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**Probability, valuations, hyperspace: three monads on Top and the support as a morphism.**  
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Measure and probability theory have been reformulated in terms of monads in category theory. All probability monads are variations on the distribution monad on [B. Jacobs, *Theor. Comput. Sci.* 412, No. 28, 3323–3336 (2011; [Zbl 1218.18003](#))]. The monad approach has two main features.

1. Conditional probabilities in the sense of Markov kernels arise as Kleisli morphisms [M. Giry, *Lect. Notes Math.* 915, 68–85 (1982; [Zbl 0486.60034](#))];
2. It provides a conceptually simple definition of integration or expectation on all algebras of the monad [<http://paoloperrone.org/phdthesis.pdf>, Chapter 1].

This paper studies three monads on the category Top of topological spaces and continuous maps. The synopsis of the paper goes as follows.

§2 deals with the monad  $H$  of closed subsets, which is a generalization of the *Hoare powerdomain* [A. Schalk, *Algebras for generalized power constructions*. Darmstadt: TH Darmstadt (1993; [Zbl 0900.68275](#)), §6.3]. It is shown that Scott-continuous modular maps from a frame  $\mathcal{O}(X)$  of open sets to  $[0, 1]$  are in canonical bijection with closed sets in  $X$  and that the topology of pointwise convergence for such functionals corresponds to the lower Vietoris topology on the space  $HX$  of closed sets (Proposition 2.1).

§3 deals with the monad  $V$  of continuous valuations, which is also known as the *extended probabilistic powerdomain* [M. Alvarez-Manilla et al., *Theor. Comput. Sci.* 328, No. 3, 221–244 (2004; [Zbl 1071.68058](#))]. It is shown that continuous valuations on a space  $X$  are in canonical bijection with Scott-continuous modular functionals on the space of lower semicontinuous function on  $X$  (Theorem 3.5).

§4 deals with the monad  $P$  of  $\tau$ -smooth Borel probability measures, which is shown to be a submonad of  $V$ . Its restriction to the subcategory of compact Hausdorff spaces is the Radon monad [K. Keimel, *Topology Appl.* 156, No. 2, 227–239 (2008; [Zbl 1161.46045](#)); T. Świrszcz, *Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys.* 22, 39–42 (1974; [Zbl 0276.46036](#))].

§5 defines the support of a continuous valuation, establishing that the operation of taking the support is a morphism of monads from  $V$  to  $H$ .

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#### MSC:

- 18C15** Monads (= standard construction, triple or triad), algebras for monads, homology and derived functors for monads  
**18F60** Categories of topological spaces and continuous mappings

#### Keywords:

monads on topological spaces; valuations; Borel measures; probability measures; hyperspaces; morphism of monads

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