Uustalu, Tarmo; **Veltri, Niccolò**; **Zeilberger, Noam**

The sequent calculus of skew monoidal categories. (English) Zbl 07440912

Casadio, Claudia (ed.) et al., Joachim Lambek: the interplay of mathematics, logic, and linguistics. Cham: Springer. Outst. Contrib. Log. 20, 377-406 (2021)

Skew monoidal categories are a variation of monoidal categories in which the unit and associativity transformations are no longer required to be natural isomorphisms but merely natural transformations in a certain direction. They first arose in [*K. Szlachányi*, Adv. Math. 231, No. 3–4, 1694–1730 (2012; Zbl 1283.18006)], being introduced to deal with bialgebroids. The authors construct free skew monoidal categories using an appropriate Gentzen sequent calculus, analyzing the decision word problem. The techniques they use are inspired by linear logic proof search [*J.-M. Andreoli*, J. Log. Comput. 2, No. 3, 297–347 (1992; Zbl 0764.03020)], based on the technique of *focusing*, having a distinguished *stoup* position in the antecedents of sequents and adaption of sequent calculus rules of inference to accommodate stoups. The resultant calculus is shown to be sound and complete with respect to the existence of maps in the free skew monoidal category. By setting up an appropriate equivalence relation on proofs and associated rewriting machinery, the authors can pick canonical representatives of each equivalence class of proofs, solving the coherence problem. Finally, the authors compare their work with [*J. Bourke*, J. Homotopy Relat. Struct. 12, No. 1, 31–81 (2017; Zbl 1417.18001); *J. Bourke* and *S. Lack*, J. Pure Appl. Algebra 222, No. 10, 3255–3281 (2018; Zbl 1428.18025); J. Algebra 506, 237–266 (2018; Zbl 1401.18019)]. A detailed review of this chapter can be seen in [Zbl 07440912].

The theory of skew monoidal categories has been investigated in an extensive series of papers by *J. Bourke* [J. Homotopy Relat. Struct. 12, No. 1, 31–81 (2017; Zbl 1417.18001)], *J. Bourke* and *S. Lack* [J. Pure Appl. Algebra 222, No. 10, 3255–3281 (2018; Zbl 1428.18025); J. Algebra 506, 237–266 (2018; Zbl 1401.18019); Theory Appl. Categ. 35, 19–63 (2020; Zbl 1431.18012)], *S. Lack* and *R. Street* [Theory Appl. Categ. 26, 385–402 (2012; Zbl 1252.18016); Adv. Math. 258, 351–396 (2014; Zbl 1350.18012); Appl. Categ. Struct. 22, No. 5–6, 789–803 (2014; Zbl 1317.18012); Theory Appl. Categ. 30, 985–1000 (2015; Zbl 1331.18007)], *R. Street* [J. Pure Appl. Algebra 217, No. 6, 973–988 (2013; Zbl 1365.18008)], with which the authors of this paper compare their work, discussing the sense in which Lambek's language of multicategories provides a better understanding of the proof-theoretic analysis given here. Finally, the authors formalize this development in the dependently typed programming language Agda.

For the entire collection see [Zbl 1470.03008].

Reviewer: Hirokazu Nishimura (Tsukuba)

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