KILLING SUBMERSIONS AND MAGNETIC CURVES

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Abstract. We investigate magnetic curves in Killing submersions and we show that the bundle curvature is constant along magnetic curves with respect to the Killing vector field if and only if all vertical tubes derived from these magnetic curves are of constant mean curvature. Then we examine magnetic Jacobi fields along horizontal Killing magnetic curves in the total space of a Killing submersion. We generalize some important results obtained by O'Neill for horizontal geodesics to horizontal magnetic curves. Finally, we study magnetic Jacobi fields along horizontal Killing magnetic trajectories in 3-dimensional Sasakian space forms.

1. Introduction

The aim of this paper is to connect the theory of magnetic trajectories on a Riemannian 3-manifold with that of Killing submersions (see Definition 4.1). In the seminal paper [41], O'Neill initiated the study of Riemannian geometry of submersions. In the next paper [42], O'Neill studied connection of geodesic theory (especially Jacobi fields) of total space and the base space. In particular, he obtained a formula which relates the index forms of horizontal geodesics and those of geodesics in the base space.

In 3-dimensional geometry, according to Thurston, there are eight model spaces (see e.g. [52]). One can see that all model spaces are total spaces of Riemannian submersions. In particular, except the hyperbolic 3-space $\mathbb{H}^3$ and the space Sol_3, the model spaces are Killing submersion over 2-dimensional model spaces. Motivated by this fundamental fact, Killing submersions with 2-dimensional base and 1-dimensional fiber are now actively studied in differential geometry (see e.g. [9], [18], [33], [48]). It should be emphasize that Killing submersions with 2-dimensional base and 1-dimensional fiber involve geometrically natural dynamical systems – *magnetic trajectories*.

In magnetism theory, a static magnetic field on an oriented Riemannian 3-manifold (M, g) is regarded as a divergence free vector field ξ. Under the influence of a magnetic vector field ξ, a particle moves along the magnetic trajectory. The magnetic trajectory is described as the curve determined by the *Lorentz equation*:

$$\nabla_{\gamma'}\gamma' = q\xi \times \gamma'.$$

Here ∇ is the Levi-Civita conection, $\times$ is the cross product of M and q is a constant, called the *charge*. Magnetic trajectories define a dynamical system (Hamiltonian system) on the

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cotangent bundle T^*M of M [6, 7, 5]. When the magnetic field is absent or the charge is zero, the magnetic trajectories reduce to geodesics. In this sense, magnetic trajectories are regarded as geometrically or physically nice perturbation of geodesic flows on T^*M. The study of the trajectories of magnetic fields on Riemannian manifolds of arbitrary dimension has grown up an active and attractive area of mathematics as well as of mathematical physics. In particular, magnetic trajectories on surfaces needed to be paid much attention to during last decades (*e.g.* [1, 8, 34, 47]). Studies on Jacobi fields along magnetic trajectories were initiated by Gouda [22] and Paternain and Paternain [44].

Let us return our attention to Killing submersions. Let $\pi : (M, g) \to (B, \bar{g})$ be a Killing submersion with $\dim M = 3$ and $\dim B = 2$ and a complete vertical unit Killing vector field ξ. Then ξ is a *static magnetic field* (called the *Killing magnetic field*) on M. Thus the total space M of a Killing submersion is naturally endowed with magnetic trajectories. This remarkable fact motivates us to develop differential geometric study of magnetic trajectories in Killing submersions (with 2-dimensional base and 1-dimensional fiber). Since geodesics are regarded as magnetic curves with charge zero, it would be expected to extend the O'Neill's work [42] to (horizontal) magnetic curves.

The aim of this article is to relate the geometry of magnetic curves in the total space M of a Killing submersion $\pi : M \to B$ with that of the base space B.

From topological point of view, the existence of special vector fields forces topological restrictions on the manifold M. The dimension 3, as is well known, is rather special. In fact, on every orientable 3-manifold M, there exists a non-vanishing vector field. Much stronger, we know a very important fact saying that every orientable Riemannian 3-manifold (M, g) admits an almost contact structure compatible to the metric (and prescribed orientation). This fact suggests us to investigate Killing submersions, by virtue of the associated almost contact structure. In fact, the use of the associated quasi-Sasakian structures enables us to avoid complicated submersion calculus.

In this paper, we show that a Killing submersion $(M^3, g) \to (B^2, \bar{g})$ with a unit vertical Killing vector field ξ induces an almost contact metric structure on M compatible with g. Moreover, with respect to the almost contact structure, M becomes a quasi-Sasakian manifold whose structure function is precisely the bundle curvature τ of the Killing submersion.

In the case when τ is non-zero everywhere, we prove that M is pseudo-conformal (CR-equivalent) to a Sasakian 3-manifold. It should be remarked that under pseudo-conformal transformations, magnetic curves are *not* preserved. However, horizontal magnetic curves are pseudo-conformal invariant.

This paper is organized as follows. After recalling prerequisite knowledges on magnetism in Section 2, and on Killing vector fields, fibering and almost contact structures in Section 3, we start our investigation on Killing submersions in Section 4. Then, in Section 5, we study magnetic curves in Killing submersions and we show, in Theorem 5.1, that the bundle curvature is constant along magnetic curves with respect to ξ if and only if all vertical tubes derived from these magnetic curves are of constant mean curvature. These assertions are also equivalent to the property for the magnetic curve of having constant second curvature.

From the stability viewpoint, the study of the Jacobi fields along magnetic trajectories (called the magnetic Jacobi fields) is important. In Section 6 we investigate the magnetic Jacobi fields along horizontal Killing magnetic curves in the total space of a Killing submersion $\pi : M \to B$. In Theorem 6.1 and Theorem 6.2 we generalize some important results by O'Neill [42] for horizontal geodesics to horizontal magnetic curves. We prove, in Theorem 6.3, the following result: *Suppose that the bundle curvature of the Killing submersion $\pi : M \to B$ is non-zero everywhere. Then, the projection of a magnetic Jacobi field E along a horizontal Killing magnetic curve γ is a magnetic Jacobi field on $\pi \circ \gamma$ if and only if its derived vector field $D(E)$ vanishes.*

In the last section, we apply our results to Killing submersions equipped with a homogeneous Sasakian structure. In the subsection 7.3 we give a complete description of the magnetic Jacobi fields along horizontal Killing magnetic curves in Sasakian space forms of dimension 3.

To develop a global study of magnetic curves, a detailed investigation on magnetic Jacobi fields is required. In our separate publications [29, 30] we obtain all magnetic Jacobi fields along contact magnetic curves on Sasakian space forms.

Throughout this article, all manifolds are assumed to be connected. We use *Alt*-convention for the differential form calculus.

2. Magnetic curves

2.1. **Magnetic fields.**

On a Riemannian manifold (M, g) consider a closed 2-form F that is regarded as a static magnetic field. The *Lorentz force* L, derived from F, is an endomorphism field defined by

$$g(LX, Y) = F(X, Y).$$

The *magnetic trajectory* of F is a curve γ satisfying the Lorentz equation

$$\nabla_{\gamma'}\gamma' = qL\gamma',$$

where q is a constant called the *charge*. One can see that every magnetic trajectory has constant speed. Unit speed magnetic curves are called *normal magnetic curves.*

Example 2.1 (Kähler magnetic fields)**.** Let $(B, \bar{g}, J)$ be a Kähler manifold with complex structure J. Then the Kähler form $F = \bar{g}(\cdot, J\cdot)$ is a magnetic field on B (called the *Kähler magnetic field*).

2.2. **Variational characterization.**

Let (M, g, F) be a Riemannian manifold with an *exact* magnetic field $F = 2dA$. Denote by $C^\infty[a, b]$ the space of smooth curves in M defined on a closed interval $[a, b]$ and satisfying the boundary condition

$$\gamma(a) = p_1, \quad \gamma(b) = p_2.$$

The *Landau-Hall functional* LH on $C^\infty[a, b]$ is defined by

$$\mathrm{LH}(\gamma) = E(\gamma) + q\int_a^b A(\gamma')ds.$$

Here $E(\gamma)$ is the *Dirichlet energy* of γ, that is

$$E(\gamma) = \int_a^b \frac{1}{2} g(\gamma'(s), \gamma'(s))\, ds.$$

The first variation formula of LH is given by

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathrm{LH}(\gamma_\varepsilon) = -\int_a^b g\big(\nabla_{\gamma'(s)}\gamma'(s) - qL\gamma'(s), V(s)\big) ds,$$

where $\gamma_\varepsilon(s)$ is a variation of γ satisfying the boundary condition

$$\gamma_\varepsilon(a) = p_1, \quad \gamma_\varepsilon(b) = p_2,$$

for any ε. The variational vector field V is defined, along $\gamma(s)$, by

$$V(s) = \frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} \gamma_\varepsilon(s),$$

which vanishes at the end points, that is it satisfies $V(a) = 0$ and $V(b) = 0$.

Gouda obtained the second variational formula of LH [22]:

$$\frac{d^2}{d\varepsilon^2}\bigg|_{\varepsilon=0} \mathrm{LH}(\gamma_\varepsilon) = -\int_a^b g(\mathcal{J}_{q,F}(W), V(s)) ds,$$

where V and W are variational vector fields along $\gamma(s)$ in a 2-parameter variation. The operator $\mathcal{J}_{q,F}$ acts on the space $\Gamma(\gamma^* TM)$ of all vector fields along γ and it is defined by

$$\mathcal{J}_{q,F}(W) = \nabla_{\gamma'}\nabla_{\gamma'}W + R(W, \gamma')\gamma' - qL(\nabla_{\gamma'}W) - q(\nabla_W L)\gamma',$$

where the Riemannian curvature R is defined by $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. The operator $\mathcal{J}_{q,F}$ will be called the *magnetic Jacobi operator* derived from the magnetic field F.

We give, for later use, the following formula

$$\mathcal{J}_{q,F}(fW) = f\mathcal{J}_{q,F}(W) + 2f'\nabla_{\gamma'}W - qf'LW + f''W, \tag{2.1}$$

where f is a smooth function defined on I.

A vector field $W(s)$, along the magnetic curve γ, is said to be a *magnetic Jacobi field* if it satisfies $\mathcal{J}_{q,F}(W) = 0$.

Remark 2.1 ([22])**.** Let $\gamma(s)$ be a magnetic curve and choose $W(s) = \gamma'(s)$. Then, since $\nabla_{\gamma'}\gamma' = qL\gamma'$, we obtain

$$\begin{aligned}\mathcal{J}_{q,F}(\gamma') =& \nabla_{\gamma'}\nabla_{\gamma'}\gamma' + R(\gamma', \gamma')\gamma' - qL(\nabla_{\gamma'}\gamma') - q(\nabla_{\gamma'}L)\gamma' \\ =& \nabla_{\gamma'}(qL\gamma') - qL(\nabla_{\gamma'}\gamma') - q(\nabla_{\gamma'}L)\gamma' = q\{(\nabla_{\gamma'}L)\gamma' - (\nabla_{\gamma'}L)\gamma'\} = 0.\end{aligned}$$

Hence $\gamma'(s)$ is a magnetic Jacobi field.

Proposition 2.1. *Let γ be a magnetic curve and W be a magnetic Jacobi field along it. Then the following formula holds along γ*

$$\frac{d}{ds} g(\nabla_{\gamma'}W, \gamma') = qg((\nabla_W L)\gamma', \gamma'). \tag{2.2}$$

Corollary 2.1 (Conservation law)**.** *The formula* (2.2) *implies that if $\nabla_W L$ is skew-adjoint with respect to g, then $g(\nabla_{\gamma'}W, \gamma')$ is constant along γ.* See also [4, Lemma 1.2].

2.3. **Three dimensional case.** Now let us focus on the situation $\dim M = 3$.

Cross product. Let (M, g) be an oriented Riemannian 3-manifold with the volume element dv_g. Then, the *cross product* $\times$ of (M, g), with respect to dv_g is defined by

$$g(X \times Y, Z) = 3!\, dv_g(X, Y, Z).$$

One can see that

$$Z \times (X \times Y) = g(Y, Z)X - g(Z, X)Y,$$

$$g(X \times Y, Z) = g(X, Y \times Z), \quad \forall X, Y, Z \in \mathfrak{X}(M).$$

Curve theory. Let $\gamma(s)$ be a unit speed curve in an oriented Riemannian 3-manifold (M, g, dv_g) with the non-vanishing acceleration $\nabla_{\gamma'}\gamma'$. Here ∇ denotes the Levi-Civita connection. We can take a unit normal vector field N by the formula $\nabla_{\gamma'}\gamma' = \kappa_1 N$. Next define a unit vector field B by $B = T \times N$. Here $T = \gamma'$. In this way we obtain an orthonormal frame field $\mathcal{F} = (T, N, B)$ along γ which is positively oriented, that is, $dv_g(T, N, B) > 0$. The orthonormal frame field $\mathcal{F}$ is called the *Frenet frame field* and it satisfies the *Frenet-Serret formula*:

$$\nabla_{\gamma'}\mathcal{F} = \mathcal{F} \begin{pmatrix} 0 & -\kappa_1 & 0 \\ \kappa_1 & 0 & -\kappa_2 \\ 0 & \kappa_2 & 0 \end{pmatrix}$$

for some function κ_2. These functions κ_1 and κ_2 are called the *first curvature* and the *second curvature* of γ, respectively. A unit speed curve γ is said to be a *helix* if both κ_1 and κ_2 are constant. In particular a helix with $\kappa_1 > 0$ and $\kappa_2 \neq 0$ are called a *proper helix*. A *Riemannian circle* is a unit speed curve with constant $\kappa_1 > 0$ and $\kappa_2 = 0$. Geodesics are regarded as the unit speed curves with $\kappa_1 = 0$.

Magnetic curves on 3-dimensional manifolds. On an oriented Riemannian 3-manifold (M, g, dv_g, F) equipped with a static magnetic field F, the magnetic field F is identified with a divergence free vector field V via the Hodge star operator $*$ as

$$V = \sharp(*F)/2.$$

Conversely, let V be a divergence free vector field. Then the corresponding closed 2-form F is given by

$$F = 2\iota_V dv_g = 2 * (\flat V).$$

Here $\sharp : T^*M \to TM$ and $\flat = \sharp^{-1} : TM \to T^*M$ are the so-called *musical isomorphisms*. The operator ι_V is the interior product by V. The Lorentz equation is rewritten as

$$\nabla_{\gamma'}\gamma' = qV \times \gamma'.$$

3. Killing vector fields and fiberings

3.1. **Regularity of vector fields.** Generally speaking, the existence of special vector fields forces topological restrictions for manifolds. Let us recall a fundamental fact.

Lemma 3.1. *Let M be an orientable manifold. Then the following two assertions are mutually equivalent*:

- *There exists a non-vanishing vector field globally defined on M.*
- *Either M is non-compact or M is compact with $\chi(M) = 0$.*

In addition we know that $\chi(M) = 0$ for any compact 3-manifold M. Therefore, we emphasize that every orientable 3-manifold admits a non-vanishing vector field.

A non-vanishing vector field ξ on a 3-manifold M is said to be *quasi-regular* (*almost regular* in the sense of [51], see also [20, p. 342]) if there exists some positive integer k, and each point $p \in M$ has a cubical coordinate neighborhood $(U; x, y, z)$ such that

(1) each integral curve of the vector field ξ passes through U at most k times, and
(2) each component of the intersection of an integral curve with U has the form $x = a$, $y = b$, with a and b constant. In case $k = 1$, ξ is called a *regular vector field* (see [36, 43]).

Tanno [49] showed that the following three conditions are mutually equivalent for regular (and complete) vector field ξ:

(1) The period function of ξ is constant (maybe infinite).
(2) There exists a 1-form η satisfying $\eta(\xi) = 1$ and $\pounds_\xi \eta = 0$.
(3) There exists a Riemannian metric g satisfying $g(\xi, \xi) = 1$ and $\pounds_\xi g = 0$.

In such a case, M is a principal bundle over the orbit space $B = M/\mathcal{G}$ under the action of $\mathcal{G} = \{\exp(t\xi)\}_{t\in\mathbb{R}}$. The prescribed vector field ξ is a *unit Killing vector field* with respect to the Riemannian metric g. In addition, there exists a Riemannian metric $\bar{g}$ on B so that $\pi : M \to B$ is a *Riemannian submersion* (see Section 4.1). The one-form η is a *connection form* of the principal bundle $\pi : M \to B$.

If the global flow $\exp(t\xi)$ has no fixed point, then M is a principal line bundle over B. On the other hand, if there exist $p \in M$ and $t \in \mathbb{R}$ such that $\exp(t\xi)(p) = p$, then M is a principal circle bundle over B. The property (2) is closely related to geodesible vector fields. See the next Remark.

Remark 3.1 (geodesible vector field)**.** A non-vanishing vector field ξ on a 3-manifold M is said to be *geodesible* if there exists a Riemannian metric g on M with respect to which ξ has unit length and the integral curves are geodesics. Wadsley and Sullivan showed that the following four conditions are mutually equivalent (see Geiges [21]):

- ξ is geodesible.
- There exists a 1-form η satisfying $\eta(\xi) = 1$ and $\pounds_\xi \eta = 0$.
- There exists a 1-form η satisfying $\eta(\xi) = 1$ and $\pounds_\xi d\eta = 0$.
- There exists a hyperplane field $\mathcal{D}$ transverse to ξ and invariant under the flows of ξ.

3.2. **Almost contact structures.** On the other hand, on an oriented Riemannian 3-manifold (M^3, g, dv_g) there exists a unit vector field ξ on M. Denote by η the metrical dual 1-form of ξ. Then we define an endomorphism field φ on M by

$$\varphi X = \xi \times X. \tag{3.1}$$

Then the structure tensor fields (φ, ξ, η, g) on M has the following properties [15]:

$$\varphi^2 = -I + \eta \otimes \xi \quad \text{and} \quad \eta(\xi) = 1, \tag{3.2}$$

$$g(\varphi X, \varphi Y) = g(X, Y) \tag{3.3}$$

for all vector fields X and Y on M and

$$dv_g = -\frac{1}{2}\eta \wedge \Phi,$$

where Φ is a two-form defined by

$$\Phi(X, Y) = g(X, \varphi Y).$$

A structure (φ, ξ, η) on an oriented Riemannian 3-manifold (M^3, g, dv_g) is called an *almost contact structure* compatible to the metric g and the orientation determined by dv_g. The cross product derived from dv_g is described as

$$X \times Y = -\Phi(X, Y)\xi + \eta(X)\varphi Y - \eta(Y)\varphi X.$$

The covariant derivate $\nabla\varphi$ of φ, the exterior derivatives $d\eta$ of η and the differential $d\Phi$ of Φ are given by (see [40]):

$$(\nabla_X \varphi)Y = g(\varphi\nabla_X \xi, Y)\xi - \eta(Y)\varphi\nabla_X \xi, \; d\eta = \eta \wedge \nabla_\xi \eta + \frac{1}{2}\mathrm{tr}(\varphi\nabla\xi)\Phi, \; d\Phi = (\mathrm{div}\,\xi)\eta \wedge \Phi, \tag{3.4}$$

respectively.

It should be remarked that the unit vector field ξ is neither regular nor Killing, in general. In the next subsection, we discuss the condition "ξ is Killing".

For later use, here we recall the notion of rank for one-forms on 3-manifolds.

Definition 3.1. A one-form η on a 3-manifold M is said to be of

- rank 1 if $d\eta = 0$.
- rank 2 if $d\eta \neq 0$ and $\eta \wedge d\eta = 0$.
- rank 3 if $\eta \wedge d\eta \neq 0$.

In particular, a one-form η of rank 3 is called a *contact form* on M.

A 3-manifold M together with a contact form η is called a *contact* 3*-manifold*. We can orient a contact 3-manifold by the volume element $-\eta \wedge d\eta/2$. On a contact 3-manifold (M, η) there exists a unique vector field ξ such that $\eta(\xi) = 1$ and $d\eta(\xi, \cdot) = 0$. This vector field is traditionally called the *Reeb vector field* of (M, η). The existence of a contact form also implies the existence of an almost contact structure.

Proposition 3.1. *On a contact* 3*-manifold* (M, η) *with the Reeb vector field* ξ*, then there exist an endomorphism field* φ *and a Riemannian metric* g *on* M *such that the* (φ, ξ, η, g) *satisfies* (3.2)–(3.3)*,* $\Phi = d\eta$ *and* $dv_g = -\eta \wedge d\eta/2$*. Such a Riemannian metric* g *is called the associated metric.*

A contact 3-manifold M together with a structure (φ, ξ, η, g) is called a *contact metric* 3*-manifold*. It should be emphasize that the Reeb vetor field ξ is divergence free and hence its corresponding two form $F := -d\eta$ is a magnetic field on M. The Lorentz force is φ.

3.3. **Unit Killing vector fields.** Now let us concentrate our attention to unit Killing vector fields on Riemannian 3-manifolds. The following results play a crucial role in this article.

Lemma 3.2 ([48])**.** *Let (M, g, dv_g) be an oriented Riemannian 3-manifold. Assume the existence of a unit Killing vector field ξ globally defined on M. Then there exists a smooth function τ on M satisfying*

$$\nabla_X \xi = \tau\, X \times \xi$$

for any vector field X on M and $d\tau(\xi) = 0$. The function τ is related to the sectional curvature function as

$$\tau^2 = K(X \wedge \xi)$$

for any tangent vector X linearly independent of ξ.

Remark 3.2. On a Riemannian manifold (M, g) with a unit Killing vector field ξ, the sectional curvature function satisfies $K(X \wedge \xi) \geq 0$, for any non-zero tangent vector linearly independent of ξ [11].

Comparing Lemma 3.2 with (3.1) and (3.4), we obtain

Theorem 3.1. *Let (M, g, dv_g, ξ) be an oriented Riemannian 3-manifold with a unit Killing vector field ξ. Then the structure (φ, ξ, η, g) defined by $\eta = \flat\xi$ and (3.1) is an almost contact metric structure on M satisfying*

$$(\nabla_X \varphi)Y = \tau(g(X, Y)\xi - \eta(Y)X), \quad \nabla_X \xi = -\tau\varphi X, \quad d\tau(\xi) = 0. \tag{3.5}$$

This almost contact metric structure satisfies $\pounds_\xi \varphi = 0$.

According to the terminology of almost contact metric geometry, the structure (φ, ξ, η, g) satisfying (3.5) is a *quasi-Sasakian structure* (see [13, 40]). The statement of the Theorem 3.1 is strongly related to Theorem 3.8 in [16]. The use of this fact will enable us to avoid complicated submersion calculus.

In particular, quasi-Sasakian 3-manifolds with $\tau = 1$ are called *Sasakian 3-manifolds.* Sasakian 3-manifolds are characterized as orientable Riemannian 3-manifolds which possess unit Killing vector fields satisfying a specific curvature property [14, 24]:

Proposition 3.2. *Let (M^3, g) be a Riemannian manifold. If there exists a unit Killing vector field ξ on M which satisfies $K(X \wedge \xi) = 1$ for any nonzero tangent vector X orthogonal to ξ, then M is a Sasakian 3-manifold. More precisely, the structure (φ, ξ, η, g) defined by $\eta =: \flat\xi$ and $\varphi := -\nabla\xi$ is Sasakian.*

4. Killing submersions

In this section we assume that $M = (M, g)$ and $B = (B, \bar{g})$ are Riemannian manifolds of dimension 3 and 2, respectively. In addition, M is suppose to be orientable.

4.1. **Submersion calculus.** Let $\pi : (M, g) \to (B, \bar{g})$ be a submersion. Then $\pi^{-1}(\bar{p})$ is a submanifold of M, for any $\bar{p} \in B$, called the *fiber* over $\bar{p}$. A vector field W on M is said to be *vertical* if it tangents to the fibers. On the other hand, vector fields orthogonal to fibers are called *horizontal vector fields*. The terminologies vertical and horizontal are used for individual tangent vectors.

A submersion π is said to be a *Riemannian submersion* if π_* preserves the length of horizontal vectors [41].

Now let $\pi : (M, g) \to (B, \bar{g})$ is a Riemannian submersion. Then, each tangent space T_pM is decomposed as

$$T_pM = \mathcal{H}_p \oplus \mathcal{V}_p,$$

where $\mathcal{V}_p$ is the tangent space of the fiber $\pi^{-1}(\pi(p))$ at p and $\mathcal{H}_p$ is the orthogonal complement of $\mathcal{V}_p$. Tangent vectors in $\mathcal{V}_p$ [resp. $\mathcal{H}_p$] are *vertical vectors* [resp. *horizontal vectors*]. The linear subspaces $\mathcal{V}_p$ and $\mathcal{H}_p$ are called the *vertical subspace* and the *horizontal subspace* at p, respectively.

Any tangent vector $X \in T_pM$ decomposes as $X = X^h + X^v$ with $X^h \in \mathcal{H}_p$ and $X^v \in \mathcal{V}_p$. The components X^h and X^v are called the *horizontal part* and the *vertical part* of X, respectively.

A vector field X on M is said to be *basic* if it is horizontal and there exists a vector field $\bar{X}$ on B such that X is π-related to $\bar{X}$, that is, $\pi_* X = \bar{X}$.

On the other hand, for any vector field $\bar{X}$ on B, there exists a basic vector field $\bar{X}^\uparrow$, which is π-related to $\bar{X}$. The basic vector field $\bar{X}^\uparrow$ is called the *horizontal lift* of $\bar{X}$.

Let ∇ be the Levi-Civita connection on M. Then the O'Neill's fundamental tensor fields $\mathcal{A}$ and $\mathcal{T}$ are respectively defined by

$$\mathcal{A}_X Y = (\nabla_{X^h} Y^h)^v + (\nabla_{X^h} Y^v)^h,$$

$$\mathcal{T}_X Y = (\nabla_{X^v} Y^v)^h + (\nabla_{X^v} Y^h)^v, \quad X, Y \in \mathfrak{X}(M).$$

The restriction $\mathcal{A}_{\mathcal{H}\times\mathcal{H}}$ of $\mathcal{A}$ to the horizontal distribution is the integrability obstruction for $\mathcal{H}$. The tensor field $\mathcal{A}$ is often refered as to the *O'Neill tensor field* or the *O'Neill's integrability tensor field*.

Let us denote by $\hat{\nabla}$ the connection on fibers induced from ∇. Then one can see that the restriction of $\mathcal{T}$ to vertical vectors gives the vector valued second fundamental form of fibers. Thus π has totally geodesic fibers if and only if $\mathcal{T}_U V = 0$ for any vertical vector fields U and V. Next, let $\bar{\nabla}$ denotes the Levi-Civita connection on B. We recall the following formula [41]:

$$\nabla_{\bar{X}^\uparrow} \bar{Y}^\uparrow = (\bar{\nabla}_{\bar{X}} \bar{Y})^\uparrow + \mathcal{A}_{\bar{X}^\uparrow} \bar{Y}^\uparrow.$$

Take a basis $\{X, Y\}$ of the horizontal subspace $\mathcal{H}_p$. Denote by κ, the Gauß curvature of $(B, \bar{g})$. Then the sectional curvature $K(\mathcal{H}_p) = K(X \wedge Y)$ (called the *horizontal curvature*) of $\mathcal{H}_p$ and κ are related by

$$\kappa = K(\mathcal{H}_p) + \frac{3\|\mathcal{A}_X Y\|^2}{\|X \times Y\|^2}. \tag{4.1}$$

For more information on Riemannian submersion, we refer to [12, 17, 41].

4.2. **Killing submersions.**

Definition 4.1. A Riemannian submersion $\pi : (M, g) \to (B, \bar{g})$ is said to be a *Killing submersion* if it admits a complete vertical unit Killing vector field ξ on M.

The one-parameter group $\mathcal{G} = \{\exp(t\xi)\}_{t\in\mathbb{R}}$ of isometries generated by ξ acts on M and B is obtained as the factor space of M by $\mathcal{G}$. An element $\exp(t\xi)$ is called a *vertical translation*. In addition M is a principal fiber bundle over B with the structure group $\mathcal{G}$ (see §3 of the present paper and [17, Theorem 1.11]). Note that the completeness of M implies that of B [17, Theorem 1.11]. The metrical dual one-form η is a connection one-form of the principal $\mathcal{G}$-bundle $\pi : M \to B$.

The unit Killing vector field ξ is identified with a magnetic field $F := 2\iota_\xi dv_g$. We call it the *Killing magnetic field* of the Killing submersion.

4.3. **Associated quasi-Sasakian structure.**

Here we collect some fundamental results on Killing submersions.

Let $\pi : (M, g) \to (B, \bar{g})$ be a Killing submersion with the unit Killing vector field ξ as before. First, since ξ is a unit Killing vector field, the connection form η and the endomorphism field φ defined by (3.1) determine a quasi-Sasakian structure (φ, ξ, η, g) on the total space M. It should be remarked that the function τ is constant along fibers since $d\tau(\xi) = 0$. Hence τ is regarded as a smooth function on B. The function τ is called the *bundle curvature* of the Killing submersion.

Next, since $\mathcal{L}_\xi\varphi = 0$, φ is invariant under the action of $\mathcal{G}$. Thus the submersion π induces a complex structure J on B as [39]:

$$J_{\pi(p)}\bar{X}_{\pi(p)} = \pi_{*p}(\varphi_p \bar{X}_p^{\uparrow}), \quad \bar{X} \in \mathfrak{X}(B).$$

One can see that J is $\bar{g}$-orthogonal. Since $\dim B = 2$, the Kähler form $\Omega = \bar{g}(\cdot, J\cdot)$ of B is closed. Thus Ω is a magnetic field on B (see Example 2.1).

Local description of Killing submersion is clarified by Manzano [33, Theorem4.2].

Theorem 4.1. *Let $(M, g) \to (B, \bar{g})$ be a Killing submersion. Then M is locally isometric to the following canonical example*:

- *The projection is given by*
$$\pi : \mathcal{D} \times \mathbb{R} \to \mathcal{D};\ \pi(x, y, z) = (x, y).$$
- *$\mathcal{D}$ is a starlike region in $\mathbb{R}^2$.*
- *The metric on $\mathcal{D}$ is $\bar{g} = \lambda(x, y)^2(dx^2 + dy^2)$.*
- *The metric on $\mathcal{D} \times \mathbb{R}$ is*
$$g = \lambda(x, y)^2(dx^2 + dy^2) + (dz + \mu(x, y)(ydx - xdy))^2\,,$$
where μ is a certain smooth function on $\mathcal{D}$. The bundle curvature τ is given by
$$\tau = \frac{2\mu + \mu_x x + \mu_y y}{2\lambda^2}.$$

Note that in the case when (M,g) is *homogeneous*, then it is locally isometric to the so-called Bianchi-Cartan-Vranceanu spaces (Example 4.1, see also [10]).

Proposition 4.1. *Let $(M,g)\to(B,\bar{g})$ be a Killing submersion with a unit Killing vector field ξ. Then the connection form η is of rank 3 if and only if the bundle curvature is non-zero everywhere on M.*

Proof. By Theorem 4.1, the connection form η is expressed as

$$\eta = dz + \mu(x,y)(ydx - xdy).$$

The exterior differential of η is computed as $d\eta = -2\tau\lambda^2 dx\wedge dy$. Hence we get

$$d\eta\wedge\eta = -2\tau\lambda^2 dx\wedge dy\wedge dz.$$

This shows the result. □

Here is a worth pointing remark: In the case when the connection η is a *contact form*, then the Killing magnetic field $F = 2\iota_\xi dv_g = -\Phi$ has a magnetic potential $-d\eta/\tau$. Namely, the contact condition implies the existence of a magnetic potential. Thus, from variational problem point of view, the contact condition of the connection η is reasonable.

4.4. **Horizontal curvature.** Let $\pi : M\to B$ be a Killing submersion. Then the O'Neill tensor field $\mathcal{A}$ is described as

$$\mathcal{A}_X Y = \frac{1}{2}\eta([X,Y])\xi$$

for all horizontal vector fields X and Y, *i.e.* vector fields orthogonal to ξ. By the definition of $d\eta$, we have

$$2d\eta(X,Y) = X\eta(Y) - Y\eta(X) - \eta([X,Y]) = -\eta([X,Y])$$

for all horizontal vector fields X and Y.

On the other hand, we have $d\eta = \tau\Phi$, so, we get

$$\mathcal{A}_X Y = -d\eta(X,Y)\xi = -\tau\Phi(X,Y)\xi.$$

This formula can be verified in another way. In fact, if X, Y are horizontal vector fields on M (that is they are orthogonal to ξ) we have

$$\mathcal{A}_X Y = (\nabla_X Y)^v.$$

Successive computations lead to

$$g(\nabla_X Y,\xi) = -g(Y,\nabla_X\xi) = -g(Y,\tau X\times\xi) = \tau g(Y,\varphi X) = -\tau\Phi(X,Y).$$

This shows that

$$\mathcal{A}_X(\varphi X) = \tau g(X,X)\xi,\quad \forall X\perp\xi.$$

Hence we obtain the following fundamental formulas:

$$\nabla_{\bar{X}^\uparrow}\bar{Y}^\uparrow = (\bar{\nabla}_{\bar{X}}\bar{Y})^\uparrow + \tau\, g(\varphi\bar{X}^\uparrow,\bar{Y}^\uparrow)\xi,\quad \text{and}\quad [\bar{X}^\uparrow,\bar{Y}^\uparrow] = [\bar{X},\bar{Y}]^\uparrow + 2\tau g(\varphi\bar{X}^\uparrow,\bar{Y}^\uparrow)\xi. \tag{4.2}$$

The Riemannian curvature R of M and $\bar{R}$ of B are related by (*cf.* [39]):

$$\begin{aligned} R(\bar{X}^{\uparrow},\bar{Y}^{\uparrow})\bar{Z}^{\uparrow} &= \big[\bar{R}(\bar{X},\bar{Y})\bar{Z}\big]^{\uparrow} + \big[d\tau(\bar{X}^{\uparrow})g(\varphi\bar{Y}^{\uparrow},\bar{Z}^{\uparrow}) - d\tau(\bar{Y}^{\uparrow})g(\varphi\bar{X}^{\uparrow},\bar{Z}^{\uparrow})\big]\xi \\ &\quad +\tau^2\big[g(\varphi\bar{X}^{\uparrow},\bar{Z}^{\uparrow})\varphi\bar{Y}^{\uparrow} - g(\varphi\bar{Y}^{\uparrow},\bar{Z}^{\uparrow})\varphi\bar{X}^{\uparrow} + 2g(\varphi\bar{X}^{\uparrow},\bar{Y}^{\uparrow})\varphi\bar{Z}^{\uparrow}\big]. \end{aligned} \tag{4.3}$$

Take a unit tangent vector X orthogonal to ξ; then the plane $X \wedge \varphi X$ coincides with the horizontal subspace. The O'Neill formula (4.1) implies that the horizontal curvature of M is given by

$$K(X \wedge \varphi X) = \kappa - 3\tau^2, \tag{4.4}$$

where κ is the Gauß curvature of B.

4.5. **Curvature properties.** Let us denote by r the scalar curvature of the total space M of the Killing submersion $\pi : M \to B$. Then the Riemannian curvature R of M is explicitly expressed as (*cf.* [40]):

$$\begin{aligned} R(X,Y)Z = &\ \frac{r-4\tau^2}{2}(g(Y,Z)X - g(Z,X)Y) \\ &+ \frac{r-6\tau^2}{2}(\eta(Z)\eta(X)Y - \eta(Y)\eta(Z)X) \\ &+ \frac{r-6\tau^2}{2}\left(g(Z,X)\eta(Y) - g(Y,Z)\eta(X)\right)\xi \\ &+ \left(d\tau(\varphi Z)\eta(X) + d\tau(\varphi X)\eta(Z)\right)Y \\ &- \left(d\tau(\varphi Z)\eta(Y) + d\tau(\varphi Y)\eta(Z)\right)X \\ &+ \left(\eta(X)g(Y,Z) - \eta(Y)g(Z,X)\right)\varphi\,\mathrm{grad}\,\tau \\ &- \left(g(Y,Z)d\tau(\varphi X) - g(Z,X)d\tau(\varphi Y)\right)\xi, \end{aligned}$$

where r is the scalar curvature. By using this formula, the horizontal curvature is computed as $K(X \wedge \varphi X) = (r - 4\tau^2)/2$. Comparing this with (4.4), we obtain

$$\begin{aligned} R(X,Y)Z = &\ (\kappa-3\tau^2)(g(Y,Z)X - g(Z,X)Y) \\ &+ (\kappa-4\tau^2)(\eta(Z)\eta(X)Y - \eta(Y)\eta(Z)X) \\ &+ (\kappa-4\tau^2)\left(g(Z,X)\eta(Y) - g(Y,Z)\eta(X)\right)\xi \\ &+ \left(d\tau(\varphi Z)\eta(X) + d\tau(\varphi X)\eta(Z)\right)Y \\ &- \left(d\tau(\varphi Z)\eta(Y) + d\tau(\varphi Y)\eta(Z)\right)X \\ &+ \left(\eta(X)g(Y,Z) - \eta(Y)g(Z,X)\right)\varphi\,\mathrm{grad}\,\tau \\ &- \left(g(Y,Z)d\tau(\varphi X) - g(Z,X)d\tau(\varphi Y)\right)\xi. \end{aligned} \tag{4.5}$$

When η is contact, then one can deform the Riemannian metric g so that the bundle curvature is 1 under preserving the horizontal distribution and the Killing vector field (up to sign). More precisely we can prove the following result [31]:

Proposition 4.2. *Assume that the bundle curvature τ has constant sign $\varepsilon = \pm 1$. Then we introduce a new structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ on the total space M, defined by*

$$\tilde{\varphi} = \varphi, \quad \tilde{\xi} = \varepsilon\xi, \quad \tilde{\eta} = \varepsilon\eta, \quad \tilde{g} = \varepsilon\tau g + (1-\varepsilon\tau)\eta \otimes \eta.$$

Then $\pi : (M, \tilde{g}) \to (B, \varepsilon\tau\bar{g})$ is still a Killing submersion with the unit *Killing vector field $\tilde{\xi}$. The resulting structure has the bundle curvature* 1 *and it is called the* pseudo-conformal deformation *of the original structure.*

$$\begin{array}{ccc} (M,g) & \longrightarrow & (M,\tilde{g}) \\ \downarrow \pi & & \downarrow \pi \\ (B,\bar{g}) & \longrightarrow & (B,\varepsilon\tau\bar{g}) \end{array}$$

Proposition 4.2 means that every total space of a Killing submersion with rank 3 connection is pseudo-conformally deformable to a Sasakian total space.

Remark 4.1 (Sasakian space forms)**.** Let M be a Sasakian 3-manifold. Take a vector $X \in T_pM$ orthogonal to ξ. Then the tangent plane $X \wedge \varphi X$ is called the *holomorphic plane* at p. The sectional curvature $K(X \wedge \varphi X)$ is called a *holomorphic sectional curvature* at p. A Sasakian 3-manifold is said to be a 3-dimensional *Sasakian space form* if it is of constant holomorphic sectional curvature.

Example 4.1 (Bianchi-Cartan-Vranceanu models)**.** Let us consider the following two-parameter family $\{M(\kappa,\tau) : (\kappa,\tau) \in \mathbb{R}^2\}$ of homogeneous Riemannian 3-spaces:

$$M(\kappa,\tau) = \left\{(x,y,z) \in \mathbb{R}^3 \, : \, 1 + \frac{\kappa}{4}(x^2+y^2) > 0\right\}$$

equipped with the Riemannian metric

$$g = \frac{dx^2 + dy^2}{\{1 + \frac{\kappa}{4}(x^2+y^2)\}^2} + \left(dz + \frac{\tau(ydx - xdy)}{1 + \frac{\kappa}{4}(x^2+y^2)}\right)^2.$$

This 2-parameter family of homogeneous Riemannian is called the *Bianchi-Cartan-Vranceanu model spaces.* The metrics as above are defined over the whole 3-space $\mathbb{R}^3$ for $\kappa \geq 0$. This two-parameter family includes the model spaces of the Thurston geometry except $\mathbb{H}^3$ and Sol_3. See e.g. [52]. More precisely, $M(\kappa,\tau)$ is isometric to the following homogeneous Riemannian 3-manifolds:

- If $\tau = \kappa = 0$, then $M(0,0) = \mathbb{E}^3$ (Euclidean 3-space),
- If $\kappa = 0$, $\tau \neq 0$, then $M(0,\tau) \cong \mathrm{Nil}_3$ (Heisenberg group),
- If $\kappa > 0$, $\tau \neq 0$, then $M(\kappa,\tau) \cong \mathrm{SU}(2) \setminus \{\infty\}$ with SO(2)-isotropic left invariant metric,
- If $\kappa < 0$, $\tau \neq 0$, then $M(\kappa,\tau) \cong \widetilde{\mathrm{SL}}_2\mathbb{R}$ with SO(2)-isotropic left invariant metric,
- If $\kappa > 0$, $\tau = 0$, then $M(\kappa,\tau) \cong \mathbb{S}^2(\kappa) \setminus \{\infty\} \times \mathbb{R}$,
- If $\kappa < 0$, $\tau = 0$, then $M(\kappa,\tau) \cong \mathbb{H}^2(\kappa) \times \mathbb{R}$,
- If $\kappa - 4\tau^2 = 0$ then $M(\kappa,\tau) \cong \mathbb{S}^3(\tau^2) \setminus \{\infty\}$.

The vector field $\xi := \partial/\partial z$ is a complete unit Killing vector field and it generates a one-parameter group of isometries on $M(\kappa,\tau)$. Furthermore, this group action is simply transitive. The orbit space $B(\kappa) = M(\kappa,\tau)/\xi$ is

$$B(\kappa) = \left\{(x,y) \in \mathbb{R}^2 \ : \ 1 + \frac{\kappa}{4}(x^2+y^2) > 0\right\}$$

equipped with Riemannian metric

$$\bar{g} = \frac{dx^2+dy^2}{\{1+\frac{\kappa}{4}(x^2+y^2)\}^2}.$$

The natural projection $\pi : M(\kappa,\tau) \to B(\kappa)$ defined by $\pi(x,y,z) = (x,y)$ is a Killing submersion with totally geodesic fibres and with *constant* bundle curvature τ. The base space $B(\kappa)$ is of constant curvature κ. With respect to the associated almost contact structure, $M(\kappa,\tau)$ is a quasi-Sasakian 3-manifold satisfying $\nabla\xi = -\tau\varphi$. In particular, $M(c,1)$ is a Sasakian space form of constant holomorphic sectional curvature c (see [10]). It is known that simply connected and complete 3-dimensional Sasakian space forms are isomorphic to one of the following spaces [50]:

- SU(2) equipped with Berger sphere metric if $c > -3$ and $c \neq 1$,
- $\mathbb{S}^3$ if $c = 1$,
- Nil_3 if $c = -3$,
- $\widetilde{\mathrm{SL}}_2\mathbb{R}$ if $c < -3$.

Note that the Killing submersion $\pi : \mathbb{S}^3 \to \mathbb{P}_1$ is nothing but the Hopf fibering. The magnetic curves in SU(2) and $\mathrm{SL}_2\mathbb{R}$ are investigated in [27] and [28], respectively.

5. Magnetic curves in Killing submersions

5.1. **Tubes.** Let $\pi : (M,g) \to (B,\bar{g})$ be a Killing submersion. Take a curve $\bar{\gamma}$ in B, then its inverse image $\Sigma_{\bar{\gamma}} := \pi^{-1}\{\bar{\gamma}\}$ of $\bar{\gamma}$ in M is a flat surface in M with mean curvature $\bar{\kappa}/2$ (see [18, 19, 26, 32]). Here $\bar{\kappa}$ denotes the geodesic curvature of $\bar{\gamma}$. The flat surface $\Sigma_{\bar{\gamma}}$ is called the *vertical cylinder* over $\bar{\gamma}$.

Example 5.1 (Hopf tubes)**.** Let $\mathbb{S}^3$ be the unit 3-sphere equipped with the standard Sasakian structure. The Hopf fibering $\pi : \mathbb{S}^3 \to \mathbb{P}_1$ is a Killing submersion. For a unit speed curve $\bar{\gamma}$ in the complex projective line $\mathbb{P}_1$, the vertical cylinder $\Sigma_{\bar{\gamma}}$ is traditionally called the *Hopf tube* or *Hopf cylinder*. When $\bar{\gamma}$ is closed, the surface $\Sigma_{\bar{\gamma}}$ is often called the *Hopf torus* over $\bar{\gamma}$.

5.2. **Magnetic curves and Killing submersions.** Let $\pi : (M,g) \to (B,\bar{g})$ be a Killing submersion from an oriented Riemannian 3-manifold onto a Riemannian 2-manifold with a unit Killing vector field ξ. Then ξ defines a quasi-Sasakian structure on M. In particular, if $\tau \neq 0$ on M, then M is quasi-Sasakian of rank 3, that is pseudo-conformal to a Sasakian 3-manifold.

Now let $\gamma(s)$ be a unit speed magnetic curve in the total space M with respect to the Killing magnetic field $F = 2\iota_\xi dv_g$. The Lorentz equation is

$$\nabla_{\gamma'}\gamma' = q\varphi\gamma'.$$

Since ξ is a Killing vector field, we obtain the following.

Proposition 5.1. *Every normal Killing magnetic curve is a loxodrome in the total space. More precisely, the contact angle function $\theta(s)$ defined by $\cos\theta(s) = g(\gamma'(s), \xi)$ is constant along $\gamma(s)$.*

Moreover, the first and second curvatures of γ are [31]:

$$\kappa_1 = |q| \sin\theta, \quad \kappa_2 = \tau + q\cos\theta.$$

We consider the projected curve $\bar{\gamma} = \pi \circ \gamma$ in B. Then the projected curve satisfies

$$\bar{\nabla}_{\bar{\gamma}'}\bar{\gamma}' = (q + 2\tau\cos\theta)J\bar{\gamma}'.$$

The proof of this fact is similar to that of [25, Theorem 2.1]. This formula shows that $\bar{\gamma}$ has the curvature $\bar{\kappa} = \frac{q+2\tau\cos\theta}{\sin\theta}$, since $\bar{s} = (\sin\theta)s$ is the arclength parameter for $\bar{\gamma}$. Hence the vertical cylinder $\Sigma_{\bar{\gamma}}$ has mean curvature $\frac{q}{2\sin\theta} + \tau\cot\theta$.

Assume that all the vertical tubes derived from magnetic curves are of constant mean curvature, then τ is constant along γ. Hence $\bar{\gamma}$ is a Kähler magnetic curve and hence it is a Riemannian circle or a geodesic.

Theorem 5.1. *Let $\pi : (M, g) \to (B, \bar{g})$ be a Killing submersion with a unit Killing vector field ξ. Then the following three statements are mutually equivalent*:

1. *The bundle curvature τ is constant along magnetic curves with respect to ξ.*
2. *All the vertical tubes derived from magnetic curves with respect to ξ are of constant mean curvature.*
3. *Every magnetic curve with respect to ξ has constant second curvature, hence, they are helices on M.*

This theorem provides the corrected version of the essential contribution of [35, Theorem 3.1].

Remark 5.1. In [46, Theorem 4.1], the authors claim that if a quasi-Sasakian 3-manifold M admits contact magnetic curves, then it is cosymplectic or an α-Sasakian 3-manifold with $\alpha \in \mathbb{R}^\times$. However, this claim is not correct. In our previous paper [31] we have exhibited explicit examples of quasi-Sasakian 3-manifold with non-constant α which admit non-trivial contact magnetic curves.

5.3. **Conservation law.** In Proposition 2.1 we emphasize a geometric condition for the Lorentz force L under which a certain quantity is constant along a magnetic curve. In the case of Killing submersion, we have

$$g((\nabla_W\varphi)\gamma', \gamma') = \tau\, g\left(g(W, \gamma')\xi - \eta(\gamma')W, \gamma'\right) = 0.$$

Proposition 5.2. *Let γ be a Killing magnetic curve in a total space of a Killing submersion. Then for any magnetic Jacobi field W along γ, the function $g(\nabla_{\gamma'}W, \gamma')$ is constant along γ.*

6. Magnetic Jacobi fields through Killing submersions

Now let us take an arc length parametrized curve $\gamma(s)$ in the total space M of a Killing submersion $\pi : M \to B$ and a vector field E along γ. We split E as $E = H + V$, where

$H = E^h$ and $V = E^v$, respectively. We use the notation (by O'Neill [42]):

$$E' = \nabla_{\gamma'} E, \quad E'' = \nabla_{\gamma'}\nabla_{\gamma'} E, \quad E_* := \pi_* E, \quad E'_* = \bar{\nabla}_{\bar{\gamma}'} E_*.$$

Then we have

$$\begin{aligned} E'(s)^h =& E'_*(s)^\uparrow + \mathcal{A}_H \gamma'(s)^v + \mathcal{A}_{\gamma'(s)^h} V(s) + \mathcal{T}_{\gamma'(s)^v} V(s), \\ E'(s)^v =& \mathcal{A}_{\gamma'(s)^h} H(s) + \mathcal{T}_{\gamma'(s)^v} H(s) + V'(s)^v, \end{aligned}$$

which implies, for $E = \gamma'(s)$, the following two formulas

$$\begin{aligned} \gamma''(s)^h =& \bar{\gamma}''(s)^\uparrow + 2\mathcal{A}_{\gamma'(s)^h}\gamma'(s)^v + \mathcal{T}_{\gamma'(s)^v}\gamma'(s)^v, \\ \gamma''(s)^v =& \mathcal{T}_{\gamma'(s)^v}\gamma'(s)^h + (\nabla_{\gamma'}\gamma'(s)^v)^v. \end{aligned}$$

Here $\bar{\gamma} = \gamma \circ \pi$ and $\bar{\gamma}'' = \bar{\nabla}_{\bar{\gamma}'}\bar{\gamma}'$.

Hereafter, we use the quasi-Sasakian structure associated to the Killing submersion $\pi : M^3 \to B^2$. First, we notice that for any vector field E on M, its vertical component E^v is given by

$$E^v = g(E, \xi)\xi = \eta(E)\xi.$$

Consequently, the horizontal component is obtained as

$$E^h = E - \eta(E)\xi.$$

Theorem 6.1. *Let $\gamma : I \to M$ be a normal Killing magnetic trajectory and E be a vector field along γ. Then we have*

$$\begin{cases} E'(s)^h = E'_*(s)^\uparrow - \tau(\cos\theta\varphi E + \eta(E)\varphi\gamma') \\ E'(s)^v = \left[\frac{d}{ds}\eta(E) + \tau g(E, \varphi\gamma')\right]\xi. \end{cases} \tag{6.1}$$

Proof. Because $\nabla_\xi \xi = 0$ and as $\mathcal{T}$ is a tensor field, it follows that $\mathcal{T}_{\gamma'(s)^v} V(s) = 0$. Then, since γ is a Killing magnetic curve, it is a *loxodrome*, that is the contact angle θ is constant. As $\cos\theta = \eta(\gamma')$, we compute

$$\begin{cases} \mathcal{A}_{H(s)}\gamma'(s)^v = \eta(\gamma')\mathcal{A}_H\xi = \cos\theta(\nabla_H\xi)^h = -\tau\cos\theta\varphi E \\ \mathcal{A}_{\gamma'(s)^h} V = \eta(V)\mathcal{A}_{\gamma'(s)^h}\xi = \eta(E)(\nabla_{\gamma'(s)^h}\xi)^h = -\tau\eta(E)\varphi\gamma'(s). \end{cases}$$

Hence, the first equation is proved. To show the second equation we write

$$\begin{cases} \mathcal{A}_{\gamma'(s)^h} H &= (\nabla_{\gamma'(s)^h} H(s))^v = \eta(\nabla_{\gamma'(s)^h} H(s))\xi = g(\nabla_{\gamma'(s)^h} H(s), \xi)\xi \\ &= -g(H, -\tau\varphi\gamma')\xi = \tau g(E, \varphi\gamma')\xi \\ \mathcal{T}_{\gamma'(s)^v} H &= \eta(\gamma')\mathcal{T}_\xi H = \cos\theta(\nabla_\xi H)^v = 0 \\ V'(s)^v &= (\nabla_{\gamma'(s)} V)^v = \left(\nabla_{\gamma'(s)}(\eta(V)\xi)\right)^v = \frac{d}{ds}\eta(E)\xi. \end{cases}$$

The conclusion is now immediate. ∎

Corollary 6.1. *When $E=\gamma'$, we have*

$$\begin{cases} \gamma''(s)^h = \bar{\gamma}''(s)^{\uparrow} - 2\tau\cos\theta\ \varphi\gamma'(s) \\ \gamma''(s)^v = 0. \end{cases} \tag{6.2}$$

In addition, we have the following *conservation lemma* (*cf.* [42, Corollary 2]):

Lemma 6.1. *If a Killing magnetic curve γ in the total space M of a Killing submersion $\pi: M^3 \to B^2$ is horizontal at the initial point, then it is horizontal at any point. In this case γ is a horizontal curve.*

Remark 6.1. An arbitrary Killing magnetic curve γ in M satisfies the Lorentz equation $\gamma''(s) = q\varphi\gamma'(s)$. Note that the two sides of the equality above are horizontal since $\varphi\gamma'(s) = (J\bar{\gamma}'(s))^{\uparrow}$, where $\bar{\gamma} = \gamma \circ \pi$ and J is the complex structure on B. The Lorentz equation induces a new "Lorentz equation" on B, that is

$$\bar{\gamma}'' = (q + 2\tau\cos\theta)J\bar{\gamma}'. \tag{6.3}$$

So, we obtained a Kähler magnetic curve on B with variable strength. However, if the bundle curvature τ is constant, or γ is horizontal, it follows that $\bar{\gamma}$ is a (true) Kähler magnetic curve on B with strength $\bar{q} = q + 2\tau\ \cos\theta$.

In analogy with the definition introduced by O'Neill along horizontal geodesic we define the following notion. Let γ be a horizontal Killing magnetic field and E a vector field along γ. Decompose E as usual, into the horizontal and the vertical parts, respectively: $E = H + V$. We define

$$D(E) = \left(\nabla_{\gamma'}V\right)^v - \mathcal{T}_V\gamma' + 2\mathcal{A}_{\gamma'}H, \tag{6.4}$$

which is called the *derived vector field* of E [42]. Note that this definition can be stated for any horizontal curve γ.

Remark 6.2. $D(E)$ is always vertical.

Our aim for the next part of this section is to give an analogue of the Theorem 2 from the O'Neill's paper [42]. More precisely, we prove the following theorem:

Theorem 6.2. *Let E be a vector field on a horizontal Killing magnetic curve γ. Then the magnetic Jacobi operator $\mathcal{J}_{q,F}$ of charge q with respect to the magnetic field $F = 2\iota_\xi dv_g$ is given by*

$$\mathcal{J}_{q,F}(E) = E'' + R(E,\gamma')\gamma' - q\varphi(\nabla_{\gamma'}E) - q(\nabla_E\varphi)\gamma'$$

and it splits as $\mathcal{J}_{q,F}(E) = \mathcal{J}_{q,F}(E)^h + \mathcal{J}_{q,F}(E)^v$ with

$$\begin{cases} \mathcal{J}_{q,F}(E)^h = (\mathcal{J}_{q,J}(E_*))^{\uparrow} - 2\tau\eta(D(E))\varphi\gamma' \\ \mathcal{J}_{q,F}(E)^v = D(E)' + \tau\eta(D(E))\varphi\gamma', \end{cases} \tag{6.5}$$

where $D(E)$ is the derived vector field of E.

Before starting the proof of this theorem note that the projected curve $\bar\gamma = \pi\circ\gamma$ of a horizontal Killing magnetic curve γ is a Kähler magnetic curve on B with the same strength q as γ. The magnetic Jacobi operator $\mathcal{J}_{q,J}$ of $\bar\gamma$ satisfies

$$\mathcal{J}_{q,J}(E_*) = E_*'' + \bar R(E_*,\bar\gamma')\bar\gamma' - qJ\bar\nabla_{\bar\gamma'}E_*, \quad E_* = \pi_*(E).$$

We can write the expression for the derived vector field $D(E)$ in a more efficient form.

Lemma 6.2. *Let E be a vector field on a horizontal Killing magnetic curve γ. Then, the derived vector field $D(E)$ can be expressed as*

$$D(E) = \left[\frac{d}{ds}\eta(E) + 2\tau g(E,\varphi\gamma')\right]\xi. \tag{6.6}$$

Proof. We compute

$$D(E) = \Big[\nabla_{\gamma'}(\eta(E)\xi) - \nabla_{\eta(E)\xi}\gamma' + 2\nabla_{\gamma'}H\Big]^v.$$

Since $\nabla_\xi\gamma'$ and $\nabla_{\gamma'}\xi$ are horizontal, we can write

$$D(E) = \frac{d}{ds}\eta(E)\xi + 2\big[\nabla_{\gamma'}H\big]^v.$$

Now, since γ is horizontal, we have

$$\nabla_{\gamma'}H = (\bar\nabla_{\bar\gamma'}E_*)^\uparrow + \mathcal{A}_{\gamma'}H = (\bar\nabla_{\bar\gamma'}E_*)^\uparrow + g(\nabla_{\gamma'}H,\xi)\xi = (\bar\nabla_{\bar\gamma'}E_*)^\uparrow + \tau g(E,\varphi\gamma')\xi.$$

Hence the formula (6.6) is proved. □

We give, without proof, the expression of the covariant derivative of the derived vector field $D(E)$.

Lemma 6.3. *Let E be a vector field on a horizontal Killing magnetic curve γ and $D(E)$ the derived vector field. Its covariant derivative is given by the formula*

$$\begin{aligned} D(E)' &= \Big\{\tfrac{d^2}{ds^2}\eta(E) - 2d\tau(\gamma')g(\varphi E,\gamma') + 2\tau g(\varphi\gamma',(\bar\nabla_{\bar\gamma'}E_*)^\uparrow) + 2\tau^2\eta(E) - 2q\tau g(E,\gamma')\Big\}\xi \\ &\quad -\tau\eta(D(E))\varphi\gamma'. \end{aligned}$$

Proof of Theorem 6.2. We will develop the four terms which appear in the expression of $\mathcal{J}_{q,F}$.

1. The term E''

We split $\nabla_{\gamma'}\nabla_{\gamma'}E$ as $\nabla_{\gamma'}\nabla_{\gamma'}E = T_1 + T_2$, where $T_1 = \nabla_{\gamma'}\nabla_{\gamma'}H$ and $T_2 = \nabla_{\gamma'}\nabla_{\gamma'}(\eta(E)\xi)$.

We compute first T_1.

$$\begin{aligned} T_1 = \ & \nabla_{\gamma'}\big[(\bar\nabla_{\bar\gamma'}E_*)^\uparrow + \tau g(\varphi\gamma',E)\xi\big] = (\bar\nabla_{\bar\gamma'}\bar\nabla_{\bar\gamma'}E_*)^\uparrow + \tau g(\varphi\gamma',(\bar\nabla_{\bar\gamma'}E_*)^\uparrow)\xi \\ & + d\tau(\gamma')g(\varphi\gamma',E)\xi + \tau\frac{d}{ds}g(\varphi\gamma',E)\xi - \tau^2 g(\varphi\gamma',E)\varphi\gamma' \\ = \ & \big[(\bar\nabla_{\bar\gamma'}\bar\nabla_{\bar\gamma'}E_*)^\uparrow + \tau^2 g(\varphi E,\gamma')\varphi\gamma'\big] + \big[\tau g(\varphi\gamma',(\bar\nabla_{\bar\gamma'}E_*)^\uparrow) - d\tau(\gamma')g(\varphi E,\gamma') + \tau\frac{d}{ds}g(\varphi\gamma',E)\big]\xi. \end{aligned}$$

To compute the last term in the previous formula we write

$$\begin{aligned}\frac{d}{ds}g(\varphi\gamma', E) =&g(\nabla_{\gamma'}(\varphi\gamma'), H) + g(\varphi\gamma', \nabla_{\gamma'}H) \qquad \text{but } \gamma' \text{ is unitary and horizontal}\\ =&g(\tau\xi - q\gamma', H) + g(\varphi\gamma', (\bar\nabla_{\bar\gamma'}E_*)^{\uparrow} + \text{vertical terms})\\ =&-qg(\gamma', E) + g(\varphi\gamma', (\bar\nabla_{\bar\gamma'}E_*)^{\uparrow}).\end{aligned}$$

Thus we obtained the horizontal and the vertical parts for T_1:

$$\begin{aligned}T_1^h &= (\bar\nabla_{\bar\gamma'}\bar\nabla_{\bar\gamma'}E_*)^{\uparrow} + \tau^2 g(\varphi E, \gamma')\varphi\gamma' = (E_*'')^{\uparrow} + \tau^2 g(\varphi E, \gamma')\varphi\gamma',\\ T_1^v &= \big[2\tau g(\varphi\gamma', (\bar\nabla_{\bar\gamma'}E_*)^{\uparrow}) - d\tau(\gamma')g(\varphi E, \gamma') - \tau q g(E, \gamma')\big]\xi.\end{aligned}$$

Next, let us compute now T_2.

$$\begin{aligned}T_2 &= \nabla_{\gamma'}\nabla_{\gamma'}(\eta(E)\xi) = \nabla_{\gamma'}\big[\tfrac{d}{ds}\eta(E)\xi - \tau\eta(E)\varphi\gamma'\big]\\ &= \tfrac{d^2}{ds^2}\eta(E)\xi - 2\alpha\tfrac{d}{ds}\eta(E)\varphi\gamma' - d\tau(\gamma')\eta(E)\varphi\gamma' - \tau\eta(E)(\tau\xi - q\gamma').\end{aligned}$$

We got the horizontal and the vertical parts for T_2:

$$T_2^h = \tau q\eta(E)\gamma' - 2\tau\frac{d}{ds}\eta(E)\varphi\gamma' - d\tau(\gamma')\eta(E)\varphi\gamma', \quad T_2^v = \big[\frac{d^2}{ds^2}\eta(E) - \tau^2\eta(E)\big]\xi.$$

2. The term $R(E, \gamma')\gamma'$

To compute the curvature term $R(E, \gamma')\gamma'$, first we decompose it as $R(E, \gamma')\gamma' = T_3 + T_4$, where $T_3 = R(H, \gamma')\gamma'$ and $T_4 = \eta(E)R(\xi, \gamma')\gamma'$ and we use the formula (4.3).

If we set $\bar X = E_*$ and $\bar Y = \bar Z = \bar\gamma'$ in (4.3) and since γ is horizontal, we obtain

$$R(H, \gamma')\gamma' = \big(\bar R(E_*, \bar\gamma')\bar\gamma'\big)^{\uparrow} - d\tau(\gamma')g(\varphi E, \gamma')\xi + 3\tau^2 g(\varphi E, \gamma')\varphi\gamma'.$$

Identifying the horizontal and the vertical parts respectively, we get

$$T_3^h = \big(\bar R(E_*, \bar\gamma')\bar\gamma'\big)^{\uparrow} + 3\tau^2 g(\varphi E, \gamma')\varphi\gamma', \quad T_3^v = -d\tau(\gamma')g(\varphi E, \gamma')\xi.$$

In order to compute T_4 we write

$$R(\xi, \gamma')\gamma' = \nabla_\xi\nabla_{\gamma'}\gamma' - \nabla_{\gamma'}\nabla_\xi\gamma' - \nabla_{[\xi,\gamma']}\gamma' = \tau^2\xi + d\tau(\gamma')\varphi\gamma'.$$

Here we extend first γ' to a basic vector field X and then we compute the brackets $[\xi, X] = 0$. This implies also $\nabla_\xi\gamma' = -\tau\varphi\gamma'$.

Note that the term T_4 can be also computed as

$$\begin{aligned}R(\xi, \gamma')\gamma' =& -\nabla_{\gamma'}\nabla_{\gamma'}\xi + \nabla_{\nabla_{\gamma'}\gamma'}\xi = \nabla_{\gamma'}(\tau\varphi\gamma') + \nabla_{q\varphi\gamma'}\xi\\ =&d\tau(\gamma')\varphi\gamma' + \tau\nabla_{\gamma'}(\varphi\gamma') + q\nabla_{\varphi\gamma'}\xi\\ =&d\tau(\gamma')\varphi\gamma' + \tau\big[\tau\xi - q\gamma'\big] + q\tau\gamma' = \tau^2\xi + d\tau(\gamma')\varphi\gamma'.\end{aligned}$$

Thus we have

$$T_4^h = \eta(E)d\tau(\gamma')\varphi\gamma', \quad T_4^v = \tau^2\eta(E)\xi.$$

3. $\boxed{\text{The term } q\varphi(\nabla_{\gamma'}E)}$

We have $T_5 := q\varphi(\nabla_{\gamma'}E) = q\varphi(\nabla_{\gamma'}H) + q\varphi(\nabla_{\gamma'}(\eta(E)\xi))$.

To compute T_5 we write

$$\varphi(\nabla_{\gamma'}H) = \varphi\big((\bar{\nabla}_{\bar{\gamma}'}E_*)^{\uparrow} + \text{vertical part}\big) = \big(J\bar{\nabla}_{\bar{\gamma}'}E_*\big)^{\uparrow}$$

and

$$\varphi(\nabla_{\gamma'}(\eta(E)\xi)) = \varphi(\frac{d}{ds}\eta(E)\xi + \eta(E)\nabla_{\gamma'}\xi) = \tau\eta(E)\gamma'.$$

We conclude that T_5 is horizontal and hence

$$T_5^h = q\big(J\bar{\nabla}_{\bar{\gamma}'}E_*\big)^{\uparrow} + q\tau\eta(E)\gamma', \quad T_5^v = 0.$$

4. $\boxed{\text{The term } q(\nabla_E\varphi)\gamma'}$

We easily get that $T_6 := q(\nabla_E\varphi)\gamma' = q\tau g(E,\gamma')\xi$, which shows that T_6 is vertical:

$$T_6^h = 0, \quad T_6^v = q\tau g(E,\gamma')\xi.$$

We now collect all terms T_1 to T_6 and consider the horizontal and the vertical parts, respectively. Combining with Lemma 6.2 and Lemma 6.3 we get the statement. □

As a consequence of the previous theorem we have the following result.

Theorem 6.3. *Let E be a magnetic Jacobi field on the horizontal Killing magnetic curve γ on M. Assume that the bundle curvature is non-zero everywhere on M. Then $E_* = \pi_*(E)$ is a magnetic Jacobi field on $\bar{\gamma} = \pi\circ\gamma$ if and only if $D(E) = 0$.*

Proof. The hypothesis says that $\mathcal{J}_{q,F}(E) = 0$. Using Theorem 6.2 we obtain

$$\begin{cases} \mathcal{J}_{q,J}(E_*) - 2\tau\eta(D(E))J\bar{\gamma}' = 0 \\ \\ D(E)' + \tau\eta(D(E))\varphi\gamma' = 0. \end{cases} \tag{6.7}$$

If $D(E)$ vanishes, $D(E)'$ also vanishes and the first equation in (6.7) implies $\mathcal{J}_{q,J}(E_*) = 0$, that is E_* is a magnetic Jacobi field on the Kähler magnetic curve $\bar{\gamma}$.

Conversely, suppose that $\mathcal{J}_{q,J}(E_*) = 0$. It follows that $\eta(D(E)) = 0$, since $\tau \neq 0$ and γ is horizontal. Recall that $D(E)$ is vertical. Thus, the condition $\eta(D(E)) = 0$ implies $D(E) = 0$.

□

Remark 6.3. In case $\tau = 0$, M is locally isometric to the Riemannian product $B\times\mathbb{R}$. In this case, the second equation in (6.7) leads to $D(E)' = 0$. Using Lemma 6.3 with $\tau = 0$, we obtain

$$\frac{d^2}{ds^2}\eta(E) = 0.$$

Hence every magnetic Jacobi field E along a horizontal Killing magnetic curve γ has the form:

$$E(s) = \overline{E}(s)^{\uparrow} + (\rho_0 s^2 + \rho_1 s + \rho_2)\xi_{\gamma(s)},$$

where $\overline{E}(s)$ is a magnetic Jacobi field along $\bar{\gamma}$ and ρ_0, ρ_1 and ρ_2 are constants.

The next result shows an application of the derived vector field. See [42, Lemma 1].

Theorem 6.4. *Let γ be a horizontal Killing magnetic curve on M. Given a vector field E_* on $\pi \circ \gamma$ and a vertical vector v at $\gamma(0)$, there exists a unique vector field E on γ such that*

$$(1)\ \pi_* E = E_*, \qquad (2)\ D(E) = 0 \quad \text{and} \quad (3)\ E(0) = v.$$

Furthermore, E is a magnetic Jacobi field if and only if E_* is.

Proof. If E is a solution of the proposed problem, then its horizontal part is uniquely determined by the horizontal lift of E_*, that is $E^h = E_*^{\uparrow}$. On the other hand, the vertical part of E is expressed as $E^v = \eta(E)\xi$ with the initial condition $E^v(0) = \eta(v)\xi$.

Now, condition (2) leads to the following differential equation

$$\frac{d}{ds}\eta(E)\xi + 2\big[\nabla_{\gamma'} E_*^{\uparrow}\big]^v = 0,$$

and so we obtain a Cauchy problem

$$\begin{cases} \frac{d}{ds}\eta(E) + 2\tau \bar{g}(E_*, J\bar{\gamma}') = 0, \\ \eta(E)(0) = \eta_{\gamma(0)}(v), \end{cases}$$

which has a unique solution.

For the second part of the statement we use Theorem 6.2. Since $D(E) = 0$, the formula (6.5) becomes

$$\mathcal{J}_{q,F}(E)^h = \big[\mathcal{J}_{q,J}(E_*)\big]^{\uparrow} \quad \text{and} \quad \mathcal{J}_{q,F}(E)^v = 0.$$

The conclusion follows immediately. □

The next result is another application of the derived vector field. Moreover, it gives the set of all vertical magnetic Jacobi fields on a horizontal Killing magnetic field on M. It is easy to prove that ξ is a vertical magnetic Jacobi fields along any horizontal Killing magnetic field on M (see also Proposition 7.1). We state the following.

Theorem 6.5. *Let γ be a horizontal Killing magnetic curve in the Killing submersion $M \to B$. If the bundle curvature τ is nowhere 0, then E is a vertical magnetic Jacobi field along γ if and only if $D(E) = 0$. In particular, the set of all vertical magnetic Jacobi fields along γ consists in $\{a\xi \,:\, a \in \mathbb{R}\}$.*

Proof. Using the formula (2.1) we find $\mathcal{J}_{q,F}(f\xi) = f''\xi - 2\tau f'\varphi\gamma'$, where f is a smooth function of M. Thus $f\xi$ is a magnetic Jacobi field along γ if and only if $f' = 0$ (since $\tau \neq 0$). Taking $f = \eta(E)$ and recalling that $D(E) = \frac{d}{ds}\eta(E)\xi$ (since E is vertical), we get the statement. □

7. Magnetic Jacobi fields along horizontal magnetic curves in Sasakian space forms

To illustrate the general theory developed in preceding section, we investigate magnetic Jacobi fields along horizontal Killing magnetic trajectories in 3-dimensional Sasakian space forms.

As we have seen before, the total space M of a Killing submersion $\pi : M \to B$ of constant bundle curvature 1 is Sasakian. In addition, a Killing submersion can be normalized to Sasakian total space if $\tau > 0$ or $\tau < 0$. One can see that Killing magnetic equation $\nabla_{\gamma'}\gamma' = q\varphi\gamma'$ is *not* preserved. However, remarkably, horizontal Killing magnetic curves are preserved under pseudo-conformal deformation. This observation suggests also us to study Killing magnetic trajectories in Killing submersions of bundle curvature 1, that is, Killing submersions with Sasakian structures.

First we should emphasize that Sasakian 3-manifolds are *not* necessarily total spaces of Killing submersions.

7.1. **The Boothby-Wang fibration.** Let $M^3 = (M, \eta)$ be a contact 3-manifold. Then M is said to be *regular* if its Reeb vector field is regular. In addition a contact metric 3-manifold is said to be a *regular contact metric 3-manifold* if its contact form η is regular.

Now let us consider a regular contact metric 3-manifold M. Then it is known that M is a Sasakian 3-manifold. Moreover the projection $\pi : M \to B = M/\mathcal{G}$ with $\mathcal{G} = \{\exp(t\xi)\}_{t\in\mathbb{R}}$ is a submersion. All the structure tensor fields of M are invariant under the action of $\mathcal{G}$. Thus the contact metric structure of M induces a Kähler structure $(\bar{g}, J)$ on B as we have seen in Section 4.3. The resulting submersion $\pi : M \to B$ is a Killing submersion.

Thus, Killing submersions with bundle curvature 1 are identified with regular Sasakian 3-manifolds. (For the quasi-Sasakian case, see [36].)

In the case when M is compact, the fibering $\pi : M \to B$ of a regular contact 3-manifold is well known as the *Boothby-Wang fibering* [14]. As we mentioned before, although every regular contact metric 3-manifold is Sasakian, the converse statement does not hold. In fact, Tanno [50, §6] constructed an example of non-regular compact Sasakian 3-manifold (of constant curvature 1). Abe gave examples of non-regular almost contact structures on exotic spheres [2]. The standard examples of *regular* Sasakian 3-manifold are 3-dimensional Sasakian space forms SU(2), $\mathbb{S}^3$, Nil_3 and $\widetilde{\mathrm{SL}}_2\mathbb{R}$ exhibited in Example 4.1.

Remark 7.1 (Quasi-regularity)**.** A contact form η is said to be *quasi-regular* if its Reeb vetor field ξ is quasi-regular. Rukimbira showed that every Sasakian 3-manifold admits a quasi-regular Sasakian structure [45]. If a compact Sasakian 3-manifold M is quasi-regular; then M is a Seifert fibration (principal $\mathbb{S}^1$-orbibundle) over a Hodge 2-orbifold, (see [14, Theorem 7.1.3]).

7.2. **Some fundamental results.** Now we prove an interesting result on the Reeb vector field ξ.

Proposition 7.1. *Let $\pi : M \to B$ be a Killing submersion of bundle curvature 1. Then the Reeb vector field is a magnetic Jacobi field along any normal Killing magnetic curve γ in a Sasakian 3-manifold.*

Proof. Since the associated quasi-Sasakian structure is Sasakian, we have

$$\eta(\gamma') = \cos\theta, \quad \nabla_{\gamma'}\xi = -\varphi\gamma',$$

where θ is the constant contact angle of γ.

In order to prove the statement we need some auxiliary computations:

$$\nabla_{\gamma'}\nabla_{\gamma'}\xi = \nabla_{\gamma'}(-\varphi\gamma') = -(1+q\cos\theta)\xi + (q+\cos\theta)\gamma'.$$

Then by the formulas (3.5) and (4.5), we get

$$(\nabla_\xi\varphi)\gamma' = 0, \quad \varphi(\nabla_{\gamma'}\xi) = \gamma' - \cos\theta\ \xi, \quad R(\xi,\gamma')\gamma' = \xi - \cos\theta\ \gamma'.$$

Hence we obtain $\mathcal{J}_{q,-d\eta}(\xi) = 0$. We conclude that ξ is a magnetic Jacobi field along γ. □

We state now the following result.

Proposition 7.2. *Let γ be a normal Killing magnetic curve in the total space of a Killing submersion of bundle curvature 1. Then $\varphi\gamma'$ is a magnetic Jacobi field along γ if and only if either it is an integral curve of the Killing vector field ξ, or the horizontal curvature of M is 1.*

Proof. We do the following computations by using the curvature formula (4.5)

$$\begin{aligned}
\nabla_{\gamma'}\nabla_{\gamma'}(\varphi\gamma') &= \nabla_{\gamma'}\big((1+q\cos\theta)\xi - (q+\cos\theta)\gamma'\big) = -(1+q^2+2q\cos\theta)\varphi\gamma',\\
R(\varphi\gamma',\gamma')\gamma' &= \big\{(\kappa-3) - ((\kappa-4))\cos^2\theta\big\}\,\varphi\gamma',\\
(\nabla_{\varphi\gamma'}\varphi)\gamma' &= -\,(\cos\theta)\varphi\gamma', \quad \varphi(\nabla_{\gamma'}(\varphi\gamma')) = -(q+\cos\theta)\varphi\gamma'.
\end{aligned}$$

Hence we get

$$\mathcal{J}_{q,-d\eta}(\varphi\gamma') = (\kappa-4)\sin^2\theta\ \varphi\gamma'.$$

Thus $\varphi\gamma'$ is an eigensection of $\mathcal{J}_{q,-d\eta}$ with the eigenvalue $\frac{\kappa-4}{2}\sin^2\theta$. The statement follows immediately because $\kappa = 4$ is equivalent to that horizontal sectional curvature is 1. □

Note that the Proposition 7.2 can be rephrased as follows:

Corollary 7.1. *Let γ be a normal Killing magnetic curve in the total space of a Killing submersion of bundle curvature 1 such that $\gamma'(s)$ and $\xi_{\gamma(s)}$ are not collinear. Then $\varphi\gamma'$ is a magnetic Jacobi field along γ if and only if the horizontal sectional curvature of M is 1.*

7.3. **Magnetic Jacobi fields.** In this subsection we assume that κ is constant. Thus as a Sasakian manifold, M is of constant holomorphic sectional curvature. We denote the holomorphic sectional curvature of M by c and the total space by $M^3(c)$.

Let γ be a normal horizontal magnetic curve in a Killing submersion of bundle curvature 1.

Let E be a magnetic Jacobi field along a horizontal normal Killing magnetic curve γ. Since $\xi_{\gamma(s)}$, $\gamma'(s)$ and $\varphi\gamma'(s)$ are unitary and linearly independent for any s, we may decompose E in this orthonormal basis as follows

$$E(s) = f(s)\xi_{\gamma(s)} + a(s)\gamma'(s) + b(s)\varphi\gamma'(s), \tag{7.1}$$

where f, a and b are smooth functions depending on s.

The aim of this section is to find magnetic Jacobi field $E(s)$.

In order to apply the formula (2.1) we recall first the following formulas

$$\mathcal{J}_{q,-d\eta}(\xi) = 0, \quad \mathcal{J}_{q,-d\eta}(\gamma') = 0 \quad \text{and} \quad \mathcal{J}_{q,-d\eta}(\varphi\gamma') = (c-1)\varphi\gamma'. \tag{7.2}$$

By straightforward computations, we can prove that the condition $\mathcal{J}_{q,-d\eta}(E)=0$ leads to the following ODE system

$$\begin{cases} f''+2b'=0 \\ a''-qb'=0 \\ b''-2f'+qa'+(c-1)b=0. \end{cases} \tag{7.3}$$

It should be remarked that the second equation is nothing but the conservation law discussed in Proposition 5.2. In fact one can check that $g(\nabla_{\gamma'}W,\gamma')=a'-qb$.

Taking the derivative in the third equation and then replacing f'' and a'' from the first two equations we obtain

$$b'''+\mu b'=0, \tag{7.4}$$

where $\mu=q^2+c+3$.

The solutions of this ODE depends on the sign of μ and hence we distinguish three cases.

The case $\mu=0$

It follows that $b(s)=b_0s^2+b_1s+b_2$, where b_0,b_1 and b_2 are real constants. Plugging this expression of b in (7.3) we obtain

$$f(s)=-\tfrac{2}{3}b_0s^3-b_1s^2+\rho_0s+\rho_1,$$

$$a(s)=\tfrac{1}{3}qb_0s^3+\tfrac{1}{2}qb_1s^2+a_0s+a_1,$$

where ρ_0, ρ_1, a_0 and a_1 are real constants such that

$$2\rho_0-qa_0=2b_0+(c-1)b_2. \tag{7.5}$$

The case $\mu>0$ Put $\mathsf{k}=\sqrt{\mu}$.

It follows that $b(s)=b_0\cos(\mathsf{k}s)+b_1\sin(\mathsf{k}s)+b_2$, where b_0,b_1 and b_2 are real constants. Plugging this expression of b in (7.3) we obtain

$$f(s)=\tfrac{2}{\mathsf{k}}\left[-b_0\sin(\mathsf{k}s)+b_1\cos(\mathsf{k}s)+\rho_0s+\rho_1\right],$$

$$a(s)=\tfrac{q}{\mathsf{k}}\left[b_0\sin(\mathsf{k}s)-b_1\cos(\mathsf{k}s)+a_0s+a_1\right],$$

where ρ_0, ρ_1, a_0 and a_1 are real constants such that

$$4\rho_0-q^2a_0=\mathsf{k}(c-1)b_2. \tag{7.6}$$

The case $\mu < 0$ Put $\mathsf{k} = \sqrt{-\mu}$.

It follows that $b(s) = b_0 \cosh(\mathsf{k}s) + b_1 \sinh(\mathsf{k}s) + b_2$, where b_0, b_1 and b_2 are real constants. Plugging this expression of b in (7.3) we obtain

$$f(s) = -\tfrac{2}{\mathsf{k}}\left[b_0 \sinh(\mathsf{k}s) + b_1 \cosh(\mathsf{k}s) + \rho_0 s + \rho_1\right],$$

$$a(s) = \tfrac{q}{\mathsf{k}}\left[b_0 \sinh(\mathsf{k}s) + b_1 \cosh(\mathsf{k}s) + a_0 s + a_1\right],$$

where ρ_0, ρ_1, a_0 and a_1 are real constants such that

$$4\rho_0 + q^2 a_0 = \mathsf{k}(1-c)b_2. \tag{7.7}$$

7.4. **Projections.** Take a horizontal normal Killing magnetic curve and a magnetic Jacobi field $E(s)$ along it. Then the horizontal part E^h and vertical part E^v of E are expressed as

$$E^h(s) = a(s)\gamma'(s) + b(s)\varphi\gamma'(s), \quad E^v(s) = f(s)\xi_{\gamma(s)}.$$

The derived vector field $D(E)$ is computed as

$$D(E) = (f'(s) + 2b(s))\xi_{\gamma(s)}.$$

From the first equation of (7.3), the coefficient $\eta(D(E))$ of $D(E)$ is constant. Thus from Theorem 6.3,

$$E_* = \pi_* E = a(s)\bar{\gamma}'(s) + b(s)J\bar{\gamma}'(s)$$

is a magnetic Jacobi filed along $\bar{\gamma} = \pi \circ \gamma$ if and only if $f' + 2b = 0$.

- The case $\mu = 0$: In this case, $f' + 2b = \rho_0 + 2b_2 = 0$. This together with (7.5), we obtain
$$2b_0 + qa_0 = -(c+3)b_2$$
- The case $\mu > 0$: In this case,
$$f' + 2b = \frac{2\rho_0}{\mathsf{k}} + 2b_2 = 0$$
This together with (7.6), we get
$$-q^2 a_0 = \mathsf{k}(c+3)b_2$$
- The case $\mu < 0$: In this case,
$$f' + 2b = -\frac{2\rho_0}{\mathsf{k}} + 2b_2 = 0$$
This together with (7.7), we get
$$-q^2 a_0 = \mathsf{k}(c+3)b_2$$
Recall that $c + 3 = \kappa$ (the Gaussian curvature of B).

Explicit expression for normal magnetic Jacobi fields on 2-dimensional space forms $\mathbb{E}^2$, $\mathbb{S}^2$ and $\mathbb{H}^2$ can be seen in [3].

Final Remarks. The study of magnetic Jacobi fields is motivated by the well known comparison theorems in Riemannian geometry. Detailed study on magnetic Jacobi fields gives us insight on how small variations in the initial conditions affect the evolution of magnetic curves.

In this direction, Adachi [4] obtained the comparison theorem for magnetic curves on Kähler manifold whose Lorentz force is a complex structure (uniform magnetic field).

Gouda [22] studied magnetic Jacobi fields on Riemannian 2-manifolds equipped with compatible Kähler structure. The parallelism of the complex structure (the Lorentz force) is crucial in these works.

On the contrary, for Sasakian 3-manifolds, the Lorentz force is non-parallel. Thus the behavior of magnetic Jacobi fields along contact magnetic curves appears complicated. Geometric and/or topological applications of magnetic Jacobi fields, *e.g.*, magnetic version of comparison theorems for geodesics, will be discussed in future publications.

As we have already mentioned before, our work is inspired by O'Neill's work [41]. There are some strong motivation to consider horizontal trajectories apart the fact this topic needs a future deep study. Legendre curves are of particular interest for contact geometry and contact topology. In terms of Killing submersion, the notion of horizontal curve coincides with that of Legendre curve. Next, for CR-geometry, Legendre magnetic trajectories are nothing but sub-Riemannian geodesics (see [29, Remark 5.1]). On the other hand, magnetic curves in some product spaces are studied in [37]. See also [38].

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