MAGNETIC JACOBI FIELDS IN SASAKIAN SPACE FORMS

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Abstract. Typical examples of uniform magnetic fields are Kähler magnetic fields on Kähler manifolds. It is very difficult to study magnetic Jacobi fields of non-uniform magnetic fields in an arbitrary Riemannian manifold. The canonical magnetic fields of Sasakian manifolds are non-uniform but exact. In this paper we completely determine magnetic Jacobi fields on Sasakian space forms of dimension greater or equal to 5.

1. Introduction

A magnetic curve on a Riemannian manifold (M, g) is a solution of the Lorentz equation

$$\nabla_{\gamma'}\gamma' = q\phi\gamma', \tag{1.1}$$

where ϕ is a $(1, 1)$ tensor field obtained from a closed 2-form F on M, q is a real number and ∇ is the Levi-Civita connection defined by g.

The study of magnetic curves is an active field of mathematics and mathematical physics as well, for several reasons:

- The original problem stems from the study of motion of a charged particle in a static magnetic field in $\mathbb{E}^3$.
- Magnetic curves obviously generalize geodesics; when the *magnetic field* F is absent, the *Lorentz force* ϕ is zero, and hence the Lorentz equation (1.1) reduces to the equation of geodesics.
- Magnetic curves are solutions of a variational problem; more precisely, they are critical points of the LH functional (on $C^\infty([a, b])$)

$$\mathtt{LH}(\gamma) = \mathtt{E}(\gamma) - q\int_a^b A(\gamma'(s))ds, \tag{1.2}$$

  where $\mathtt{E}(\gamma) = \displaystyle\int_a^b \frac{1}{2}g(\gamma'(s), \gamma'(s))ds$ is the Dirichlet energy of γ and A is the potential 1-form generating the magnetic field F.

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A second variational formula for the integral $\mathtt{LH}$ leads to the concept of *magnetic Jacobi field*. Maybe the first researchers who investigated Jacobi fields along magnetic trajectories on a Riemannian manifold were Gouda [11] and Paternain & Paternain [19]. See also [1].

In our previous papers we investigated several geometric properties of magnetic curves in almost contact metric manifolds when the magnetic field is given by the fundamental 2-form. We have been interested in finding the maximum osculating order of such a curve, in obtaining some reduction results for the codimension of these curves and to get conditions under which the magnetic curves are periodic. See e.g. [8, 17, 14, 16, 18] as well as [6, 13].

Detailed study on magnetic Jacobi fields gives us insight on how small variations affect the evolution of magnetic curves. One needs to point out that Adachi [1] and Gouda [12] studied magnetic Jacobi fields with respect to uniform magnetic fields, that is when the Lorentz force is parallel, i.e. $\nabla\phi = 0$. The parallelism is determinative in their study. However, when the ambient space is a Sasakian manifold, the Lorentz force comes naturally obtained from the contact magnetic field $F = -\Omega = -d\eta$ and hence $\phi = \varphi$. Therefore, due to (3.2), the Lorentz force is no longer parallel and hence the magnetic field is not uniform.

The study of Jacobi fields on geodesics in Sasakian space forms is done in [3, 4], where the Jacobi equation is completely solved. When the magnetic field is involved, the problem becomes more complicated.

In a very recent paper [15] we obtained all magnetic Jacobi fields along contact magnetic curves on 3-dimensional Sasakian space forms. In particular, we constructed explicit examples of magnetic Jacobi fields on the unit 3-sphere $\mathbb{S}^3$, on the Heisenberg group Nil_3 and on $\mathrm{SL}_2\mathbb{R}$, respectively.

In this paper we continue the study initiated in [15] and we investigate magnetic Jacobi fields in Sasakian space forms of dimension greater or equal to 5.

2. Preliminaries

On a Riemannian manifold (M, g) with a closed 2-form F (regarded as a static magnetic field), the Lorentz force ϕ derived from F is an endomorphism field defined by

$$g(\phi X, Y) = F(X, Y).$$

The *magnetic trajectory* of F is a curve γ satisfying the Lorentz equation (1.1), where the constant q is called the *charge*. One can see that every magnetic trajectory has constant speed. Unit speed magnetic curves are called *normal magnetic curves*.

We have already pointed out that magnetic curves are critical points of the $\mathtt{LH}$ functional (1.2), where the magnetic field F is obtained from the potential 1-form A, that is $F = 2dA$. Gouda obtained in [11] the second variational formula of $\mathtt{LH}$:

$$\frac{d^2}{d\varepsilon^2}\bigg|_{\varepsilon=0} \mathtt{LH}(\gamma_\varepsilon) = -\int_a^b g(\mathcal{J}_{q,F}(W(s)), V(s))ds,$$

where V and W are variational vector fields along $\gamma(s)$ in a 2-parameter variation. The operator $\mathcal{J}_{q,F}$ acts on the space $\Gamma(\gamma^*TM)$ of all vector fields along γ and it is defined by

$$\mathcal{J}_{q,F}(W) = \nabla_{\gamma'}\nabla_{\gamma'}W + R(W,\gamma')\gamma' - q\phi(\nabla_{\gamma'}W) - q(\nabla_W\phi)\gamma'.$$

We call the operator $\mathcal{J}_{q,F}$ the *magnetic Jacobi operator* derived from the magnetic field F.

A vector field $W(s)$ along γ is said to be a *magnetic Jacobi field* if it satisfies $\mathcal{J}_{q,F}(W) = 0$.

Proposition 2.1. ([11]) On a magnetic curve $\gamma(s)$, the velocity $\gamma'(s)$ is a magnetic Jacobi field along $\gamma(s)$.

Proof. Let $\gamma(s)$ be a magnetic curve and choose $W(s) = \gamma'(s)$. Since γ is a solution of the Lorentz equation (1.1), we have

$$\begin{aligned}\mathcal{J}_{q,-d\eta}(\gamma') =& \nabla_{\gamma'}\nabla_{\gamma'}\gamma' + R(\gamma',\gamma')\gamma' - q\phi(\nabla_{\gamma'}\gamma') - q(\nabla_{\gamma'}\phi)\gamma' \\ =& \nabla_{\gamma'}(q\phi\gamma') - q\phi(\nabla_{\gamma'}\gamma') - q(\nabla_{\gamma'}\phi)\gamma' = 0.\end{aligned}$$

□

Let us give now some motivation for our investigation.

Proposition 2.2. Let (M,φ,ξ,η,g) be a Sasakian manifold and let W be a vector field on M such that $\mathcal{L}_\xi W = p\xi$, where $p \in \mathbb{R}$. Here, by $\mathcal{L}$ we denote the Lie differentiation. Then W is a magnetic Jacobi field along any integral curve of ξ (thought as a magnetic curve with arbitrary strength q), i.e. $\mathcal{J}_{q,-d\eta}(W) = 0$.

Proof. Since $[\xi, W] = \nabla_\xi W - \nabla_W \xi$ we easily get $\nabla_\xi W = p\xi - \varphi W$. A second covariant derivative along ξ leads to $\nabla_\xi\nabla_\xi W = -W + \eta(W)\xi$. Hence $\mathcal{J}_{q,-d\eta}(W) = 0$ for any q. □

Remark 2.1. The condition $\mathcal{L}_\xi W = p\xi$ is not artificial; for example it occurs when the Sasakian manifold M is a non-trivial (i.e. non-Einstein) Ricci soliton. See e.g. [10, 9].

Remark 2.2. The notion of Jacobi field (along geodesics) was extended also in other direction. A vector field W on a Riemannain manifold (M,g) is called a *Jacobi-type vector field* if it satisfies the equation

$$\nabla_X\nabla_X W - \nabla_{\nabla_X X}W + R(W,X)X = 0, \quad \text{for any} \quad X \in \mathfrak{X}(M).$$

In this regard, it is proved that any Killing vector field on M is a Jacobi-type vector field. The converse is not true, that is there are Jacobi-type vector fields that are not Killing vector fields. See [7]. Moreover, in the same paper [7], the author provides a sufficient condition for a Ricci soliton to be an Einstein manifold. This condition is that the potential field is a Jacobi-type vector field.

Remark 2.3. In [9] the author states that the soliton vector of a non-Einstein Sasakian manifold is a Jacobi field along the integral curves (geodesics) of the Reeb vector field ξ. The statement given in Proposition 2.2 extends this result to arbitrary q. This is a good motivation to study magnetic Jacobi fields in Sasakian manifolds.

3. Magnetic Jacobi fields in Sasakian space forms

Our aim is to study magnetic Jacobi fields along contact magnetic curves in Sasakian space forms of arbitrary dimension.

3.1. **Sasakian space forms.**
Let us recall some notions in the (almost) contact (metric) geometry.

A (φ, ξ, η) *structure* on a manifold M is defined by

- a field φ of endomorphisms of tangent spaces,
- a vector field ξ and
- a 1-form η

satisfying

$$\eta(\xi) = 1, \quad \varphi^2 = -\mathrm{I} + \eta \otimes \xi, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0.$$

When (M, φ, ξ, η) is endowed with a compatible Riemannian metric g, namely

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \text{ for all } X, Y \in \mathfrak{X}(M),$$

then M is said to have an *almost contact metric structure*, and $(M, \varphi, \xi, \eta, g)$ is called an *almost contact metric manifold*.

As an immediate consequence we have that ξ is unitary and $\eta(X) = g(\xi, X)$, $\forall X \in \mathfrak{X}(M)$.

We define a 2-form Ω on $(M, \varphi, \xi, \eta, g)$ by

$$\Omega(X, Y) = g(X, \varphi Y), \text{ for all } X, Y \in \mathfrak{X}(M), \tag{3.1}$$

called *the fundamental* 2*-form* of the almost contact metric structure (φ, ξ, η, g).

If $\Omega = d\eta$, then $(M, \varphi, \xi, \eta, g)$ is called a *contact metric manifold*. Here $d\eta$ is defined by $d\eta(X, Y) = \frac{1}{2}\big(X\eta(Y) - Y\eta(X) - \eta([X, Y])\big)$, for any $X, Y \in \mathfrak{X}(M)$. On a contact metric manifold M, the 1-form η is a contact form (see Blair's book [2]). The vector field ξ is called the *Reeb vector field* of M and it is characterized by $\iota_\xi \eta = 1$ and $\iota_\xi d\eta = 0$. Here ι denotes the interior product.

An almost contact metric manifold M is said to be *normal* if the normality tensor $S(X, Y) = N_\varphi(X, Y) + 2d\eta(X, Y)\xi$ vanishes, where N_φ is the *Nijenhuis torsion* of φ defined by $N_\varphi(X, Y) = [\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]$, for any $X, Y \in \mathfrak{X}(M)$.

A *Sasakian manifold* is defined as a normal contact metric manifold. Denote by ∇ the Levi-Civita connection associated to g and adopt the definition given in [2] to characterize the Sasakian manifold $(M, \varphi, \xi, \eta, g)$ by the equation

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \quad \text{for any } \ X, Y \in \mathfrak{X}(M). \tag{3.2}$$

A straightforward consequence, but very useful in calculations, is the next formula

$$\nabla_X \xi = -\varphi X, \ \forall X \in \mathfrak{X}(M). \tag{3.3}$$

A plane section Π at $p \in M^{2n+1}$ is called a φ-section if it is invariant under φ_p. The sectional curvature $K(\Pi)$ of a φ-section is called the *φ-sectional curvature* of M^{2n+1} at p. A Sasakian manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ is said to be a *Sasakian space form* if it has constant φ-sectional curvature. Sasakian space forms are invariant under *D-homothetic deformations*. See [2]. Therefore, we emphasize that for every value of c there exist Sasakian space forms, as follows: the elliptic Sasakian space forms, also known as the *Berger spheres* if $c > -3$, the *Heisenberg space* $\mathbb{R}^{2n+1}(-3)$, if $c = -3$, and $B^{2n} \times \mathbb{R}$ when $c < -3$. In the third model B^{2n} is complex space form of negative holomorphic sectional curvature. See, for further details, [2, Theorem 7.15]. Note that the case $c > -3$ includes the standard unit sphere $\mathbb{S}^{2n+1}(1)$.

In the following we give some formulas for the curvature of a Sasakian manifold $(M, \varphi, \xi, \eta, g)$. If R denotes the curvature tensor of type $(1, 3)$ defined by $R(X, Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$ we have:

- If M is an arbitrary Sasakian manifold, then
$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \quad R(X, Y)\xi = -(\nabla_X\varphi)Y + (\nabla_Y\varphi)X.$$
- If M is a Sasakian space form of constant holomorphic sectional curvature c and $\dim M \geq 5$, then
$$\begin{aligned} R(X,Y)Z =& \frac{c+3}{4}(g(Y,Z)X - g(Z,X)Y) \\ &+ \frac{c-1}{4}\{\eta(Z)\eta(X)Y - \eta(Y)\eta(Z)X \\ &+ g(Z,X)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ &- g(Y,\varphi Z)\varphi X - g(Z,\varphi X)\varphi Y + 2g(X,\varphi Y)\varphi Z\}. \end{aligned}$$

3.2. **Magnetic Jacobi fields.** On a Sasakian manifold M, the contact form η naturally defines a magnetic field $F = -d\eta$ and so, the Lorentz force $\phi = \varphi$. A curve $\gamma : I \subseteq \mathbb{R} \to M$ is called a *contact magnetic curve* on M if it is a solution of the Lorentz equation
$$\nabla_{\gamma'}\gamma' = q\varphi\gamma'.$$
Note that the curve γ is parametrized by the arclength.

Let now W be a magnetic vector field along the contact magnetic curve γ.

① Start with the easiest case when γ is an integral curve of ξ, case in which γ is a geodesic on M. Then, W is a magnetic Jacobi field along γ if and only if it satisfies the equation
$$\nabla_\xi\nabla_\xi W + R(W, \xi)\xi - q\varphi(\nabla_\xi W) - q(\nabla_W\varphi)\xi = 0. \tag{3.4}$$
Since M is a Sasakian space form, the equation (3.4) becomes
$$\nabla_\xi\nabla_\xi W + (1 + q)\big(W - \eta(W)\xi\big) - q\varphi(\nabla_\xi W) = 0. \tag{3.5}$$
Note that we can take an orthonormal basis along $\gamma(s) = \mathrm{Exp}(s\xi)$ of the following form:
$$\{\xi(\gamma(s)), E_a(s), \varphi E_a(s)\}_{a=1,\dots,n}$$
and such that it is parallel. This can be done by virtue of the property $\nabla_\xi\varphi = 0$.

Decompose $W(s)$ with respect to this frame as
$$W(s) = f(s)\xi(\gamma(s)) + \sum_{a=1}^{n}\big[\alpha_a(s)E_a(s) + \beta_a(s)\varphi E_a(s)\big],$$

where f, α_a and β_a are smooth functions on I.

The equation (3.5) leads to the following system of differential equations:

$$f'' = 0 \tag{3.6a}$$

$$\alpha_a'' + (q+1)\alpha_a + q\beta_a' = 0 \tag{3.6b}$$

$$\beta_a'' + (q+1)\beta_a - q\alpha_a' = 0, \quad a = 1, \ldots, n. \tag{3.6c}$$

Our aim is to obtain explicit solutions for this system. Some special situations, which are analogue to those in dimension 3 (see [15]), are briefly described in the next few lines.

Case $\mathbf{q = 0}$. This case corresponds to Jacobi fields along geodesics. We obtain easily the solution

$$\begin{aligned} f(s) &= f_1 s + f_0 \\ \alpha_a(s) &= A_a \cos s + B_a \sin s \\ \beta_a(s) &= \bar{A}_a \cos s + \bar{B}_a \sin s, \end{aligned} \tag{3.7}$$

where f_0, f_1, A_a, B_a, $\bar{A}_a$ and $\bar{B}_a$ are real constant which can be obtained from the initial conditions. See also [5].

Case $\mathbf{q = -1}$. The solution of the system (3.6) is given by

$$\begin{aligned} f(s) &= f_1 s + f_0 \\ \alpha_a(s) &= C_a + A_a \cos s + B_a \sin s \\ \beta_a(s) &= D_a + B_a \cos s - A_a \sin s, \end{aligned} \tag{3.8}$$

where f_0, f_1, A_a, B_a, C_a and D_a are real constants.

Case $\mathbf{q = -2}$. The solution of the system (3.6) is given by

$$\begin{aligned} f(s) &= f_1 s + f_0 \\ \alpha_a(s) &= (A_a + \bar{A}_a s) \cos s + (B_a + \bar{B}_a s) \sin s \\ \beta_a(s) &= (B_a + \bar{B}_a s) \cos s - (A_a + \bar{A}_a s) \sin s, \end{aligned} \tag{3.9}$$

where f_0, f_1, A_a, $\bar{A}_a$, B_a and $\bar{B}_a$ are real constants.

Case $\mathbf{q \notin \{-2, -1, 0\}}$. The first equation (3.6a) yields immediately

$$f(s) = f_1 s + f_0, \tag{3.10}$$

where f_0 and f_1 are real constants. So, we analyze the next $2n$ differential equations.

From (3.6b) we get $\beta_a' = -\frac{\alpha_a'' + (q+1)\alpha_a}{q}$. In (3.6c) we take the derivative and then we substitute the expression of β_a', previously obtained. Thus, we obtain a fourth order differential equation in α_a, namely

$$\alpha_a^{(4)} + (q^2 + 2q + 2)\alpha_a'' + (q+1)^2 \alpha_a = 0.$$

The general solution of this equation is

$$\alpha_a(s) = A_a \cos s + \bar{A}_a \sin s + B_a \cos(q+1)s + \bar{B}_a \sin(q+1)s, \tag{3.11}$$

where A_a, $\bar{A}_a$, B_a and $\bar{B}_a$ are constants. Going back to the system (3.6) we find

$$\beta_a(s) = \bar{A}_a \cos s - A_a \sin s - \bar{B}_a \cos(q+1)s + B_a \sin(q+1)s. \tag{3.12}$$

It is not clear for us if the two very particular situations $q = -1$ and $q = -2$ have any geometrical or physical interpretation. Moreover, we have to point out that when γ is an integral curve of ξ, it is not necessary assuming M to be a Sasakian space form. Indeed, the equation (3.5) holds in any Sasakian manifold.

② Next, let γ be a non-geodesic magnetic curve on M such that $\gamma'(s)$ and $\xi(\gamma(s))$ are not collinear. Denote by θ the contact angle, that is $\cos\theta = \eta(\gamma')$. It is known, see e.g. [8], that θ is constant.

Remark 3.1. The 3-space $\{\xi(\gamma(s)), \gamma'(s), \varphi\gamma'(s)\}$ is parallel along $\gamma(s)$. Moreover, if $E(s)$ is parallel along γ and orthogonal to $\xi(\gamma(s))$ then $\varphi E(s)$ is also parallel.

Proof. To show the second part of the statement we compute

$$\nabla_{\gamma'}(\varphi E) = (\nabla_{\gamma'}\varphi)E + \varphi \underbrace{\nabla_{\gamma'}E}_{=0} = g(\gamma', E)\xi - \eta(E)\gamma' = 0.$$

□

In order to sustain the first part of the previous remark we give the following proposition.

Proposition 3.1. *The following vector fields along γ are parallel:*

(i) $E_1(s) = f_1\xi + A_1\gamma' + B_1\varphi\gamma'$, *where*

$$f_1(s) = \tfrac{q(1+q\cos\theta)}{k^2}(1-\cos ks)$$
$$A_1(s) = 1 + \tfrac{q(q+\cos\theta)}{k^2}(\cos ks - 1)$$
$$B_1(s) = -\tfrac{q}{k}\sin ks;$$

(ii) $E_{2n}(s) = f_{2n}\xi + A_{2n}\gamma' + B_{2n}\varphi\gamma'$, *where*

$$f_{2n}(s) = \tfrac{1}{\sin\theta} + \tfrac{(1+q\cos\theta)^2}{k^2\sin\theta}(\cos ks - 1)$$
$$A_{2n}(s) = -\cot\theta + \tfrac{(q+\cos\theta)(1+q\cos\theta)}{k^2\sin\theta}(1-\cos ks)$$
$$B_{2n}(s) = \tfrac{1+q\cos\theta}{k\sin\theta}\sin ks;$$

(iii) $E_{2n+1}(s) = f_{2n+1}\xi + A_{2n+1}\gamma' + B_{2n+1}\varphi\gamma'$, *where*

$$f_{2n+1}(s) = -\tfrac{1+q\cos\theta}{k\sin\theta}\sin ks$$
$$A_{2n+1}(s) = \tfrac{q+\cos\theta}{k\sin\theta}\sin ks$$
$$B_{2n+1}(s) = \tfrac{1}{\sin\theta}\cos ks.$$

Here $k = \sqrt{(q+\cos\theta)^2 + \sin^2\theta}$.

Set, at some point p

$$e_1 = \gamma', \quad e_{2n} = \frac{1}{\sin\theta}\left(\xi - \cos\theta\gamma'\right) \quad \text{and} \quad e_{2n+1} = \frac{1}{\sin\theta}\varphi\gamma'$$

and $e_2, \ldots, e_{2n-1}$ such that they are unitary, mutually orthogonal and $e_{a+n-1} = \varphi e_a$, for $a = 2, \ldots, n$.

We will consider the parallel transport of the vectors $e_2, \ldots, e_{2n-1}$ along γ to obtain $E_a(s), \varphi E_a(s)$ for $a = 2, \ldots, n$ and hence to have a basis at $\gamma(s)$. This idea arises from [4].

Now a Jacobi magnetic vector field $W(s)$ along γ may be decomposed as

$$W(s) = f(s)\xi(\gamma(s)) + A(s)\gamma'(s) + B(s)\varphi\gamma'(s) + \sum_{a=2}^{n}\left[\alpha_a E_a + \beta_a \varphi E_a\right], \tag{3.13}$$

where f, A, B, α_a and β_a are smooth functions on I.

The equation $J_{q,-d\eta}(W) = 0$ implies

$$\begin{aligned} &(2B' + qB'\cos\theta + f'')\,\xi + (A'' - 2B'\cos\theta - qB')\,\gamma' \\ &+\Big(B'' + qA' + (c-1)B\sin^2\theta - 2f'\Big)\varphi\gamma' \\ &+\sum_{a=2}^{n}\left(\alpha_a'' + \ell\alpha_a + q\beta_a'\right)E_a + \sum_{a=2}^{n}\left(\beta_a'' + \ell\beta_a - q\alpha_a'\right)\varphi E_a = 0, \end{aligned} \tag{3.14}$$

where $\ell = \frac{(c+3)-(c-1)\cos^2\theta}{4} + q\cos\theta$. Note that ℓ is the analogue of $\frac{l}{4}$ from the paper [5]. We obtain the following system of differential equations

$$f'' + (2 + q\cos\theta)B' = 0, \tag{3.15a}$$

$$A'' - (q + 2\cos\theta)B' = 0, \tag{3.15b}$$

$$B'' + qA' + (c-1)B\sin^2\theta - 2f' = 0, \tag{3.15c}$$

$$\alpha_a'' + \ell\alpha_a + q\beta_a' = 0, \tag{3.15d}$$

$$\beta_a'' + \ell\beta_a - q\alpha_a' = 0. \tag{3.15e}$$

We integrate the first two equations, (3.15a) and (3.15b) respectively, that is, there exist two constants λ_1 and λ_2 such that

$$f' = \lambda_1 - (2 + q\cos\theta)B \quad \text{and} \quad A' = \lambda_2 + (q + 2\cos\theta)B. \tag{3.16}$$

From (3.16) and (3.15c) we find a second order ODE in B:

$$B''(s) + \mu B(s) + \lambda_0 = 0, \tag{3.17}$$

where $\mu = q^2 + 4q\cos\theta + 4 + (c-1)\sin^2\theta$ and $\lambda_0 = q\lambda_2 - 2\lambda_1$.

The solution of the equation (3.17) can be described, briefly, as follows:

Case $\mu = 0$. We obtain

$$B(s) = -\frac{\lambda_0}{2}s^2 + B_0 s + \bar{B}_0, \tag{3.18}$$

where $B_0, \bar{B}_0 \in \mathbb{R}$.

Using (3.16) we get

$$\begin{aligned} f(s) &= \lambda_1 s - (2+q\cos\theta)\left(-\tfrac{\lambda_0}{6}s^3 + \tfrac{B_0}{2}s^2 + \bar{B}_0 s\right) + f_0 \\ A(s) &= \lambda_2 s + (q+2\cos\theta)\left(-\tfrac{\lambda_0}{6}s^3 + \tfrac{B_0}{2}s^2 + \bar{B}_0 s\right) + A_0, \end{aligned} \tag{3.19}$$

where $f_0, A_0 \in \mathbb{R}$.

Case $\mu > 0$. Set $k = \sqrt{\mu}$. The solution for the differential equation (3.17) is

$$B(s) = B_0 \cos ks + \bar{B}_0 \sin ks - \frac{\lambda_0}{\mu}, \tag{3.20}$$

where $B_0,\ \bar{B}_0 \in \mathbb{R}$.

Using (3.16) we get

$$\begin{aligned} f(s) &= \lambda_1 s - (2+q\cos\theta)\left(\tfrac{B_0}{k}\sin ks - \tfrac{\bar{B}_0}{k}\cos ks - \tfrac{\lambda_0}{\mu}s\right) + f_0 \\ A(s) &= \lambda_2 s + (q+2\cos\theta)\left(\tfrac{B_0}{k}\sin ks - \tfrac{\bar{B}_0}{k}\cos ks - \tfrac{\lambda_0}{\mu}s\right) + A_0, \end{aligned} \tag{3.21}$$

where $f_0,\ A_0 \in \mathbb{R}$.

Case $\mu < 0$. Set $k = \sqrt{-\mu}$. The solution for the differential equation (3.17) is

$$B(s) = B_0 \cosh ks + \bar{B}_0 \sinh ks - \frac{\lambda_0}{\mu}, \tag{3.22}$$

where $B_0,\ \bar{B}_0 \in \mathbb{R}$.

Using (3.16) we get

$$\begin{aligned} f(s) &= \lambda_1 s - (2+q\cos\theta)\left(\tfrac{B_0}{k}\sinh ks + \tfrac{\bar{B}_0}{k}\cosh ks - \tfrac{\lambda_0}{\mu}s\right) + f_0 \\ A(s) &= \lambda_2 s + (q+2\cos\theta)\left(\tfrac{B_0}{k}\sinh ks + \tfrac{\bar{B}_0}{k}\cosh ks - \tfrac{\lambda_0}{\mu}s\right) + A_0, \end{aligned} \tag{3.23}$$

where f_0 and $A_0 \in \mathbb{R}$.

Obviously, all the constants that appeared above can be obtained from the initial conditions.

We still have to analyze the differential equations (3.15d) and (3.15e). In order to get the solution $\{\alpha_a, \beta_a\}$ we adopt the same strategy as we did for (3.6b) and (3.6c).

First we obtain $\beta_a' = -\frac{1}{q}\left(\alpha_a'' + \ell\alpha_a\right)$. Then take the derivative in (3.15e). Combining the two results we get a fourth order differential equation in α_a, namely

$$\alpha_a^{(4)} + (2\ell + q^2)\alpha_a'' + \ell^2\alpha_a = 0. \tag{3.24}$$

The associated algebraic equation is $t^4 + (2\ell + q^2)t^2 + \ell^2 = 0$. Compute $\Delta_{t^2} = q^2(q^2+4\ell) = q^2\mu$.

We will distinguish (again) the three cases according to the sign of μ.

Case $\mu = 0$. We get

$$\begin{aligned} \alpha_a(s) &= \ (A_a + \bar{A}_a s)\cos\tfrac{qs}{2} + (B_a + \bar{B}_a s)\sin\tfrac{qs}{2} \\ \beta_a(s) &= -(B_a + \bar{B}_a s)\cos\tfrac{qs}{2} + (A_a + \bar{A}_a s)\sin\tfrac{qs}{2}, \end{aligned} \tag{3.25}$$

where $A_a, \bar{A}_a, B_a$ and $\bar{B}_a$ are constants.

Case $\mu > 0$.

Subcase $\ell = 0$.
We obtain

$$\begin{aligned} \alpha_a(s) &= C_a + A_a \cos qs + B_a \sin qs \\ \beta_a(s) &= D_a - B_a \cos qs + A_a \sin qs, \end{aligned} \tag{3.26}$$

where A_a, B_a, C_a and D_a are real constants.

Subcase $\ell \neq 0$.
The solutions of the algebraic equation in t are $t_{1,2} = \pm\frac{|k+q|\sqrt{-1}}{2}$ and $t_{3,4} = \pm\frac{|k-q|\sqrt{-1}}{2}$.
Hence we obtain

$$\begin{aligned} \alpha_a(s) &= A_a \cos \tfrac{k+q}{2}s + \bar{A}_a \sin \tfrac{k+q}{2}s + B_a \cos \tfrac{k-q}{2}s + \bar{B}_a \sin \tfrac{k-q}{2}s \\ \beta_a(s) &= A_a \sin \tfrac{k+q}{2}s - \bar{A}_a \cos \tfrac{k+q}{2}s - B_a \sin \tfrac{k-q}{2}s + \bar{B}_a \cos \tfrac{k-q}{2}s, \end{aligned} \tag{3.27}$$

where $A_a, \bar{A}_a$, B_a and $\bar{B}_a$ are constants.

Case $\mu < 0$.

The solutions of the algebraic equation in t are $t_{1,2} = \frac{k\pm q\sqrt{-1}}{2}$ and $t_{3,4} = \frac{-k\pm q\sqrt{-1}}{2}$. We obtain

$$\begin{aligned} \alpha_a(s) &= \left(A_a \cos \tfrac{qs}{2} + \bar{A}_a \sin \tfrac{qs}{2}\right) \cosh \tfrac{ks}{2} + \left(B_a \cos \tfrac{qs}{2} + \bar{B}_a \sin \tfrac{qs}{2}\right) \sinh \tfrac{ks}{2} \\ \beta_a(s) &= \left(A_a \sin \tfrac{qs}{2} - \bar{A}_a \cos \tfrac{qs}{2}\right) \cosh \tfrac{ks}{2} + \left(B_a \sin \tfrac{qs}{2} - \bar{B}_a \cos \tfrac{qs}{2}\right) \sinh \tfrac{ks}{2}, \end{aligned} \tag{3.28}$$

where $A_a, \bar{A}_a$, B_a and $\bar{B}_a$ are constants.

Remark 3.2. We have pointed out that $\mu = (q + 2\cos\theta)^2 + (c+3)\sin^2\theta$, which implies:

- If $c > -3$ then $\mu > 0$ (since γ' and ξ are not collinear).
- If $c = -3$ then $\mu \geq 0$. The equality holds if and only if the slant angle of the magnetic curve and the strength are related by $q + 2\cos\theta = 0$.
- If $c < -3$ then the sign of μ depends on the strength. More precisely, there exist $q_1 < q_2$ such that
  - $\mu > 0$ if $q < q_1$ or $q > q_2$;
  - $\mu = 0$ if $q = q_1$ or $q = q_2$;
  - $\mu < 0$ if $q \in (q_1, q_2)$.

To conclude, we note that the condition $q + 2\cos\theta = 0$, that appears in the case $c = -3$, can be geometrically interpreted saying that the curve is a geodesic with respect to the Tanaka-Webster connection. Indeed, on a Sasakian manifold, the Tanaka-Webster connection $\bar{\nabla}$ is given by ([2])

$$\bar{\nabla}_X Y = \nabla_X Y + g(X, \varphi Y)\xi + \eta(X)\varphi Y + \eta(Y)\varphi X.$$

Consequently, we find

$$\bar{\nabla}_{\gamma'}\gamma' = q\varphi\gamma' + 2\eta(\gamma')\varphi\gamma' = (q + 2\cos\theta)\varphi\gamma'.$$

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