiming that a quasi-separated scheme can be reconstructed, up to isomorphism of schemes, solely from the abelian category of quasi-coherent sheaves on the scheme, which is a Grothendieck category [https://stacks.math.columbia.edu/]. The theorem was initially established for noetherian schemes by *P. Gabriel* [Bull. Soc. Math. Fr. 90, 323–448 (1962; Zbl 0201.35602)] and generalized to quasi-separated schemes by Rosenberg [file:///C:/Users/user/Downloads/chptr%208-Spectra%20of%20spaces%20represented%20by%20abelian%20categories_with%20title.pdf]. The Gabriel-Popescu theorem [*N. Popesco* and *P. Gabriel*, C. R. Acad. Sci., Paris 258, 4188–4190 (1964; Zbl 0126.03304)] allows of interpreting Grothendieck categories as a linear version of Grothendieck topoi [*W. Lowen*, J. Pure Appl. Algebra 190, No. 1–3, 197–211 (2004; Zbl 1051.18007)].

Different 2-categories obtain, depending on the choice of morphisms.

- Grt the 2-category of Grothendieck categories and left adjoints as morphisms
- $\mathsf{Grt}_\flat$ the 2-category of Grothendieck categories and left exact left adjoints as morphisms

The 2-category $\mathsf{Grt}_\flat$ is the main object of study in this paper. It is shown that $\mathsf{Grt}_\flat$ can be endowed with a monoidal structure, where the exponentiable objects are characterized. From an algebro-geometric standpoint, this can be seen as a contribution to the understanding of exponentiable schemes or Hom-schemes when restricted to the flat case.

The synopsis of the paper goes as follows.

- §2 aims to $\mathsf{Grt}_\flat$ can simulate flat algebraic geometry via a collection of examples.
- §3 shows that the monoidal structure $\boxtimes$ on Grt [*W. Lowen* et al., Int. Math. Res. Not. 2018, No. 21, 6698–6736 (2018; Zbl 1408.18024)] nicely restricts to $\mathsf{Grt}_\flat$, which is easy on the level of objects, but a highly non-trivial task on the level of morphisms. The problem of exponentiability is introduced. It is shown that the category of linear presheaves Mod $(\mathfrak{a})$ are exponentiable (Proposition 3.15).
- §4 investigates the properties of the forgetful functor
$$: \mathsf{Grt}_\flat^\circ \to \mathsf{Cat}_k$$
showing that it is representable (Prposition 4.2).
- §5 introduces and investigates quasi-injective Grothendieck categories (§5.1), continuous linear categories (§5.2) and then connect the two concepts (§5.3). These are technical tools for the main theorem.
- §6 gives the main theorem:

  Theorem 6.1. A Grothendieck category is exponentiable in $\mathsf{Grt}_\flat$ iff it is continuous. In particular, every finitely presentable Grothendieck category is exponentiable.
- §7 is a collecction of examples and instances of the main theorem. The most relevant is the following:

  Proposition 7.2. Let X be a quasi-compact quasi-separated scheme over k, then $\mathsf{Qcoh}\,(X)$ is exponentiable.

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MSC:

14A22	Noncommutative algebraic geometry
14F06	Sheaves in algebraic geometry
18B25	Topoi
18C35	Accessible and locally presentable categories
18F10	Grothendieck topologies and Grothendieck topoi
18M05	Monoidal categories, symmetric monoidal categories
18E10	Abelian categories, Grothendieck categories

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Keywords:

Grothendieck categories; noncommutative algebraic geometry; flatness; monoidal structures; exponentiability; continuous categories; quasi-coherent sheaves

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References:

[1] Anel, Mathieu; Lejay, Damien, Exponentiable higher toposes (2018), Preprint

[2] Adámek, Jiří; Rosický, Jiří, Locally Presentable and Accessible Categories, London Mathematical Society Lecture Note Series (1994), Cambridge University Press · Zbl 0795.18007

[3] Artin, Michael; Grothendieck, Alexander; Verdier, Jean-Louis, (Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos. Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos, Séminaire de Géométrie Algébrique du Bois-Marie 1963-1964 (SGA 4). Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos. Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos, Séminaire de Géométrie Algébrique du Bois-Marie 1963-1964 (SGA 4), Lecture Notes in Mathematics, vol. 269 (1972), Springer-Verlag: Springer-Verlag Berlin-New York), Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat

[4] Artin, Michael; Zhang, James J., Noncommutative projective schemes, Adv. Math., 109, 2, 228-287 (1994) · Zbl 0833.14002

[5] Barr, Michael, Exact categories, (Exact Categories and Categories of Sheaves. Exact Categories and Categories of Sheaves, Lecture Notes in Math., vol. 236 (1971), Springer: Springer Berlin), 1-120 · Zbl 0223.18010

[6] Bird, Gregory J., Limits in 2-categories of locally-presented categories (1984), University of Sidney, PhD thesis

[7] Borceux, Francis, Handbook of Categorical Algebra. 3, Encyclopedia of Mathematics and Its Applications, vol. 52 (1994), Cambridge University Press: Cambridge University Press Cambridge, Categories of sheaves · Zbl 0911.18001

[8] Borceux, Francis; Quinteiro, Carmen, A theory of enriched sheaves, Cah. Topol. Géom. Différ. Catég., 37, 2, 145-162 (1996) · Zbl 0883.18006

[9] Borceux, Francis; Quinteriro, Carmen, Enriched accessible categories, Bull. Aust. Math. Soc., 54, 3, 489-501 (1996) · Zbl 0881.18011

[10] Brandenburg, Martin, Tensor categorical foundations of algebraic geometry (2014), University of Münster, PhD thesis · Zbl 1351.14001

[11] Brandenburg, Martin, Rosenberg's reconstruction theorem, Expo. Math., 36, 1, 98-117 (2018) · Zbl 1415.18004

[12] Brandenburg, Martin, Bicategorical colimits of tensor categories (2020), Preprint

[13] Brandenburg, Martin, Localizations of tensor categories and fiber products of schemes (2020), Preprint

[14] Brandenburg, Martin; Chirvasitu, Alexandru, Tensor functors between categories of quasi-coherent sheaves, J. Algebra, 399, 675-692 (2014) · Zbl 1327.14079

[15] Brandenburg, Martin; Chirvasitu, Alexandru; Johnson-Freyd, Theo, Reflexivity and dualizability in categorified linear algebra, Theory Appl. Categ., 30, 23, 808-835 (2015) · Zbl 1374.18003

[16] Brown, Harrison, Why are flat morphisms flat? MathOverflow, (version: 2009-11-25), visited on 2021-05-10

[17] Chirvasitu, Alex; Johnson-Freyd, Theo, The fundamental pro-groupoid of an affine 2-scheme, Appl. Categ. Struct., 21, 5, 469-522 (2013) · Zbl 1285.18008

[18] Chorny, Boris; Rosický, Jiří, Class-locally presentable and class-accessible categories, J. Pure Appl. Algebra, 216, 10, 2113-2125 (2012) · Zbl 1260.18005

[19] Day, Brian J.; Kelly, Gregory M., On topological quotient maps preserved by pullbacks or products, Proc. Camb. Philos. Soc., 67, 553-558 (1970) · Zbl 0191.20801

[20] Day, Brian J.; Lack, Stephen, Limits of small functors, J. Pure Appl. Algebra, 210, 3, 651-663 (2007) · Zbl 1120.18001

[21] Deligne, Pierre, Catégories tannakiennes, (The Grothendieck Festschrift, Vol. II. The Grothendieck Festschrift, Vol. II, Progr. Math., vol. 87 (1990), Birkhäuser Boston: Birkhäuser Boston Boston, MA), 111-195 · Zbl 0727.14010

[22] Di Liberti, Ivan, General facts on the Scott adjunction, Appl. Categ. Struct. (2022)

[23] Di Liberti, Ivan, The Scott adjunction (2020), Preprint

[24] Gabriel, Pierre, Des catégories abéliennes, Bull. Soc. Math. Fr., 90, 323-448 (1962) · Zbl 0201.35602

[25] Grothendieck, Alexander, Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II, Publ. Math. Inst. Hautes Études Sci., 24, 231 (1965) · Zbl 0135.39701

[26] Grothendieck, Alexander, Techniques de construction et théorèmes d'existence en géométrie algébrique. IV. Les schémas de Hilbert, (Séminaire Bourbaki, Vol. 6, Exp. No. 221 (1995), Soc. Math. France: Soc. Math. France Paris), 249-276

[27] Grothendieck, Alexander; Dieudonné, Jean A., Eléments de géométrie algébrique. I, Grundlehren der Mathematischen Wissenschaften, vol. 166 (1971), Springer-Verlag: Springer-Verlag Berlin · Zbl 0203.23301

[28] Hyland, J. Martin E., Function spaces in the category of locales, (Banaschewski, B.; Hoffmann, R.-E., Proceedings of the Conference on Topological and Categorical Aspects of Continuous Lattices (Workshop IV). Proceedings of the Conference on Topological and Categorical Aspects of Continuous Lattices (Workshop IV), University of Bremen, Bremen, November 9-11, 1979. Proceedings of the Conference on Topological and Categorical Aspects of Continuous Lattices (Workshop IV). Proceedings of the Conference on Topological and Categorical Aspects of Continuous Lattices (Workshop IV), University of Bremen, Bremen, November 9-11, 1979, Lecture Notes in Mathematics, vol. 871 (1981), Springer-Verlag: Springer-Verlag Berlin-New York), 264-281

[29] Johnstone, Peter T., Sketches of an Elephant: A Topos Theory Compendium. Vol. 1, Oxford Logic Guides, vol. 43 (2002), The Clarendon Press, Oxford University Press: The Clarendon Press, Oxford University Press New York · Zbl 1071.18001

[30] Johnstone, Peter T., Sketches of an Elephant: A Topos Theory Compendium. Vol. 2, Oxford Logic Guides, vol. 44 (2002), The Clarendon Press, Oxford University Press: The Clarendon Press, Oxford University Press Oxford · Zbl 1071.18001

[31] Johnstone, Peter T.; Joyal, André, Continuous categories and exponentiable toposes, J. Pure Appl. Algebra, 25, 3, 255-296 (1982) · Zbl 0487.18003

[32] Kelly, Gregory M., Structures defined by finite limits in the enriched context. I, Third Colloquium on Categories, Part VI. Third Colloquium on Categories, Part VI, Amiens, 1980. Third Colloquium on Categories, Part VI. Third Colloquium on Categories, Part VI, Amiens, 1980, Cah. Topol. Géom. Différ. Catég., 23, 1, 3-42 (1982) · Zbl 0538.18006

[33] Kelly, Gregory M., Basic concepts of enriched category theory, Repr. Theory Appl. Categ., 10 (2005), vi+137, Reprint of the 1982 original [Cambridge Univ. Press, Cambridge; MR0651714] · Zbl 1086.18001

[34] Kock, Anders, Monads for which structures are adjoint to units, J. Pure Appl. Algebra, 104, 1, 41-59 (1995) · Zbl 0849.18008

[35] Kontsevich, Maxim; Rosenberg, Alexander L., Noncommutative smooth spaces, (The Gelfand Mathematical Seminars, 1996-1999. The Gelfand Mathematical Seminars, 1996-1999, Gelfand Math. Sem. (2000), Birkhäuser Boston: Birkhäuser Boston Boston, MA), 85-108 · Zbl 1003.14001

[36] Maxim Kontsevich, Alexander L. Rosenberg, Noncommutative spaces and flat descent, MPIM preprint series, 2004.

[37] Krause, Henning, Deriving Auslander's formula, Doc. Math., 20, 669-688 (2015) · Zbl 1348.18018

[38] López Franco, Ignacio, Tensor products of finitely cocomplete and abelian categories, J. Algebra, 396, 207-219 (2013) · Zbl 1305.18042

[39] Lowen, Wendy, A generalization of the Gabriel-Popescu theorem, J. Pure Appl. Algebra, 190, 1-3, 197-211 (2004) · Zbl 1051.18007

[40] Lowen, Wendy, Linearized topologies and deformation theory, Topol. Appl., 200, 176-211 (2016) · Zbl 1342.18024

[41] Lowen, Wendy; Ramos González, Julia; Shoikhet, Boris, On the tensor product of linear sites and Grothendieck categories, Int. Math. Res. Not., 21, 6698-6736 (2018) · Zbl 1408.18024

[42] Lurie, Jacob, Derived algebraic geometry II: noncommutative algebra (2007)

[43] Lurie, Jacob, Higher Topos Theory, Annals of Mathematics Studies, vol. 170 (2009), Princeton University Press: Princeton University Press Princeton, NJ · Zbl 1175.18001

[44] MacLane, Saunders, Categories for the Working Mathematician, Graduate Texts in Mathematics, vol. 5 (1971), Springer-Verlag: Springer-Verlag New York-Berlin · Zbl 0705.18001

[45] Murfet, Daniel, Modules over a scheme (2006)

[46] Pitts, Andrew, On product and change of base for toposes, Cah. Topol. Géom. Différ. Catég., 26, 1, 43-61 (1985) · Zbl 0574.18003

[47] Popesco, Nicolae; Gabriel, Pierre, Caractérisation des catégories abéliennes avec générateurs et limites inductives exactes, C. R. Acad. Sci. Paris, 258, 4188-4190 (1964) · Zbl 0126.03304

[48] Positselski, Leonid E.; Rosický, Jiří, Covers, envelopes, and cotorsion theories in locally presentable abelian categories and contramodule categories, J. Algebra, 483, 83-128 (2017) · Zbl 1402.18011

[49] González, Julia Ramos, Grothendieck categories as a bilocalization of linear sites, Appl. Categ. Struct., 26, 4, 717-745 (2018) · Zbl 1408.18025

[50] González, Julia Ramos, Grothendieck categories and their tensor product as filtered colimits (2020)

[51] Rosenberg, Alexander L., Noncommutative schemes, Compos. Math., 112, 1, 93-125 (1998) · Zbl 0936.14002

[52] Alexander L. Rosenberg, Spectra of 'spaces' represented by abelian categories, MPIM preprint series, 2004.

[53] Schäppi, Daniel, Ind-abelian categories and quasi-coherent sheaves, Math. Proc. Camb. Philos. Soc., 157, 3, 391-423 (2014) · Zbl 1305.18043

[54] Scott, Dana, Continuous lattices, (Toposes, Algebraic Geometry and Logic. Toposes, Algebraic Geometry and Logic, Dalhousie Univ. Halifax 1971. Toposes, Algebraic Geometry and Logic. Toposes, Algebraic Geometry and Logic, Dalhousie Univ. Halifax 1971, Lect. Notes Math., vol. 274 (1972)), 97-136

[55] The Stacks Project Authors, Stacks Project (2021)

[56] Stafford, J. Toby; Van den Bergh, Michel, Noncommutative curves and noncommutative surfaces, Bull. Am. Math. Soc. (N.S.), 38, 2, 171-216 (2001) · Zbl 1042.16016

[57] Street, Ross, Consequences of splitting idempotents (1996)

[58] Wood, Richard J., Some remarks on total categories, J. Algebra, 75, 2, 538-545 (1982) · Zbl 0504.18001

