

# A Matsumoto type theorem for linear groups over rings of non-commutative Laurent polynomials

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## Introduction

Many researchers have studied the structure of the general linear group and its elementary subgroup over a field  $F$  or a commutative ring  $R$ . They also have analyzed associated lower  $K$ -groups, for example [8] and [16]. Needless to say, the general linear groups are important objects and have many applications in various areas of mathematics, but they particularly have much to do with Lie theory; Lie groups, Lie algebras and their representations. Based on Chevalley's study on linear groups over any field  $F$ , Iwahori–Matsumoto [4] generalized a Bruhat decomposition: for a  $\mathfrak{p}$ -adic Chevalley group  $G$ , they showed that there exists a decomposition  $G = \bigcup_{w \in W_a} BwB$  with  $W_a$  the corresponding affine Weyl group, where  $B$  is a generalized Borel subgroup called the “Iwahori subgroup”. Using this notion, Moody–Teo [19], Marcuson [6] and Peterson–Kac [12] discussed a Tits system for Kac–Moody groups (generalized Chevalley groups). As a special case of affine Kac–Moody groups, Morita [9] proved that loop groups have Tits systems with the affine Weyl groups. Linear groups corresponding to extended affine Lie algebras, which are an affinization of [9], were studied in [11]. There is a direct relationship between the nullity of extended affine Lie algebras and the number of variables of the Laurent polynomials. Morita–Sakaguchi [11] researched groups over a completed quantum torus with two variables. In order to generalize [11] to the case of nullity  $n \geq 3$ , we need a ring  $D_\tau = D[t, t^{-1}]$  of non-commutative Laurent polynomials over a division ring  $D$  (see Section 1).

Our main object in this paper is the following exact sequence [17]:

$$1 \rightarrow K_2(n, D_\tau) \rightarrow St(n, D_\tau) \xrightarrow{\phi} GL(n, D_\tau) \rightarrow K_1(n, D_\tau) \rightarrow 1.$$

In Section 1, we reveal the structure of the groups in the above sequence, except for the presentation of  $K_2(n, D_\tau)$ . We first describe an existence of the Tits system in the elementary subgroup  $E(n, D_\tau)$  of the general linear group  $GL(n, D_\tau)$  and the associated Steinberg group  $St(n, D_\tau)$  in Subsections 1.2 and 1.3, respectively. Using these facts, we show that the above homomorphism  $\phi$  is a central extension of  $E(n, D_\tau)$ , that is, we confirm that  $\text{Ker } \phi = K_2(n, D_\tau)$  is a central subgroup of the Steinberg group in Subsection 1.4. It is proved in Subsection 1.5 that  $\phi$  is universal when the center  $Z(D)$  of  $D$  has at least five elements. Meanwhile, we discuss the structure of the associated  $K_1$ -group and  $K_2$ -group in Subsections 1.4 and 1.6. In particular, we check that the  $K_2$ -group is generated by the Steinberg symbols.

In Section 2, we give a presentation of  $K_2(n, D_\tau)$ . The structure of linear groups is important object, but the associated lower  $K$ -groups are also remarkable ones. Indeed, it is a well-known fact that the  $K_2$ -group is an invariant which measures the size of central extensions of the group. For the Chevalley group over any field, the presentation of the  $K_2$ -group has already been given by Matsumoto [7]. The  $K_2$ -group derived from the loop group was studied in Tomie [19], and Sakaguchi [15] gave the Matsumoto type presentation for linear groups over the quantum torus in two variables. Moreover, Rehmann [13] [14] has already determined the presentation of the  $K_2$ -group over a division ring  $D$ .

As mentioned in Matsumoto [7], there exist two types of the  $K_2$ -group, which are called “symplectic type” when  $n = 2$ , and “non-symplectic type” when  $n \geq 3$ . In fact, the  $K_2$ -group changes its group structure depending on the size of the elementary subgroup: the symplectic type  $K_2$ -group  $K_2(2, F)$  is presented by the symbols  $c(u, v)$ ,  $u, v \in F^\times$ , and the following defining relations:

$$\begin{aligned} c(u, v)c(uv, w) &= c(u, vw)c(v, w), \\ c(1, 1) &= 1, \\ c(u, v) &= c(u^{-1}, v^{-1}), \\ c(u, v) &= c(u, (1 - u)v) \quad (1 - u \in F^\times). \end{aligned}$$

On the other hand, the non-symplectic type  $K_2$ -groups  $K_2(n, F)$  is presented by the symbols  $c(u, v)$ ,  $u, v \in F^\times$ , and the following defining relations:

$$\begin{aligned} c(uv, w) &= c(u, w)c(v, w), \\ c(u, vw) &= c(u, v)c(u, w), \\ c(u, 1 - u) &= 1 \quad (1 - u \in F^\times). \end{aligned}$$

In this paper, we determine the presentation of the  $K_2$ -groups of symplectic type and non-symplectic type, respectively (see Section 2), which is a non-commutative version of Tomie’s result for loop groups [19]. Our main idea is due to Rehmann’s approach in the case of division rings [13], [14]. In Subsection 2.1, we give the group presentation of the symplectic  $K_2$ -group  $K_2(2, D_\tau)$ . Let  $P$  be the group presented by generators  $c(u, v)$ ,  $u, v \in D_\tau^\times$ , with the following defining relations:

$$\begin{aligned} (\text{P1}) \quad & c(u, v)c(vu, w) = c(u, vw)c(v, w), \\ (\text{P2}) \quad & c(u, v) = c(uvu, u^{-1}), \\ (\text{P3}) \quad & c(x, y)c(u, v)c(x, y)^{-1} = c([x, y]u, v)c(v, [x, y]), \\ (\text{P4}) \quad & c(u, v) = c(u, v(1 - u)) \quad (1 - u \in D_\tau^\times), \\ (\text{P5}) \quad & c(u, v) = c(u, -vu). \end{aligned}$$

Then, there exists a natural homomorphism  $\varphi$  of  $P$  onto  $[D_\tau^\times, D_\tau^\times]$ , whose kernel is isomorphic to  $K_2(2, D_\tau)$ . The presentation of non-symplectic  $K_2(n, D_\tau)$  is given in Subsection 2.2. Let  $Q$  be the group presented by generators  $c(u, v)$ ,  $u, v \in D_\tau^\times$  with the following defining relations:

$$\begin{aligned} (\text{Q1}) \quad & c(uv, w) = c({}^u v, {}^u w)c(u, w), \\ (\text{Q2}) \quad & c(u, vw) = c(u, v)c({}^v u, {}^v w), \\ (\text{Q3}) \quad & c(u, 1 - u) = 1 \quad (1 - u \in D_\tau^\times), \end{aligned}$$

where  ${}^u v = uvu^{-1}$ . Then, there exists a natural homomorphism  $\varphi_0$  of  $Q$  onto  $[D_\tau^\times, D_\tau^\times]$ , whose kernel is isomorphic to  $K_2(n, D_\tau)$ .

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# 1 Universal central extensions of linear groups over rings of non-commutative Laurent polynomials $D_\tau$

## 1.1 Preliminaries.

Let  $D$  be a division ring, and fix an automorphism  $\tau \in \text{Aut}(D)$ . In the following, we denote by  $D_\tau = D[t, t^{-1}]$  the ring of Laurent polynomials with coefficients in  $D$ , where  $t$  is an indeterminate with the relation  $tat^{-1} = \tau(a)$  for  $a \in D$ ; note that  $D_\tau^\times = \{st^k = t^k\tau^{-k}(s) \mid s \in D \setminus \{0\}, k \in \mathbb{Z}\}$ , where  $R^\times$  denotes the group of invertible elements in a ring  $R$ . Let  $M(n, D_\tau)$  be the ring of  $n \times n$  matrices whose entries are in  $D_\tau$ , and define  $GL(n, D_\tau) = M(n, D_\tau)^\times$ .

Let  $\Delta = \{\epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n\}$  be the root system of type  $A_{n-1}$ , where  $\{\epsilon_i\}_{1 \leq i \leq n}$  is an orthonormal basis with respect to an inner product  $(\cdot, \cdot)$  in an  $n$ -dimensional vector space over  $\mathbb{R}$ . We see that for  $1 \leq i < j \leq n$ ,

$$\epsilon_i - \epsilon_j = (\epsilon_i - \epsilon_{i+1}) + (\epsilon_{i+1} - \epsilon_{i+2}) + \cdots + (\epsilon_{j-1} - \epsilon_j).$$

Let  $\Pi = \{\alpha_i = \epsilon_i - \epsilon_{i+1} \mid 1 \leq i \leq n-1\}$  be the simple system of  $\Delta$ . Let  $\Delta^+ = (\text{Span}_{\mathbb{Z}_{\geq 0}} \Pi) \cap \Delta$  be the set of positive roots in  $\Delta$ , and  $\Delta^- = -\Delta^+$  the set of negative roots in  $\Delta$ . Also, let  $\Delta_a = \Delta \times \mathbb{Z}$  be the (abstract) affine root system of type  $A_{n-1}^{(1)}$ , and let  $\Pi_a = \{\dot{\alpha}_i = (\alpha_i, 0) \mid 1 \leq i \leq n-1\} \cup \{\dot{\alpha}_0 = (-\theta, 1)\}$  be the simple system of  $\Delta_a$ , where  $\theta = \epsilon_i - \epsilon_j = \alpha_1 + \cdots + \alpha_{n-1}$  is the highest root in  $\Delta$ . Let  $\Delta_a^+ = (\Delta^+ \times \mathbb{Z}_{\geq 0}) \cup (\Delta^- \times \mathbb{Z}_{> 0})$  (resp.  $\Delta_a^- = (\Delta^+ \times \mathbb{Z}_{< 0}) \cup (\Delta^- \times \mathbb{Z}_{\leq 0})$ ) be the set of positive roots (resp. negative roots) in  $\Delta_a$ . We identify  $\beta \in \Delta$  with  $(\beta, 0) \in \Delta_a$ , and regard  $\Delta$  as a subset of  $\Delta_a$ . For  $\dot{\beta} = (\beta, m) \in \Delta_a$ , we set  $-\dot{\beta} = (-\beta, -m)$ .

By definition, the Weyl group  $W$  of  $\Delta$  is generated by all reflections  $\sigma_\beta$  for  $\beta \in \Delta$ , and the Weyl group  $W_a$  of  $\Delta_a$  (the affine Weyl group of  $\Delta$ ) is generated by all reflections  $\sigma_{\dot{\beta}}$  for  $\dot{\beta} \in \Delta_a$ , where the action of  $\sigma_{\dot{\beta}}$  is defined by the following way; for  $\dot{\beta} = (\beta, m)$ ,  $\dot{\gamma} = (\gamma, n) \in \Delta_a$ ,

$$\sigma_{\dot{\beta}}(\dot{\gamma}) = (\sigma_\beta(\gamma), n - \langle \gamma, \beta \rangle m),$$

where  $\langle \gamma, \beta \rangle = 2(\gamma, \beta)/(\beta, \beta)$  for  $\beta, \gamma \in \Delta$ . Define  $\xi_{\dot{\beta}} = \sigma_{\dot{\beta}}\sigma_\beta$  for each  $\dot{\beta} = (\beta, m) \in \Delta_a$ , and let  $T_a$  be the subgroup of  $W_a$  generated by  $\xi_{\dot{\beta}}$  for all  $\dot{\beta} \in \Delta_a$ . We know the following lemma and proposition from [9, Lemma 1.1, Proposition 1.2].

### Lemma 1.1.1.

- (1) Let  $\dot{\beta} = (\beta, m)$ , and  $\dot{\gamma} = (\gamma, n)$ . Then,  $\xi_{\dot{\beta}}(\dot{\gamma}) = (\gamma, n + \langle \gamma, \beta \rangle m)$ .
- (2) The subgroup  $T_a$  is the free abelian group with  $\{\xi_{(\alpha, 1)} \mid \alpha \in \Pi\}$  a free  $\mathbb{Z}$ -basis.
- (3) The subgroup  $T_a$  is a normal subgroup of  $W_a$ .

**Proposition 1.1.2.** It holds that  $W_a = T_a W = T_a \rtimes W$ .

## 1.2 Linear groups over rings of non-commutative Laurent polynomials $D_\tau$

For  $\beta = \epsilon_i - \epsilon_j \in \Delta$  and  $f \in D_\tau$ , we define

$$x_\beta(f) = x_{ij}(f) = I + fE_{ij},$$

where  $I \in GL(n, D_\tau)$  is the identity matrix, and  $E_{ij} \in M(n, D_\tau)$  is the matrix unit. For  $\beta = \epsilon_i - \epsilon_j \in \Delta$  and  $u \in D_\tau^\times$ , we set

$$\begin{aligned} w_\beta(u) &= w_{ij}(u) = x_\beta(u)x_{-\beta}(-u^{-1})x_\beta(u), \\ h_\beta(u) &= h_{ij}(u) = w_\beta(u)w_\beta(-1). \end{aligned}$$

Also, for  $\dot{\beta} = (\beta, m) \in \Delta_a$ ,  $f \in D$ , and  $s \in D^\times$ , we set

$$\begin{aligned} x_{\dot{\beta}}(f) &= \begin{cases} x_\beta(ft^m) & \text{if } \beta \in \Delta^+, \\ x_\beta(t^m f) & \text{if } \beta \in \Delta^-, \end{cases} \\ w_{\dot{\beta}}(s) &= x_{\dot{\beta}}(s)x_{-\dot{\beta}}(-s^{-1})x_{\dot{\beta}}(s), \\ h_{\dot{\beta}}(s) &= w_{\dot{\beta}}(s)w_{\dot{\beta}}(-1), \end{aligned}$$

It can be easily checked that  $x_{\dot{\beta}}(f)^{-1} = x_{\dot{\beta}}(-f)$ ,  $w_{\dot{\beta}}(s)^{-1} = w_{\dot{\beta}}(-s)$ . The elementary subgroup  $E(n, D_\tau)$  is defined to be the subgroup of  $GL(n, D_\tau)$  generated by  $x_{\dot{\beta}}(f)$  for  $\dot{\beta} \in \Delta_a$  and  $f \in D$ , that is,

$$E(n, D_\tau) = \langle x_{\dot{\beta}}(f) \mid \dot{\beta} \in \Delta_a, f \in D \rangle.$$

Notice that

$$\begin{aligned} E(n, D_\tau) &= \langle x_\beta(g) \mid \beta \in \Delta, g \in D_\tau \rangle \\ &= \langle x_{ij}(g) \mid 1 \leq i \neq j \leq n, g \in D_\tau \rangle. \end{aligned}$$

As subgroups of  $E(n, D_\tau)$ , we put

$$\begin{aligned} U_{\dot{\beta}} &= \{x_{\dot{\beta}}(f) \mid f \in D\} \quad \text{for each } \dot{\beta} \in \Delta_a, \\ U^\pm &= \langle U_{\dot{\beta}} \mid \dot{\beta} \in \Delta_a^\pm \rangle, \\ N &= \langle w_{\dot{\beta}}(u) \mid \dot{\beta} \in \Delta_a, u \in D^\times \rangle, \\ T &= \langle h_{\dot{\beta}}(u) \mid \dot{\beta} \in \Delta_a, u \in D^\times \rangle, \end{aligned}$$

Since  $h_{\dot{\beta}}(u)$  is a diagonal matrix for every  $\dot{\beta}$  and  $u \in D^\times$ , an element  $h \in T$  is of the form  $h = \text{diag}(u_1, u_2, \dots, u_n)$  with some  $u_i \in D_\tau^\times$ . For each  $1 \leq i \leq n$ , if  $u_i = s_i t^{m_i}$  with  $s_i \in D^\times$  and  $m_i \in \mathbb{Z}$ , then we define

$$\deg_i(h) = \deg(u_i) = m_i.$$

We set

$$\begin{aligned} T_0 &= \langle h \mid h \in T, \deg_i(h) = 0 \text{ for all } i = 1, 2, \dots, n \rangle, \\ B^\pm &= \langle U^\pm, T_0 \rangle, \\ S &= \{w_{\dot{\beta}}(1) \bmod T_0 \mid \dot{\beta} \in \Delta_a\}. \end{aligned}$$

**Theorem 1.2.1.** The quadruple  $(E(n, D_\tau), B^\pm, N, S)$  is a Tits system with  $W_a$  the corresponding affine Weyl group.

In order to prove Theorem 1.2.1, we need some lemmas and equalities. We can easily check that for  $\beta = \epsilon_i - \epsilon_j$ ,  $\gamma = \epsilon_k - \epsilon_l \in \Delta$  with  $(i, j) \neq (l, k)$ , and  $f, g \in D_\tau$ ,

$$\begin{aligned} \text{(R1)} \quad & x_\beta(f)x_\beta(g) = x_\beta(f+g), \\ \text{(R2)} \quad & [x_\beta(f), x_\gamma(g)] = \begin{cases} x_{\beta+\gamma}(fg) & \text{if } \beta + \gamma \in \Delta, j = k, \\ x_{\beta+\gamma}(-gf) & \text{if } \beta + \gamma \in \Delta, i = l, \\ I & \text{otherwise.} \end{cases} \end{aligned}$$

Also, for  $\beta = \epsilon_i - \epsilon_j, \gamma = \epsilon_k - \epsilon_l \in \Delta$ ,  $f, g \in D_\tau$ , and  $s, u \in D_\tau^\times$ ,

$$\text{(R3)} \quad w_\beta(u)x_\gamma(f)w_\beta(u)^{-1} = \begin{cases} x_\gamma(f) & \text{if } (\beta, \gamma) = 0, \\ x_{\mp\beta}(-u^{\mp 1}fu^{\mp 1}) & \text{if } \gamma = \pm\beta, \\ x_{\sigma_\beta(\gamma)}(-u^{-1}f) & \text{if } \beta \pm \gamma \neq 0 \text{ and } i = k, \\ x_{\sigma_\beta(\gamma)}(-fu) & \text{if } \beta \pm \gamma \neq 0 \text{ and } i = l, \\ x_{\sigma_\beta(\gamma)}(uf) & \text{if } \beta \pm \gamma \neq 0 \text{ and } j = k, \\ x_{\sigma_\beta(\gamma)}(fu^{-1}) & \text{if } \beta \pm \gamma \neq 0 \text{ and } j = l, \end{cases}$$

$$\text{(R4)} \quad h_\beta(u)x_\gamma(f)h_\beta(u)^{-1} = \begin{cases} x_\gamma(f) & \text{if } (\beta, \gamma) = 0, \\ x_{\pm\beta}(-u^{\pm 1}fu^{\pm 1}) & \text{if } \gamma = \pm\beta, \\ x_\gamma(uf) & \text{if } \beta \pm \gamma \neq 0 \text{ and } i = k, \\ x_\gamma(fu^{-1}) & \text{if } \beta \pm \gamma \neq 0 \text{ and } i = l, \\ x_\gamma(u^{-1}f) & \text{if } \beta \pm \gamma \neq 0 \text{ and } j = k, \\ x_\gamma(fu) & \text{if } \beta \pm \gamma \neq 0 \text{ and } j = l, \end{cases}$$

$$\text{(R5)} \quad w_\beta(u)w_\gamma(s)w_\beta(u)^{-1} = \begin{cases} w_\gamma(s) & \text{if } (\beta, \gamma) = 0, \\ w_{\mp\beta}(-u^{\mp 1}su^{\mp 1}) & \text{if } \gamma = \pm\beta, \\ w_{\sigma_\beta(\gamma)}(-u^{-1}s) & \text{if } \beta \pm \gamma \neq 0 \text{ and } i = k, \\ w_{\sigma_\beta(\gamma)}(-su) & \text{if } \beta \pm \gamma \neq 0 \text{ and } i = l, \\ w_{\sigma_\beta(\gamma)}(us) & \text{if } \beta \pm \gamma \neq 0 \text{ and } j = k, \\ w_{\sigma_\beta(\gamma)}(su^{-1}) & \text{if } \beta \pm \gamma \neq 0 \text{ and } j = l, \end{cases}$$

$$\text{(R6)} \quad w_\beta(u)h_\gamma(s)w_\beta(u)^{-1} = \begin{cases} h_\gamma(s) & \text{if } (\beta, \gamma) = 0, \\ h_{\mp\beta}(u^{\mp 1}su^{\mp 1})h_{\mp\beta}(u^{\pm 2}) & \text{if } \gamma = \pm\beta, \\ h_{\sigma_\beta(\gamma)}(u^{-1}s)h_{\sigma_\beta(\gamma)}(u) & \text{if } \beta \pm \gamma \neq 0 \text{ and } i = k, \\ h_{\sigma_\beta(\gamma)}(su)h_{\sigma_\beta(\gamma)}(u^{-1}) & \text{if } \beta \pm \gamma \neq 0 \text{ and } i = l, \\ h_{\sigma_\beta(\gamma)}(us)h_{\sigma_\beta(\gamma)}(u^{-1}) & \text{if } \beta \pm \gamma \neq 0 \text{ and } j = k, \\ h_{\sigma_\beta(\gamma)}(su^{-1})h_{\sigma_\beta(\gamma)}(u) & \text{if } \beta \pm \gamma \neq 0 \text{ and } j = l. \end{cases}$$



**Lemma 1.2.2.** The following hold in  $E(n, D_\tau)$ .

- (1)  $B^\pm = U^\pm \rtimes T_0$ ,
- (2)  $T_0 \triangleleft N$  and  $T \triangleleft N$ ,
- (3)  $B^\pm \cap N = T_0$ ,
- (4)  $N/T_0 \cong W_a$ .

*Proof.* From the definition, we know that  $B^\pm = \langle U^\pm, T_0 \rangle$ , and see that  $U^\pm \cap T_0 = \{I\}$  if we consider the degree of an element in  $U^\pm$ .  $T_0$  normalizes  $U^\pm$  by (R4), and hence (1) holds. (2) has already been proven by (R6). We prove (4) along with [9, Proposition 1.2]. If we put  $N_0 = \langle w_\alpha(u) \mid \alpha \in \Pi, u \in D^\times \rangle$ , then we have  $N = TN_0$  by the definition of  $T$ . Also, we know that  $T_0 \triangleleft T$  and  $T_0 \triangleleft N$  from (2). Therefore

$$N/T_0 = (T/T_0)(N_0/T_0) = (T/T_0)W.$$

Recall that  $T_a$  is the free abelian subgroup of  $W_a$  with  $\{\xi_{(\alpha,1)} \mid \alpha \in \Pi\}$  a free  $\mathbb{Z}$ -basis. We deduce that the map  $T_a \rightarrow T/T_0$  defined by  $\xi_{(\alpha,1)} \mapsto h_\alpha(t)T_0$  is an isomorphism of groups, and hence we have  $N/T_0 \cong W_a$  from Proposition 1.1.2. By  $U^\pm \cap T_0 = \{I\}$  and (4), we see that (3) holds.  $\square$

To discuss these subgroups more explicitly we put

$$U'_{\pm\dot{\alpha}} = \langle x_{\pm\dot{\alpha}}(g)U_{\dot{\beta}}x_{\pm\dot{\alpha}}(g)^{-1} \mid g \in D, \dot{\beta} \in \Delta_a^\pm \setminus \{\pm\dot{\alpha}\} \rangle$$

for each  $\dot{\alpha} \in \Pi_a$ .

**Proposition 1.2.3.** Let  $\dot{\alpha} \in \Pi_a$ . Then the following hold.

- (1)  $w_{\pm\dot{\alpha}}(u)U'_{\pm\dot{\alpha}}w_{\pm\dot{\alpha}}(u)^{-1} = U'_{\pm\dot{\alpha}}$  for all  $u \in D^\times$ ,
- (2)  $U^\pm = U'_{\pm\dot{\alpha}} \rtimes U_{\pm\dot{\alpha}}$ .

*Proof.* (1) It suffices to show that

$$w_{\dot{\alpha}}(u)x_{\dot{\alpha}}(g)x_{\dot{\beta}}(f')x_{\dot{\alpha}}(g)^{-1}w_{\dot{\alpha}}(u)^{-1} \in U'_{\dot{\alpha}}$$

for all  $\dot{\beta} \in \Delta_a^+ \setminus \{\dot{\alpha}\}$  and  $f' \in D$ . There are two cases as follows:

Case 1. If  $\dot{\beta} = (\beta, m) \in \Delta_a^+ \setminus \{\dot{\alpha}\}$  with  $\beta \neq -\alpha$ , then we see by (R2) and (R3) that

$$\begin{aligned} & w_{\dot{\alpha}}(u)x_{\dot{\alpha}}(g)x_{\dot{\beta}}(f')x_{\dot{\alpha}}(g)^{-1}w_{\dot{\alpha}}(u)^{-1} \\ &= \begin{cases} w_\alpha(u')x_{\alpha+\beta}(g'f'')x_\beta(f'')w_\alpha(u')^{-1} & \text{if } \alpha + \beta \in \Delta, j = k, \\ w_\alpha(u')x_{\alpha+\beta}(-f''g')x_\beta(f'')w_\alpha(u')^{-1} & \text{if } \alpha + \beta \in \Delta, i = l, \\ w_\alpha(u')x_\beta(f'')w_\alpha(u')^{-1} & \text{otherwise,} \end{cases} \end{aligned}$$

where  $u' \in D_\tau^\times$  and  $g', f'' \in D_\tau$ . In the first case, the element can be written as  $x_{\sigma_\alpha(\alpha+\beta)}(s)x_{\sigma_\alpha(\beta)}(u)$  for some  $s, u \in D_\tau^\times$ , but these belong to  $U'_\alpha$  because  $m \in \mathbb{Z}_{\geq 0}$ . The second and third cases are similar.

Case 2. If  $\dot{\beta} = (\beta, m) \in \Delta_a^+ \setminus \{\dot{\alpha}\}$  and  $\beta = -\alpha$ , then we see by the definition of  $w_\alpha(u)$  and (R3) that

$$\begin{aligned} & w_\alpha(u)x_\alpha(g)x_{\dot{\beta}}(f')x_\alpha(g)^{-1}w_\alpha(u)^{-1} \\ &= w_\alpha(u)x_\alpha(g)w_\alpha(u)^{-1}w_\alpha(u)x_{\dot{\beta}}(f')w_\alpha(u)^{-1}w_\alpha(u)x_\alpha(g)^{-1}w_\alpha(u)^{-1} \\ &= x_\alpha(g')w_\alpha(-g')x_\alpha(g')x_{(-\beta, m)}(f'')x_\alpha(g')^{-1}w_\alpha(-g')^{-1}x_\alpha(g')^{-1} \\ &= x_\alpha(g')x_{\dot{\beta}}(f''')x_\alpha(g')^{-1} \in U'_\alpha. \end{aligned}$$

(2) From the definition, we see that  $U'_{\pm\alpha} < U^\pm$  and  $U_{\pm\alpha} < U^\pm$ , and that  $U_{\pm\alpha}$  normalizes  $U'_{\pm\alpha}$ . Suppose that  $U'_{\pm\alpha} \cap U_{\pm\alpha} \neq \{I\}$ . Then, for each  $x_{\pm\alpha}(f) \in U_{\pm\alpha}$ , there exist  $x'_i \in U'_{\pm\alpha}$  such that  $x_{\pm\alpha}(f) = x'_1 x'_2 \dots x'_r$ . By (1), we have

$$U^\mp \ni w_{\pm\alpha}(u)x_{\pm\alpha}(f)w_{\pm\alpha}(u)^{-1} = w_{\pm\alpha}(u)x'_1 x'_2 \dots x'_r w_{\pm\alpha}(u)^{-1} \in U^\pm.$$

Thus, we get  $U'_{\pm\alpha} \cap U_{\pm\alpha} = \{I\}$ .  $\square$

*PROOF OF THEOREM 1.2.1.* We check the axiom of a Tits system. We known from Lemma 1.2.2 that  $B^\pm$  and  $N$  are subgroups of  $E(n, D_\tau)$ , and that  $S$  is a subset of  $N/(B^\pm \cap N)$ . In addition,  $E(n, D_\tau) = \langle B^\pm, N \rangle$  and  $B^\pm \cap N \triangleleft N$ . Moreover,  $N/(B^\pm \cap N) \cong W_a = \langle S \rangle$ , and all elements in  $S$  are of order two (modulo  $T_0$ ). We need to check the following two conditions:

$$\begin{aligned} \text{(TS1)} \quad & sB^\pm w \subset B^\pm wB^\pm \cup B^\pm swB^\pm \quad \text{for all } s \in S \text{ and } w \in W_a, \\ \text{(TS2)} \quad & sB^\pm s \not\subset B^\pm \quad \text{for each } s \in S. \end{aligned}$$

It is enough to check only the case of  $B^+$ . In the following, we write  $B = B^+$ .

Define the length, called  $l(w)$ , of an element  $w$  in  $W_a$  as usual. For  $w \in W_a$  and  $s = w_\alpha(1) \in S$  with  $\alpha \in \Pi_a$ , if  $l(w) < l(sw)$ , then  $w(\alpha) \in \Delta_a^+$ . Therefore we have

$$\begin{aligned} wBs &= wU_\alpha U'_\alpha T_0 s \\ &= wU_\alpha w^{-1}wsU'_\alpha sT_0 s \\ &= U_{w(\alpha)}wsU'_\alpha T_0 \\ &\subset BwsB. \end{aligned}$$

If  $l(ws) < l(w)$ , then  $w(\alpha) \in \Delta_a^-$ , and we have  $l(w') < l(w's)$  for  $w' = ws$ . Thus we obtain

$$\begin{aligned} wBs &= w'sBs \\ &\subset w'(B \cup BsB) \\ &= w'B \cup w'BsB \\ &\subset Bw'B \cup Bw'sBB \\ &= BwsB \cup BwB. \end{aligned}$$

On the other hand, we can easily see that (TS2) holds by direct calculation.  $\square$

**Corollary 1.2.4.** We get three decompositions of  $E(n, D_\tau)$  as follows:

(1)  $E(n, D_\tau)$  has a Bruhat decomposition:

$$E(n, D_\tau) = \bigcup_{w \in W_a} B^\pm w B^\pm = U^\pm N U^\pm.$$

(2)  $E(n, D_\tau)$  has a Birkhoff decomposition:

$$E(n, D_\tau) = \bigcup_{w \in W_a} B^\mp w B^\pm = U^\mp N U^\pm.$$

(3)  $E(n, D_\tau)$  has a Gauss decomposition:

$$E(n, D_\tau) = \bigcup_{X \in U^\pm} X B^\mp B^\pm X^{-1} = U^\pm B^\mp U^\pm.$$

### 1.3 Steinberg groups over rings of non-commutative Laurent polynomials $D_\tau$

Let  $St(n, D_\tau)$  be the Steinberg group, which is defined by the generators  $\hat{x}_\beta(f) = \hat{x}_{ij}(f)$  for  $f \in D_\tau$ ,  $\beta = \epsilon_i - \epsilon_j \in \Delta$ , and the following defining relations:

$$(ST1) \quad \hat{x}_\beta(f) \hat{x}_\beta(g) = \hat{x}_\beta(f + g)$$

$$(ST2) \quad [\hat{x}_\beta(f), \hat{x}_\gamma(g)] = \begin{cases} \hat{x}_{\beta+\gamma}(fg) & \text{if } \beta + \gamma \in \Delta \text{ and } j = k, \\ \hat{x}_{\beta+\gamma}(-gf) & \text{if } \beta + \gamma \in \Delta \text{ and } i = l, \\ 1 & \text{otherwise,} \end{cases}$$

where  $f, g \in D_\tau$  and  $\beta = \epsilon_i - \epsilon_j, \gamma = \epsilon_k - \epsilon_l \in \Delta$  with  $\beta \neq \gamma$ . Then, we can easily check that  $\hat{x}_\beta(f)^{-1} = \hat{x}_\beta(-f)$  from (ST1). When  $n = 2$ , we use

$$(ST2)' \quad \hat{w}_\beta(u) \hat{x}_\beta(f) \hat{w}_\beta(-u) = \hat{x}_{-\beta}(-u^{-1} f u^{-1})$$

instead of (ST2), where  $\hat{w}_\beta(u) = \hat{x}_\beta(u) \hat{x}_{-\beta}(-u^{-1}) \hat{x}_\beta(u)$  for  $u \in D_\tau^\times$ . There exists a natural homomorphism  $\phi$  from  $St(n, D_\tau)$  onto  $E(n, D_\tau)$  defined by  $\phi(\hat{x}_\beta(f)) = x_\beta(f)$  for  $f \in D_\tau$  and  $\beta \in \Delta$  from (R1), (R2) (or (R3) if  $n = 2$ ), and (ST1), (ST2) (or (ST2)' if  $n = 2$ ).

For  $\dot{\beta} = (\beta, m) \in \Delta_a$  and  $f \in D$ , we put

$$\hat{x}_{\dot{\beta}}(f) = \begin{cases} \hat{x}_\beta(f t^m) & \text{if } \beta \in \Delta^+, \\ \hat{x}_\beta(t^m f) & \text{if } \beta \in \Delta^-, \end{cases}$$

and we also put  $\hat{w}_{\dot{\beta}}(u) = \hat{x}_{\dot{\beta}}(u) \hat{x}_{-\dot{\beta}}(-u^{-1}) \hat{x}_{\dot{\beta}}(u)$  and  $\hat{h}_{\dot{\beta}}(u) = \hat{w}_{\dot{\beta}}(u) \hat{w}_\beta(-1)$  for  $u \in D^\times$ . As subgroups of  $St(n, D_\tau)$ , we set

$$\begin{aligned} \hat{U}_{\dot{\beta}} &= \{\hat{x}_{\dot{\beta}}(f) \mid f \in D\} \quad \text{for each } \dot{\beta} \in \Delta_a, \\ \hat{U}^\pm &= \langle \hat{U}_{\dot{\beta}} \mid \dot{\beta} \in \Delta_a^\pm \rangle, \\ \hat{N} &= \langle \hat{w}_{\dot{\beta}}(u) \mid \dot{\beta} \in \Delta_a, u \in D^\times \rangle, \\ \hat{T} &= \langle \hat{h}_{\dot{\beta}}(u) \mid \dot{\beta} \in \Delta_a, u \in D^\times \rangle. \end{aligned}$$

Also, we set

$$\begin{aligned} \hat{T}_0 &= \langle h \mid h \in \hat{T}, \deg_i(\phi(h)) = 0 \text{ for all } i = 1, \dots, n \rangle, \\ \hat{B}^\pm &= \langle \hat{U}^\pm, \hat{T}_0 \rangle, \\ \hat{S} &= \{ \hat{w}_{\dot{\beta}}(1) \bmod \hat{T}_0 \mid \dot{\beta} \in \Delta_a \}. \end{aligned}$$

We also need several relations between  $\hat{x}_{\dot{\beta}}(f)$ ,  $\hat{w}_{\dot{\beta}}(u)$  and  $\hat{h}_{\dot{\beta}}(s)$  like (R1)–(R6) in  $St(n, D_\tau)$ , but those are the same except for (R6). In the Steinberg group  $St(n, D_\tau)$ , the relation (R6) is given as follows: for  $\beta = \epsilon_i - \epsilon_j$ ,  $\gamma = \epsilon_k - \epsilon_l \in \Delta$ , and  $u, s \in D_\tau^\times$ ,

$$\begin{aligned} (\hat{R}6) \quad & \hat{w}_\beta(u) \hat{h}_\gamma(s) \hat{w}_\beta(u)^{-1} \\ &= \begin{cases} \hat{h}_\gamma(s) & \text{if } (\beta, \gamma) = 0, \\ \hat{h}_{\mp\beta}(-u^{\mp 1} s u^{\mp 1}) \hat{h}_{\mp\beta}(-u^{\pm 2})^{-1} & \text{if } \gamma = \pm\beta, \\ \hat{h}_{\sigma_\beta(\gamma)}(-u^{-1} s) \hat{h}_{\sigma_\beta(\gamma)}(-u^{-1})^{-1} & \text{if } \beta \pm \gamma \neq 0 \text{ and } i = k, \\ \hat{h}_{\sigma_\beta(\gamma)}(-s u) \hat{h}_{\sigma_\beta(\gamma)}(-u)^{-1} & \text{if } \beta \pm \gamma \neq 0 \text{ and } i = l, \\ \hat{h}_{\sigma_\beta(\gamma)}(u s) \hat{h}_{\sigma_\beta(\gamma)}(u)^{-1} & \text{if } \beta \pm \gamma \neq 0 \text{ and } j = k, \\ \hat{h}_{\sigma_\beta(\gamma)}(s u^{-1}) \hat{h}_{\sigma_\beta(\gamma)}(u^{-1})^{-1} & \text{if } \beta \pm \gamma \neq 0 \text{ and } j = l. \end{cases} \end{aligned}$$

We have the following lemma (see Lemma 1.2.2).

**Lemma 1.3.1.** The following hold in  $St(n, D_\tau)$ :

- (1)  $\hat{B}^\pm = \hat{U}^\pm \rtimes \hat{T}_0$ ,
- (2)  $\hat{T}_0 \triangleleft \hat{N}$  and  $\hat{T} \triangleleft \hat{N}$ ,
- (3)  $\hat{B}^\pm \cap \hat{N} = \hat{T}_0$ ,
- (4)  $\hat{N}/\hat{T}_0 \cong W_a$ .

We put

$$\hat{U}'_{\pm\dot{\alpha}} = \langle \hat{x}_{\pm\dot{\alpha}}(g) \hat{U}_{\dot{\beta}} \hat{x}_{\pm\dot{\alpha}}(g)^{-1} \mid g \in D, \dot{\beta} \in \Delta_a^\pm \setminus \{\pm\dot{\alpha}\} \rangle$$

for each  $\dot{\alpha} \in \Pi_a$ . Then the following proposition holds (see Proposition 1.2.3).

**Proposition 1.3.2.** The following hold in  $St(n, D_\tau)$ : for  $\dot{\alpha} \in \Pi$ ,

- (1)  $\hat{w}_{\pm\dot{\alpha}}(u) \hat{U}'_{\pm\dot{\alpha}} \hat{w}_{\pm\dot{\alpha}}(u)^{-1} = \hat{U}'_{\pm\dot{\alpha}}$  for  $u \in D^\times$ ,
- (2)  $\hat{U}^\pm = \hat{U}'_{\pm\dot{\alpha}} \hat{U}_{\pm\dot{\alpha}} = \hat{U}_{\pm\dot{\alpha}} \hat{U}'_{\pm\dot{\alpha}}$ ,
- (3)  $\hat{s} \hat{B}^\pm w \subset \hat{B}^\pm w \hat{B}^\pm \cup \hat{B}^\pm \hat{s} w \hat{B}^\pm$  for all  $\hat{s} \in \hat{S}$  and  $w \in W_a$ ,
- (4)  $\hat{s} \hat{B}^\pm \hat{s} \not\subset \hat{B}^\pm$  for each  $\hat{s} \in \hat{S}$ .

**Theorem 1.3.3.** The quadruple  $(St(n, D_\tau), \hat{B}^\pm, \hat{N}, \hat{S})$  is a Tits system with  $W_a$  the corresponding affine Weyl group.

**Corollary 1.3.4.** The following hold:

(1)  $St(n, D_\tau)$  has a Bruhat decomposition:

$$St(n, D_\tau) = \bigcup_{w \in W_a} \hat{B}^\pm w \hat{B}^\pm = \hat{U}^\pm \hat{N} \hat{U}^\pm.$$

(2)  $St(n, D_\tau)$  has a Birkhoff decomposition:

$$St(n, D_\tau) = \bigcup_{w \in W_a} \hat{B}^\mp w \hat{B}^\pm = \hat{U}^\mp \hat{N} \hat{U}^\pm.$$

(3)  $St(n, D_\tau)$  has a Gauss decomposition:

$$St(n, D_\tau) = \bigcup_{\hat{X} \in \hat{U}^\pm} \hat{X} \hat{B}^\mp \hat{B}^\pm \hat{X}^{-1} = \hat{U}^\pm \hat{B}^\mp \hat{U}^\pm.$$

## 1.4 Central extensions and $K_2$ -groups

In this subsection, we show that  $\phi$  is a central extension, that is,  $\text{Ker } \phi$  is a central subgroup of  $St(n, D_\tau)$ . For this homomorphism  $\phi$ , we define our  $K_2$ -groups as  $K_2(n, D_\tau) = \text{Ker } \phi$  (cf. [8, page 40]). We also give some relations in  $K_2(n, D_\tau)$  for later use (see Section 2).

Let  $\tilde{E}(2, D_\tau)$  be the group generated by  $\tilde{x}_\beta(f)$  for  $f \in D_\tau$ ,  $\beta \in \Delta$  with the defining relations that (ST1), (ST2)' and

$$(ST3) \quad \tilde{c}_\beta(u_1, v_1) \tilde{c}_\beta(u_2, v_2) \cdots \tilde{c}_\beta(u_r, v_r) = 1$$

for  $r \geq 0$  and  $u_i, v_i \in D_\tau^\times$  such that  $[u_1, v_1][u_2, v_2] \cdots [u_r, v_r] = 1$ , where for  $u, v \in D_\tau^\times$  we put

$$\begin{aligned} \tilde{w}_\beta(u) &= \tilde{x}_\beta(u) \tilde{x}_{-\beta}(-u^{-1}) \tilde{x}_\beta(u), \\ \tilde{h}_\beta(u) &= \tilde{w}_\beta(u) \tilde{w}_\beta(-1), \\ \tilde{c}_\beta(u, v) &= \tilde{h}_\beta(u) \tilde{h}_\beta(v) \tilde{h}_\beta(vu)^{-1}, \end{aligned}$$

and where we change  $\hat{x}$  and  $\hat{w}$  in the defining relations of  $St(2, D_\tau)$  to  $\tilde{x}$  and  $\tilde{w}$  respectively. Thus,  $\tilde{E}(2, D_\tau)$  is the quotient group of  $St(2, D_\tau)$  by the group generated by the corresponding elements  $\hat{c}_\beta(u_1, v_1) \hat{c}_\beta(u_2, v_2) \cdots \hat{c}_\beta(u_r, v_r)$ . Here, we note that

$$\phi(\hat{c}_{\alpha_1}(u, v)) = \begin{pmatrix} [u, v] & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \phi(\hat{c}_{-\alpha_1}(u, v)) = \begin{pmatrix} 1 & 0 \\ 0 & [u, v] \end{pmatrix}$$

for  $u, v \in D_\tau^\times$ . In particular, we have  $\hat{c}_\beta(u, v) \in \hat{T}_0$  for  $\beta \in \Delta$ . Also, we see that for  $\beta = \alpha_1$ ,

$$\begin{aligned} & \phi(\hat{c}_{\alpha_1}(u_1, v_1) \hat{c}_{\alpha_1}(u_2, v_2) \cdots \hat{c}_{\alpha_1}(u_r, v_r)) \\ &= \begin{pmatrix} [u_1, v_1] & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} [u_2, v_2] & 0 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} [u_r, v_r] & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} [u_1, v_1][u_2, v_2] \cdots [u_r, v_r] & 0 \\ 0 & 1 \end{pmatrix} \\ &= I \end{aligned}$$

if  $[u_1, v_1][u_2, v_2] \cdots [u_r, v_r] = 1$ .

**Theorem 1.4.1.** It holds that  $\tilde{E}(2, D_\tau) \cong E(2, D_\tau)$ .

*PROOF OF THEOREM 1.4.1.* We show this along with [11, Theorem 8]. The homomorphism  $\phi$  induces two canonical homomorphisms called  $\hat{\phi}$  and  $\tilde{\phi}$ :

$$\hat{\phi} : St(2, D_\tau) \rightarrow \tilde{E}(2, D_\tau) \quad \text{and} \quad \tilde{\phi} : \tilde{E}(2, D_\tau) \rightarrow E(2, D_\tau)$$

such that the following diagram commutes:

$$\begin{array}{ccc} & St(2, D_\tau) & \\ \hat{\phi} \swarrow & \circlearrowright & \searrow \phi \\ \tilde{E}(2, D_\tau) & \xrightarrow{\tilde{\phi}} & E(2, D_\tau) \end{array}$$

Note that for  $1 \leq i \neq j \leq 2$  and  $f \in D_\tau$ ,

$$\hat{\phi}(\hat{x}_{ij}(f)) = \tilde{x}_{ij}(f) \quad \text{and} \quad \tilde{\phi}(\tilde{x}_{ij}(f)) = x_{ij}(f).$$

We use the same notation in subgroups of  $\tilde{E}(2, D_\tau)$  as in  $St(2, D_\tau)$  changing  $\hat{\cdot}$  to  $\tilde{\cdot}$ . Then, for any  $\tilde{x} \in \text{Ker } \tilde{\phi}$ , we can take  $\tilde{y} \in \tilde{U}$  and  $\tilde{z} \in \tilde{T}_0$  satisfying  $\tilde{x} = \tilde{y}\tilde{z}$  because  $St(2, D_\tau)$  has the Bruhat decomposition by Theorem 1.3.3 and  $\text{Ker } \tilde{\phi} \subset \tilde{B} = \tilde{U}\tilde{T}_0$ . We know that  $\tilde{\phi}(\tilde{x}) = \tilde{\phi}(\tilde{y})\tilde{\phi}(\tilde{z}) = I$  and  $U \cap T_0 = \{I\}$ , so that we have  $\tilde{\phi}(\tilde{y}) = I$  and  $\tilde{\phi}(\tilde{z}) = I$ . In particular,  $\tilde{y}$  and  $\tilde{z}$  belong to the kernel of  $\tilde{\phi}$ . Therefore we need to prove  $\tilde{y} = 1$  and  $\tilde{z} = 1$ .

**STEP 1.** We claim that  $\tilde{z} = 1$ .

We know  $\tilde{w}_{-\alpha_1}(u) = \tilde{w}_{\alpha_1}(-u^{-1})$  from (R3), and this implies that  $\tilde{h}_{-\alpha_1}(u) = \tilde{h}_{\alpha_1}(u)^{-1}$ . In addition, we see that  $1 = \tilde{c}_{\alpha_1}(u, u^{-1}) = \tilde{h}_{\alpha_1}(u)\tilde{h}_{\alpha_1}(u^{-1})$  for  $u \in D_\tau^\times$ , that is,  $\tilde{h}_{\alpha_1}(u)^{-1} = \tilde{h}_{\alpha_1}(u^{-1})$  by (ST3). Therefore, we find that  $\tilde{z}$  is of the form

$$\tilde{z} = \tilde{h}_{\alpha_1}(u_1) \cdots \tilde{h}_{\alpha_1}(u_r)$$

for  $u_i \in D_\tau^\times$ . We note that  $\tilde{\phi}(\tilde{z}) = I$  implies  $u_1 u_2 \cdots u_r = 1$  and  $u_r u_{r-1} \cdots u_1 = 1$ . These give  $[u_1, u_2][u_2 u_1, u_3] \cdots [u_{r-1} \cdots u_1, u_r] = 1$ . Then, by (ST3), we obtain

$$\begin{aligned} \tilde{z} &= \tilde{h}_{\alpha_1}(u_1)\tilde{h}_{\alpha_1}(u_2) \cdots \tilde{h}_{\alpha_1}(u_r) \\ &= \tilde{c}_{\alpha_1}(u_1, u_2)\tilde{h}_{\alpha_1}(u_2 u_1)\tilde{h}_{\alpha_1}(u_3) \cdots \tilde{h}_{\alpha_1}(u_r) \\ &= \tilde{c}_{\alpha_1}(u_1, u_2)\tilde{c}_{\alpha_1}(u_2 u_1, u_3)\tilde{h}_{\alpha_1}(u_3 u_2 u_1)\tilde{h}_{\alpha_1}(u_4) \cdots \tilde{h}_{\alpha_1}(u_r) \\ &\quad \vdots \\ &= \tilde{c}_{\alpha_1}(u_1, u_2)\tilde{c}_{\alpha_1}(u_2 u_1, u_3) \cdots \tilde{c}_{\alpha_1}(u_{r-2} \cdots u_1, u_{r-1})\tilde{h}_{\alpha_1}(u_{r-1} \cdots u_1)\tilde{h}_{\alpha_1}(u_r) \\ &= \tilde{c}_{\alpha_1}(u_1, u_2)\tilde{c}_{\alpha_1}(u_2 u_1, u_3) \cdots \tilde{c}_{\alpha_1}(u_{r-1} \cdots u_1, u_r)\tilde{h}_{\alpha_1}(u_r \cdots u_1) \\ &= \tilde{c}_{\alpha_1}(u_1, u_2)\tilde{c}_{\alpha_1}(u_2 u_1, u_3) \cdots \tilde{c}_{\alpha_1}(u_{r-1} \cdots u_1, u_r) \\ &= 1. \end{aligned}$$

**STEP 2.** We claim that  $\tilde{y} = 1$ .

We set subgroups of  $U$  as follows:

$$U_1 = \{x_{\alpha_1}(f) \mid f \in D[t]\} \quad \text{and} \quad U_2 = \{x_{-\alpha_2}(g) \mid g \in D[t]t\}.$$

Then we see that  $U = \langle U_1, U_2 \rangle$ . We show that this is a free product. We introduce a “degree map”  $\deg : D[t] \rightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$ , which is defined by  $\deg(f) = m$  for  $f = \sum_{i=0}^m f_i t^i$  and  $\deg(0) = \infty$ , where  $f_i \in D$ . Let  $x \in U$ , and assume that

$$x = x_{\beta_1}(q_1)x_{\beta_2}(q_2) \cdots x_{\beta_r}(q_r)$$

for  $r \geq 1$ ,  $\beta_i \in \Delta$ ,  $\beta_i \neq \beta_{i+1}$ ,  $q_i \neq 0$ , and

$$q_i \in \begin{cases} D[t] & \text{if } \beta_i \in \Delta^+, \\ D[t]t & \text{if } \beta_i \in \Delta^-. \end{cases}$$

For each  $1 \leq i \leq r$ , we put

$$\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} = x_{\beta_1}(q_1)x_{\beta_2}(q_2) \cdots x_{\beta_i}(q_i).$$

Assume that  $\beta_1 = \epsilon_1 - \epsilon_2$ . Then we have

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} 1 & q_1 \\ 0 & 1 \end{pmatrix}$$

and  $0 = \deg(a_1) \leq \deg(b_1)$ . Since

$$\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q_2 & 1 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 q_2 & b_1 \\ c_1 + d_1 q_2 & d_1 \end{pmatrix},$$

we obtain  $\deg(a_2) > \deg(b_2) = \deg(b_1)$ . Moreover, since

$$\begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} 1 & q_3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_2 & a_2 q_3 + b_2 \\ c_2 & c_2 q_3 + d_2 \end{pmatrix},$$

we have  $\deg(a_2) = \deg(a_3) \leq \deg(b_3)$ . Continuing this process, we can reach

$$\begin{aligned} \deg(a_1) &\leq \deg(b_1) = \deg(b_2) < \deg(a_2) = \\ \deg(a_3) &\leq \deg(b_3) = \deg(b_4) < \deg(a_4) = \\ \deg(a_5) &\leq \deg(b_5) = \deg(b_6) < \deg(a_6) = \cdots \end{aligned}$$

Next, assume that  $\beta_1 = \epsilon_2 - \epsilon_1$ . Then, in the same way as above, we can get

$$\begin{aligned} \deg(b_1) &\leq \deg(a_1) = \deg(a_2) < \deg(b_2) = \\ \deg(b_3) &\leq \deg(a_3) = \deg(a_4) < \deg(b_4) = \\ \deg(b_5) &\leq \deg(a_5) = \deg(a_6) < \deg(b_6) = \cdots \end{aligned}$$

In any case we can show  $x \neq I$ , which means that  $U = U_1 \star U_2$  is a free product, and we see that  $\tilde{\phi}(\tilde{y}) = I$  implies  $\tilde{y} = 1$ .  $\square$

**Proposition 1.4.2.** It holds that

$$K_2(2, D_\tau) = \langle \hat{c}_{\alpha_1}(u_1, v_1) \hat{c}_{\alpha_1}(u_2, v_2) \cdots \hat{c}_{\alpha_1}(u_r, v_r) \mid r \geq 1, u_i, v_i \in D_\tau^\times, \\ [u_1, v_1][u_2, v_2] \cdots [u_r, v_r] = 1 \rangle,$$

where  $\hat{c}_{\alpha_1}(u, v) = \hat{h}_{\alpha_1}(u) \hat{h}_{\alpha_1}(v) \hat{h}_{\alpha_1}(vu)^{-1}$ . Moreover,  $K_2(2, D_\tau)$  is a central subgroup of  $St(2, D_\tau)$ .

*Proof.* It suffices to check that  $\hat{c}_{\alpha_1}(u_1, v_1) \cdots \hat{c}_{\alpha_1}(u_r, v_r) \in K_2(2, D_\tau)$  is a central element when  $[u_1, v_1] \cdots [u_r, v_r] = 1$ . We easily see that for  $f \in D_\tau$  and  $\beta \in \Delta$ ,

$$\begin{aligned} & \{ \hat{c}_{\alpha_1}(u_1, v_1) \cdots \hat{c}_{\alpha_1}(u_r, v_r) \} \hat{x}_\beta(f) \{ \hat{c}_{\alpha_1}(u_1, v_1) \cdots \hat{c}_{\alpha_1}(u_r, v_r) \}^{-1} \\ &= \hat{x}_\beta(\{ [u_1, v_1] \cdots [u_r, v_r] \}^{\pm 1} f \{ [u_1, v_1] \cdots [u_r, v_r] \}^{\mp 1}) \\ &= \hat{x}_\beta(f). \end{aligned}$$

□

Using Proposition 1.4.2 and [3, Theorem], we construct a central extension of higher rank. Assume that  $n \geq 3$ . We consider the following commutative diagram:

$$\begin{array}{ccccccc} 1 & \rightarrow & K_2(2, D_\tau) & \rightarrow & St(2, D_\tau) & \rightarrow & E(2, D_\tau) \rightarrow 1 & \text{(exact)} \\ & & \downarrow & & \downarrow & & \downarrow & \\ 1 & \rightarrow & K_2(n, D_\tau) & \rightarrow & St(n, D_\tau) & \rightarrow & E(n, D_\tau) \rightarrow 1 & \text{(exact)} \end{array}$$

Then we know from [3, Theorem] that the canonical homomorphism of  $K_2(2, D_\tau)$  into  $K_2(n, D_\tau)$  is surjective since  $D_\tau$  is a euclidean ring. Using this and Theorem 1.4.1, we obtain the following theorem.

**Theorem 1.4.3.** Let  $\tilde{E}(n, D_\tau)$  be the group generated by  $\tilde{x}_\beta(f)$  for  $\beta \in \Delta$  and  $f \in D_\tau$  with the defining relations (ST1), (ST2) and (ST3). Then  $\tilde{E}(n, D_\tau) \cong E(n, D_\tau)$ .

Put  $\bar{\xi}_{\alpha_i} = \hat{c}_{\alpha_i}(u_1, v_1) \hat{c}_{\alpha_i}(u_2, v_2) \cdots \hat{c}_{\alpha_i}(u_r, v_r) \in K_2(n, D_\tau)$  for  $\alpha_i \in \Pi$ . If  $n = 3$ , we see by (R6) that

$$\bar{\xi}_{\alpha_2} = \hat{w}_{\alpha_2}(1) \hat{w}_{\alpha_1}(1) \bar{\xi}_{\alpha_2} \hat{w}_{\alpha_1}(1)^{-1} \hat{w}_{\alpha_2}(1)^{-1} = \bar{\xi}_{\alpha_1}$$

Similarly, we see by simply computation that  $\bar{\xi}_{\alpha_1} = \bar{\xi}_{\alpha_2} = \cdots = \bar{\xi}_{\alpha_{n-1}} \in K_2(n, D_\tau)$  for  $n \geq 3$ . Therefore, we write

$$\hat{c}(u_1, v_1) \hat{c}(u_2, v_2) \cdots \hat{c}(u_r, v_r) = \hat{c}_{\alpha_i}(u_1, v_1) \hat{c}_{\alpha_i}(u_2, v_2) \cdots \hat{c}_{\alpha_i}(u_r, v_r)$$

for simplicity.

**Proposition 1.4.4.** Let  $n \geq 2$ . We have

$$K_2(n, D_\tau) = \langle \hat{c}(u_1, v_1) \hat{c}(u_2, v_2) \cdots \hat{c}(u_r, v_r) \mid r \geq 1, u_i, v_i \in D_\tau^\times, \\ [u_1, v_1][u_2, v_2] \cdots [u_r, v_r] = 1 \rangle.$$



Moreover,  $K_2(n, D_\tau)$  is a central subgroup of  $St(n, D_\tau)$ .

We give some relations in  $\hat{T}$  for later use (see Section 2). We first assume that  $n = 2$ .

**Lemma 1.4.5.** In  $\hat{T} \subset St(2, D_\tau)$ , the following hold for  $1 \leq i \neq j \leq 2$  and  $u, v, x, y \in D_\tau^\times$ :

$$\begin{aligned}
(\text{T1}) \quad & \hat{h}_{ij}(u)\hat{h}_{ij}(v) = \hat{h}_{ij}(uvu)\hat{h}_{ij}(u^{-1}), \\
(\text{T1}') \quad & \hat{h}_{ij}(u)\hat{h}_{ij}(v) = \hat{h}_{ij}(v^{-1})\hat{h}_{ij}(vuv), \\
(\text{T2}) \quad & \hat{c}(u, v) = \hat{h}_{ij}(v^{-1}u^{-1})^{-1}\hat{h}_{ij}(u^{-1})\hat{h}_{ij}(v^{-1}), \\
(\text{T2}') \quad & \hat{h}_{ij}(u^{-1}v^{-1})^{-1}\hat{h}_{ij}(u^{-1})\hat{h}_{ij}(v^{-1}) = \hat{h}_{ij}(u)\hat{h}_{ij}(v)\hat{h}_{ij}(uv)^{-1}, \\
(\text{T3}) \quad & \hat{c}_{ij}(u, v) = \hat{c}_{ij}(uvu, u^{-1}) = \hat{c}_{ij}(v^{-1}, vuv), \\
(\text{T4}) \quad & \hat{c}_{ij}(u, v)\hat{c}_{ij}(vu, w) = \hat{c}_{ij}(u, vw)\hat{c}_{ij}(v, w) \\
& \quad = \hat{h}_{ij}(u)\hat{c}_{ij}(v, w)\hat{h}_{ij}(u)^{-1}\hat{c}_{ij}(u, vw), \\
(\text{T5}) \quad & \hat{c}_{ij}(x, y)\hat{h}_{ij}(u)\hat{c}_{ij}(x, y)^{-1} = \hat{h}_{ij}([x, y]u)\hat{h}_{ij}([x, y])^{-1} \\
& \quad = \hat{h}_{ij}([y, x])^{-1}\hat{h}_{ij}(u[y, x]), \\
(\text{T6}) \quad & \hat{c}_{ij}(x, y)\hat{c}_{ij}(u, v)\hat{c}_{ij}(x, y)^{-1} = \hat{c}_{ij}(u, [x, y])^{-1}\hat{c}_{ij}(u, [x, y]v) \\
& \quad = \hat{c}_{ij}([x, y]u, v)\hat{c}_{ij}([x, y], v)^{-1}, \\
(\text{T7}) \quad & \hat{c}_{ij}(u, v) = \hat{c}_{ij}(u, v(1-u)) \quad \text{if } 1-u \in D_\tau^\times, \\
(\text{T8}) \quad & \hat{c}_{ij}(u, v) = \hat{c}_{ij}(u, -vu).
\end{aligned}$$

*Proof.* Equalities (T1) and (T1)' can be shown by direct calculation. Let us show equalities (T2) and (T2)'. By (T1), we see that

$$\begin{aligned}
\hat{h}_{ij}(u)\hat{h}_{ij}(u^{-1}v) &= \hat{h}_{ij}(vu)\hat{h}_{ij}(u^{-1}), \\
\hat{h}_{ij}(v)\hat{h}_{ij}(v^{-1}u^{-1}) &= \hat{h}_{ij}(u^{-1}v)\hat{h}_{ij}(v^{-1}),
\end{aligned}$$

and hence  $\hat{h}_{ij}(vu)^{-1}\hat{h}_{ij}(v)\hat{h}_{ij}(u) = \hat{h}_{ij}(u^{-1})\hat{h}_{ij}(v^{-1})\hat{h}_{ij}(v^{-1}u^{-1})^{-1}$ . Replacing  $u$  and  $v$  by  $u^{-1}$  and  $v^{-1}$ , respectively, we obtain

$$\hat{h}_{ij}(v^{-1}u^{-1})^{-1}\hat{h}_{ij}(v^{-1})\hat{h}_{ij}(u^{-1}) = \hat{h}_{ij}(u)\hat{h}_{ij}(v)\hat{h}_{ij}(vu)^{-1} = \hat{c}_{ij}(u, v),$$

as desired. Equality (T3) is a straightforward consequence of (T1) and (T1)'.

We show the first equality in (T4) as follows:

$$\begin{aligned}
& \hat{c}_{ij}(u, v)\hat{c}_{ij}(vu, w) \\
&= \hat{h}_{ij}(u)\hat{h}_{ij}(v)\hat{h}_{ij}(w)\hat{h}_{ij}(wvu)^{-1} \text{ (by the definitions of } \hat{c}_{ij}(u, v) \text{ and } \hat{c}_{ij}(vu, w)) \\
&= \hat{c}_{ij}(u, vw)\hat{h}_{ij}(vwu)\hat{h}_{ij}(vw)^{-1}\hat{h}_{ij}(v)\hat{h}_{ij}(w)\hat{h}_{ij}(wvu)^{-1} \\
&= \hat{c}_{ij}(u, vw)\hat{h}_{ij}(vwu)\hat{h}_{ij}(v^{-1})\hat{h}_{ij}(w^{-1})\hat{h}_{ij}(v^{-1}w^{-1})^{-1}\hat{h}_{ij}(wvu)^{-1} \\
&= \hat{c}_{ij}(u, vw)\hat{h}_{ij}(v)\hat{h}_{ij}(wuv^{-1})\hat{h}_{ij}(v^{-1})^{-1}\hat{h}_{ij}(v^{-1}) \quad \text{(by (T1)')} \\
&\quad \times \hat{h}_{ij}(w^{-1})\hat{h}_{ij}(v^{-1}w^{-1})^{-1}\hat{h}_{ij}(wvu)^{-1} \\
&= \hat{c}_{ij}(u, vw)\hat{h}_{ij}(v)\hat{h}_{ij}(w)\hat{h}_{ij}(uv^{-1}w^{-1})\hat{h}_{ij}(w^{-1})^{-1} \quad \text{(by (T1))} \\
&\quad \times \hat{h}_{ij}(w^{-1})\hat{h}_{ij}(v^{-1}w^{-1})^{-1}\hat{h}_{ij}(wvu)^{-1} \\
&= \hat{c}_{ij}(u, vw)\hat{c}_{ij}(v, w)\hat{h}_{ij}(wv)\hat{h}_{ij}(uv^{-1}w^{-1})\hat{h}_{ij}(v^{-1}w^{-1})^{-1}\hat{h}_{ij}(wvu)^{-1} \\
&= \hat{c}_{ij}(u, vw)\hat{c}_{ij}(v, w). \quad \text{(by (T1))}
\end{aligned}$$

For the second equality in (T4), we compute

$$\begin{aligned}
\hat{c}_{ij}(u, v)\hat{c}_{ij}(vu, w) &= \hat{h}_{ij}(u)\hat{h}_{ij}(v)\hat{h}_{ij}(w)\hat{h}_{ij}(wvu)^{-1} \\
&= \hat{h}_{ij}(u)\hat{c}_{ij}(v, w)\hat{h}_{ij}(u)^{-1}\hat{h}_{ij}(u)\hat{h}_{ij}(wv)\hat{h}_{ij}(wvu)^{-1} \\
&= \hat{h}_{ij}(u)\hat{c}_{ij}(v, w)\hat{h}_{ij}(u)^{-1}\hat{c}_{ij}(u, wv).
\end{aligned}$$

For equality (T5), we need some extra identities. Putting  $v = 1$  in (T1) and (T1)', we find that

$$\hat{h}_{ij}(u) = \hat{h}_{ij}(u^2)\hat{h}_{ij}(u^{-1}) = \hat{h}_{ij}(u^{-1})\hat{h}_{ij}(u^2),$$

which implies that  $\hat{h}_{ij}(u^2)^{-1} = \hat{h}_{ij}(u^{-2})$ . Thus we get

$$(T5)' \hat{h}_{ij}(u)^{\pm 1}\hat{h}_{ij}(v)\hat{h}_{ij}(u)^{\mp 1} = \hat{h}_{ij}(u^{\pm 1}vu^{\pm 1})\hat{h}_{ij}(u^{\pm 2})^{-1}.$$

Applying (T5)' three times, we obtain the first equality in (T5) as follows:

$$\begin{aligned}
& \hat{c}_{ij}(x, y)\hat{h}_{ij}(u)\hat{c}_{ij}(x, y)^{-1} \\
&= \hat{h}_{ij}(x)\hat{h}_{ij}(y)\hat{h}_{ij}(yx)^{-1}\hat{h}_{ij}(u)\hat{h}_{ij}(yx)\hat{h}_{ij}(v)^{-1}\hat{h}_{ij}(x)^{-1} \\
&= \hat{h}_{ij}(x)\hat{h}_{ij}(y)\hat{h}_{ij}(x^{-1}y^{-1}ux^{-1}y^{-1})\hat{h}_{ij}(x^{-1}y^{-1}x^{-1}y^{-1})^{-1}\hat{h}_{ij}(y)^{-1}\hat{h}_{ij}(x)^{-1} \\
&= \hat{h}_{ij}(x)\hat{h}_{ij}(yx^{-1}y^{-1}ux^{-1})\hat{h}_{ij}(yx^{-1}y^{-1}x^{-1})^{-1}\hat{h}_{ij}(x)^{-1} \\
&= \hat{h}_{ij}([x, y]u)\hat{h}_{ij}([x, y])^{-1}.
\end{aligned}$$

Also, by using (T1), we get the second equality in (T5) as follows:

$$\begin{aligned}
& \hat{c}_{ij}(x, y)\hat{h}_{ij}(u)\hat{c}_{ij}(x, y)^{-1} \\
&= \hat{h}_{ij}([x, y]u)\hat{h}_{ij}([x, y])^{-1} \\
&= \hat{h}_{ij}([y, x])^{-1}\hat{h}_{ij}(u[y, x])\hat{h}_{ij}(u[y, x])^{-1}\hat{h}_{ij}([y, x])\hat{h}_{ij}([x, y]u)\hat{h}_{ij}([x, y])^{-1} \\
&= \hat{h}_{ij}([y, x])^{-1}\hat{h}_{ij}(u[y, x])\hat{h}_{ij}(u[y, x])^{-1}\hat{h}_{ij}(u[y, x]) \\
&= \hat{h}_{ij}([y, x])^{-1}\hat{h}_{ij}(u[y, x]).
\end{aligned}$$

Equality (T6) is a consequence of (T4) and (T5). Equality (T7) is equivalent to

$$\hat{w}_{ij}(vu - vu^2)\hat{w}_{ij}(v - vu)^{-1} = \hat{w}_{ij}(vu)\hat{w}_{ij}(v)^{-1},$$

which follows from

$$\begin{aligned} & \hat{w}_{ij}(vu - vu^2)\hat{w}_{ij}(v - vu)^{-1} \\ &= \hat{x}_{ij}(vu^2)\hat{x}_{ij}(-vu^2)\hat{w}_{ij}(vu - vu^2)\hat{w}_{ij}(v - vu)^{-1} \\ &= \hat{x}_{ij}(vu^2)\hat{w}_{ij}(vu(1 - u))\hat{w}_{ij}(-v(1 - u))\hat{x}_{ij}(-v) \quad (\text{by (ST2)}) \\ &= \hat{x}_{ij}(vu)\hat{x}_{ji}(-(1 - u)^{-1}u^{-1}v^{-1})\hat{w}_{ji}((1 - u)^{-1}v^{-1}) \\ &\quad \times \hat{x}_{ji}(-(1 - u)^{-1}uv^{-1})\hat{x}_{ij}(-v) \quad (\text{by (R1)}) \\ &= \hat{x}_{ij}(vu)\hat{x}_{ji}(-u^{-1}v^{-1})\hat{x}_{ij}(vu - u)\hat{x}_{ji}(v^{-1})\hat{x}_{ij}(-v) \quad (\text{by (R1)}) \\ &= \hat{w}_{ij}(vu)\hat{w}_{ij}(v)^{-1}. \end{aligned}$$

Equality (T8) is equivalent to

$$\hat{w}_{ij}(vu)\hat{w}_{ij}(v)\hat{w}_{ij}(-vu) = \hat{w}_{ij}(vu^2),$$

and which follows from (R3) and  $\hat{w}_{ij}(u) = \hat{w}_{ji}(-u^{-1})$ . Thus we have proved Lemma 1.4.5.  $\square$

Next, assume that  $n \geq 3$ . We know from [8, Lemmas 9.8 and 9.10] that for  $1 \leq i \neq j \leq n$ ,  $u \in D_\tau^\times$ , and  $s \in D^\times$  with  $s \neq 1$ ,

$$\begin{aligned} (\text{TT1}) \quad & \hat{h}_{ij}(u)\hat{h}_{ji}(u) = 1, \\ (\text{TT2}) \quad & \hat{h}_{ij}(u)\hat{h}_{ki}(u)\hat{h}_{jk}(u) = 1 \quad (k \neq i, k \neq j), \\ (\text{TT3}) \quad & \hat{c}_{ij}(s, 1 - s) = 1. \end{aligned}$$

Also, we know from [13, Proposition 2.1] that

$$(\text{TT0}) \quad \hat{c}_{ij}(u, v) = \hat{c}_{ik}(u, v) \text{ for } k \neq i \text{ and } u, v \in D_\tau^\times.$$

**Lemma 1.4.6.** In  $\hat{T}$ , the following hold for  $1 \leq i \neq j \leq n$ ,  $1 \leq k \leq n$  with  $k \neq i$ , and  $u, v, w, x \in D_\tau^\times$ :

$$\begin{aligned} (\text{TT4}) \quad & \hat{c}_{ij}(u, v) = [\hat{h}_{ij}(u), \hat{h}_{ik}(v)], \\ (\text{TT4}') \quad & \hat{c}_{ij}(u, v)^{-1} = \hat{c}_{ij}(v, u), \\ (\text{TT5}) \quad & \hat{h}_{ij}(x)\hat{c}_{ik}(u, v)\hat{h}_{ij}(u)^{-1} \\ &= \hat{c}_{ik}(u, x)^{-1}\hat{c}_{ik}(u, xv) = \hat{c}_{ik}(xu, v)\hat{c}_{ik}(x, v)^{-1} = \hat{c}_{ik}(xu, x, v), \\ (\text{TT6}) \quad & \hat{c}_{ij}(uv, w) = \hat{c}_{ij}(uv, u)w\hat{c}_{ij}(u, w), \\ (\text{TT7}) \quad & \hat{c}_{ij}(u, vw) = \hat{c}_{ij}(u, v)\hat{c}_{ij}(v, w), \end{aligned}$$

where  $uv = uvu^{-1}$  for  $u, v \in D_\tau^\times$ .

*Proof.* (TT4) can be easily checked by (R6). (TT4)' is computed as follows by (TT0) and (TT4):

$$\hat{c}_{ik}(u, v)^{-1} = [\hat{h}_{ik}(u), \hat{h}_{ij}(v)]^{-1} = \hat{c}_{ij}(v, u) = \hat{c}_{ik}(v, u).$$

By (R6), we see that

$$\begin{aligned}\hat{h}_{ij}(x)\hat{c}_{ik}(u,v)\hat{h}_{ij}(u)^{-1} &= \hat{h}_{ij}(x)\hat{h}_{ik}(u)\hat{h}_{ik}(v)\hat{h}_{ik}(vu)^{-1}\hat{h}_{ij}(x)^{-1} \\ &= \hat{h}_{ik}(xu)\hat{h}_{ik}(x)^{-1}\hat{h}_{ik}(xv)\hat{h}_{ik}(xvu)^{-1} \\ &= \hat{c}_{ik}(u,x)^{-1}\hat{c}_{ik}(u,xv).\end{aligned}$$

Hence the first equation of (TT5) holds. Similarly, we get

$$\hat{h}_{ij}(x)\hat{c}_{ik}(u,v)\hat{h}_{ij}(u)^{-1} = \hat{c}_{ik}(xu,v)\hat{c}_{ik}(x,v)^{-1}$$

by (TT4)'. In order to prove the last equality of (TT5), we need the next relation; for  $u, v, w \in D^\times$ ,

$$(TT5)' \quad \hat{c}_{ij}(u,vw) = \hat{c}_{ij}(uv,w)\hat{c}_{ij}(wu,v).$$

We compute

$$\begin{aligned}\hat{c}_{ij}(u,vw) &= [\hat{h}_{ij}(u), \hat{h}_{ik}(vw)] \\ &= [\hat{h}_{ij}(u), \hat{c}_{ik}(v,w)\hat{h}_{ik}(w)\hat{h}_{ik}(v)] \\ &= \hat{h}_{ij}(u)\hat{c}_{ik}(v,w)\hat{h}_{ik}(w)\hat{h}_{ik}(v)\hat{h}_{ij}(u)^{-1}\hat{h}_{ik}(v)^{-1}\hat{h}_{ik}(w)^{-1}\hat{c}_{ik}(v,w)^{-1} \\ &= \hat{h}_{ij}(u)\hat{c}_{ik}(v,w)\hat{h}_{ik}(w)\hat{h}_{ik}(v)\hat{h}_{ij}(u)^{-1}\hat{h}_{ik}(vw)^{-1} \\ &= \hat{c}_{ik}(uv,w)\hat{c}_{ik}(u,w)^{-1}\hat{h}_{ij}(u)\hat{h}_{ik}(w)\hat{h}_{ik}(v)\hat{h}_{ij}(u)^{-1}\hat{h}_{ik}(vw)^{-1} \quad (\text{by (TT5)}) \\ &= \hat{c}_{ik}(uv,w)\hat{h}_{ik}(w)\hat{h}_{ij}(u)\hat{h}_{ik}(v)\hat{h}_{ij}(u)^{-1}\hat{h}_{ik}(vw)^{-1} \\ &= \hat{c}_{ik}(uv,w)\hat{h}_{ik}(w)\hat{c}_{ij}(u,v)\hat{h}_{ik}(v)\hat{h}_{ik}(vw)^{-1} \\ &= \hat{c}_{ik}(uv,w)\hat{c}_{ij}(wu,v)\hat{c}_{ij}(w,v)^{-1}\hat{h}_{ik}(w)\hat{h}_{ik}(v)\hat{h}_{ik}(vw)^{-1} \quad (\text{by (TT5)}) \\ &= \hat{c}_{ij}(uv,w)\hat{c}_{ij}(wu,v) \quad (\text{by (TT0)}).\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}\hat{c}_{ik}(x^u, x^v) &= \hat{c}_{ik}(xu, vx^{-1})\hat{c}_{ik}(vux^{-1}, x) \quad (\text{by (TT5)'}) \\ &= \hat{c}_{ik}(xu, v)\hat{h}_{ij}(v)\hat{c}_{ik}(xu, x^{-1})\hat{c}_{ik}(ux^{-1}, x)\hat{h}_{ij}(v)\hat{c}_{ik}(v, x) \quad (\text{by (TT5)}) \\ &= \hat{c}_{ik}(xu, v)\hat{c}_{ik}(v, x) \quad (\text{by (TT5)'}) \\ &= \hat{h}_{ij}(x)\hat{c}_{ik}(u, v)\hat{h}_{ij}(x)^{-1} \quad (\text{by (TT5)}).\end{aligned}$$

Hence (TT5) holds. (TT6) and (TT7) follow from (TT5).  $\square$

## 1.5 Universal central extensions

We show the universality of  $\phi : St(n, D_\tau) \rightarrow E(n, D_\tau)$  when  $|Z(D)| \geq 5$  and  $|Z(D)| \neq 9$ , where  $Z(D)$  is the center of  $D$ . We first fix four central elements  $a, b, c, d \in D^\times$ , which satisfy

$$a^2 - 1 \neq 0, \quad b = (a^2 - 1)^{-1}, \quad c - 1 \neq 0, \quad c^2 - c + 1 \neq 0, \quad d^3 - 1 \neq 0.$$

Since  $Z(D)$  is a field, we can choose these elements (cf. [16, Proof of Theorem 10, page 51]). We note that  $E(n, D_\tau)$  is perfect since

$$x_\beta(f) = [h_\beta(a), x_\beta(bf)]$$

for  $\beta \in \Delta$  and  $f \in D_\tau$ . Similarly, we see that  $St(n, D_\tau)$  is also perfect.

Now, let  $\phi^* : E^* \rightarrow E(n, D_\tau)$  be any central extension, and set

$$M(z) = \{z^* \in E^* \mid \phi^*(z^*) = z\} = (\phi^*)^{-1}(\{z\})$$

for  $z \in E(n, D_\tau)$ . Define

$$x_\beta^*(f) = [h^*, x^*] \in E^*$$

for  $h^* \in M(h_\beta(a))$  and  $x^* \in M(x_\beta(bf))$ . We see that this is well-defined from the following lemma, which is called “central trick”. By direct calculation, we get the following lemma.

**Lemma 1.5.1** (central trick). Let  $\psi : K \rightarrow G$  be a central extension of a group  $G$ . For  $X, X', Y, Y' \in K$ , if  $\psi(X) = \psi(X')$  and  $\psi(Y) = \psi(Y')$ , then  $[X, Y] = [X', Y']$ .

For  $u \in D_\tau$ , we set

$$\begin{aligned} w_\beta^*(u) &= x_\beta^*(u)x_\beta^*(-u^{-1})x_\beta^*(u), \\ h_\beta^*(u) &= w_\beta^*(u)w_\beta^*(-1). \end{aligned}$$

**Lemma 1.5.2.** The following equations hold for  $\beta \in \Delta$ ,  $f \in D_\tau$ , and  $u \in D_\tau^\times$ :

$$\begin{aligned} (1) \quad & w_\beta^*(u)x_\beta^*(f)w_\beta^*(u)^{-1} = x_{-\beta}^*(-u^{-1}fu^{-1}), \\ (2) \quad & h_\beta^*(u)x_\beta^*(f)h_\beta^*(u)^{-1} = x_\beta^*(ufu). \end{aligned}$$

*Proof.*

$$\begin{aligned} (1) \quad & w_\beta^*(u)x_\beta^*(f)w_\beta^*(u)^{-1} = w_\beta^*(u)[h_\beta^*(a), x_\beta^*(bf)]w_\beta^*(u)^{-1} \\ &= [w_\beta^*(u)h_\beta^*(a)w_\beta^*(u)^{-1}, w_\beta^*(u)x_\beta^*(bf)w_\beta^*(u)^{-1}] \\ &= [h_{-\beta}^*(a), x_{-\beta}^*(-bu^{-1}fu^{-1})] \text{ (by central trick)} \\ &= x_{-\beta}^*(-u^{-1}fu^{-1}). \end{aligned}$$

$$\begin{aligned} (2) \quad & h_\beta^*(u)x_\beta^*(f)h_\beta^*(u)^{-1} = h_\beta^*(u)[h_\beta^*(a), x_\beta^*(bf)]h_\beta^*(u)^{-1} \\ &= [h_\beta^*(u)h_\beta^*(a)h_\beta^*(u)^{-1}, h_\beta^*(u)x_\beta^*(bf)h_\beta^*(u)^{-1}] \\ &= [h_\beta^*(a), x_\beta^*(bufu)] \text{ (by central trick)} \\ &= x_\beta^*(ufu). \end{aligned}$$

□

For  $\beta = \epsilon_i - \epsilon_j, \gamma = \epsilon_k - \epsilon_l \in \Delta$  with  $i \neq j$  and  $k \neq l$ , and  $f, g \in D_\tau$ , we define  $\pi_{\beta, \gamma}(f, g)$  by

$$\pi_{\beta, \gamma}(f, g) = \begin{cases} [x_\beta^*(f), x_\gamma^*(g)]x_{\beta+\gamma}^*(fg)^{-1} & \text{if } j = k, \\ [x_\beta^*(f), x_\gamma^*(g)]x_{\beta+\gamma}^*(-gf)^{-1} & \text{if } i = l, \\ [x_\beta^*(f), x_\gamma^*(g)] & \text{otherwise.} \end{cases}$$

We note that  $\pi_{\beta, \gamma}(f, g)$  is central in  $E^*$ .

**Lemma 1.5.3.** Let  $\beta = \epsilon_i - \epsilon_j, \gamma = \epsilon_k - \epsilon_l \in \Delta$  with  $(i, j) \neq (l, k)$  and  $f, f', g, g' \in D_\tau$ . Then

- (1)  $\pi_{\beta, \gamma}(f + f', g) = \pi_{\beta, \gamma}(f, g)\pi_{\beta, \gamma}(f', g)$ ,
- (2)  $\pi_{\beta, \gamma}(f, g + g') = \pi_{\beta, \gamma}(f, g)\pi_{\beta, \gamma}(f, g')$ ,
- (3)  $\pi_{\beta, \gamma}(f, g) = 1$ ,
- (4)  $x_\beta^*(f)x_\beta^*(g) = x_\beta^*(f + g)$ .

*Proof.* We prove this lemma in three steps. In Step 1, we first check the equalities (1), (2) for  $j \neq k$  and  $i \neq l$ , and then prove (3) except for the cases  $j = k$  and  $i = l$ . Using these facts, we show that (4) holds for all  $\beta \in \Delta, f, g \in D_\tau$  in Step 2. Finally, we check (1), (2) and (3) for the others.

**Step 1.** Assume that  $j \neq k, i \neq l$ . Then, by the definition of  $\pi_{\beta, \gamma}(f, g)$ , we have

$$\begin{aligned} \pi_{\beta, \gamma}(f + f', g) &= [x_\beta^*(f + f'), x_\gamma^*(g)] \\ &= [x_\beta^*(f)x_\beta^*(f'), x_\gamma^*(g)] \\ &= x_\beta^*(f)x_\beta^*(f')x_\gamma^*(g)x_\beta^*(f')^{-1}x_\beta^*(f)^{-1}x_\gamma^*(g)^{-1} \\ &= x_\beta^*(f)\pi_{\beta, \gamma}(f', g)x_\gamma^*(g)x_\beta^*(f)^{-1}x_\gamma^*(g)^{-1} \\ &= \pi_{\beta, \gamma}(f, g)\pi_{\beta, \gamma}(f', g), \end{aligned}$$

and

$$\begin{aligned} \pi_{\beta, \gamma}(f, g + g') &= [x_\beta^*(f), x_\gamma^*(g + g')] \\ &= [x_\beta^*(f), x_\gamma^*(g)x_\gamma^*(g')] \\ &= x_\beta^*(f)x_\gamma^*(g)x_\gamma^*(g')x_\beta^*(f)^{-1}x_\gamma^*(g')^{-1}x_\gamma^*(g)^{-1} \\ &= x_\beta^*(f)x_\gamma^*(g)x_\beta^*(f)^{-1}\pi_{\beta, \gamma}(f, g')x_\gamma^*(g)^{-1} \\ &= \pi_{\beta, \gamma}(f, g)\pi_{\beta, \gamma}(f, g'). \end{aligned}$$

If  $i \neq k$  and  $j \neq l$ , then

$$\begin{aligned} \pi_{\beta, \gamma}(bf, g) &= h_\beta^*(a)\pi_{\beta, \gamma}(bf, g)h_\beta^*(a)^{-1} \\ &= [h_\beta^*(a)x_\beta^*(bf)h_\beta^*(a)^{-1}, h_\beta^*(a)x_\gamma^*(g)h_\beta^*(a)^{-1}] \\ &= [x_\beta^*(a^2bf), x_\gamma^*(g)] \quad (\text{by the central trick and (R4)}) \\ &= \pi_{\beta, \gamma}(a^2bf, g), \end{aligned}$$

and

$$\begin{aligned}
\pi_{\beta,\gamma}(f, g) &= \pi_{\beta,\gamma}(a^2bf - bf, g) \\
&= [x_\beta^*(a^2bf)x_\beta^*(bf)^{-1}, x_\gamma^*(g)] \\
&= x_\beta^*(a^2bf)x_\beta^*(bf)^{-1}\pi_{\beta,\gamma}(bf, g)^{-1}x_\beta^*(bf)x_\gamma^*(g)x_\beta^*(a^2bf)^{-1}x_\gamma^*(g)^{-1} \\
&= \pi_{\beta,\gamma}(a^2bf, g)\pi_{\beta,\gamma}(bf, g)^{-1}.
\end{aligned}$$

Therefore, we obtain  $\pi_{\beta,\gamma}(f, g) = 1$ .

If  $i = k$  and  $j \neq l$ , then

$$\begin{aligned}
\pi_{\beta,\gamma}(f, g) &= h_\beta^*(d)\pi_{\beta,\gamma}(f, g)h_\beta^*(d)^{-1} \\
&= \pi_{\beta,\gamma}(d^2f, dg) \quad (\text{by the central trick and (R4)}),
\end{aligned}$$

and

$$\begin{aligned}
\pi_{\beta,\gamma}(f, g) &= h_{\beta-\gamma}^*(d)\pi_{\beta,\gamma}(f, g)h_{\beta-\gamma}^*(d)^{-1} \\
&= \pi_{\beta,\gamma}(df, d^{-1}g) \quad (\text{by the central trick and (R4)}).
\end{aligned}$$

Therefore, we obtain  $\pi_{\beta,\gamma}(f, g) = \pi_{\beta,\gamma}(d^2f, dg) = \pi_{\beta,\gamma}(d^3f, g)$ , hence  $\pi_{\beta,\gamma}((d^3 - 1)f, g) = 1$ . Thus  $\pi_{\beta,\gamma}(f, g) = 1$  for all  $f, g \in D_\tau$ .

If  $i \neq k$  and  $j = l$ , then using the same way as the previous case, we see that  $\pi_{\beta,\gamma}((d^3 - 1)f, g) = 1$ , and hence,  $\pi_{\beta,\gamma}(f, g) = 1$  for all  $f, g \in D_\tau$ .

If  $i = k$  and  $j = l$ , then

$$\pi_{\beta,\beta}(f, g) = h_\beta^*(c)\pi_{\beta,\beta}(f, g)h_\beta^*(c)^{-1} = [x_\beta^*(c^2f), x_\beta^*(c^2g)] = \pi_{\beta,\beta}(c^2f, c^2g).$$

Therefore, we see that

$$\begin{aligned}
\pi_{\beta,\beta}(f, g) &= \pi_{\beta,\beta}(cf + (1 - c)f, g) \\
&= \pi_{\beta,\beta}(cf, g)\pi_{\beta,\gamma}((1 - c)f, g) \\
&= \pi_{\beta,\beta}(f, gc^{-1})\pi_{\beta,\gamma}(f, g(1 - c)^{-1}) \\
&= \pi_{\beta,\beta}(f, gc^{-1} + g(1 - c)^{-1}) \\
&= \pi_{\beta,\beta}(f, g\{c(1 - c)\}^{-1}) \\
&= \pi_{\beta,\beta}(c(1 - c)f, g).
\end{aligned}$$

Thus we get  $\pi_{\beta,\beta}((c^2 - c + 1)f, g) = 1$ . Therefore we obtain  $\pi_{\beta,\beta}(f, g) = 1$  for all  $f, g \in D_\tau$ , in particular,

$$x_\beta^*(f)x_\beta^*(g) = x_\beta^*(g)x_\beta^*(f)$$

for all  $\beta \in \Delta$  and  $f, g \in D_\tau$ .

**Step 2.** If we put  $(f, g) = x_\beta^*(f)x_\beta^*(g)x_\beta^*(f + g)^{-1}$  for  $\beta \in \Delta$  and  $f, g \in D_\tau$ ,

then  $(f, g)$  is central in  $E^*$ . Therefore, by Step 1 and the central trick, we obtain

$$\begin{aligned}
(bf, bg) &= h_\beta^*(a)(bf, bg)h_\beta^*(a)^{-1} \\
&= h_\beta^*(a)x_\beta^*(bf)x_\beta^*(bg)x_\beta^*(b(f+g))^{-1}h_\beta^*(a)^{-1} \\
&= [h_\beta^*(a), x_\beta^*(bf)]x_\beta^*(bf)h_\beta^*(a)x_\beta^*(bg)x_\beta^*(b(f+g))^{-1}h_\beta^*(a)^{-1} \\
&= [h_\beta^*(a), x_\beta^*(bf)]x_\beta^*(bf)[h_\beta^*(a), x_\beta^*(bg)]x_\beta^*(bg) \\
&\quad \cdot [h_\beta^*(a), x_\beta^*(b(f+g))^{-1}]x_\beta^*(b(f+g))^{-1} \\
&= x_\beta^*(f)x_\beta^*(bf)x_\beta^*(g)x_\beta^*(bg)x_\beta^*(f+g)^{-1}x_\beta^*(b(f+g))^{-1} \\
&= (f, g)(bf, bg).
\end{aligned}$$

Thus  $(f, g) = 1$  for all  $f, g \in D\tau$ , hence  $x_\beta^*(f)x_\beta^*(g) = x_\beta^*(f+g)$ . In particular, we get  $x_\beta^*(f)^{-1} = x_\beta^*(-f)$ .

**Step 3.** Assume that  $j = k$ . By Step 1 and Step 2, we obtain that

$$\begin{aligned}
\pi_{\beta, \gamma}(f + f', g) &= [x_\beta^*(f + f'), x_\gamma^*(g)]x_{\beta+\gamma}^*(-(f + f')g) \\
&= x_\beta^*(f)x_\beta^*(f')x_\gamma^*(g)x_\beta^*(-f')x_\beta^*(-f)x_\gamma^*(-g)x_{\beta+\gamma}^*(-fg)x_{\beta+\gamma}^*(-f'g) \\
&= x_\beta^*(f)[x_\beta^*(f'), x_\gamma^*(g)]x_\gamma^*(g)x_\beta^*(-f)x_\gamma^*(-g)x_{\beta+\gamma}^*(-fg)x_{\beta+\gamma}^*(-f'g) \\
&= x_\beta^*(f)\pi_{\beta, \gamma}(f', g)x_{\beta+\gamma}^*(f'g)x_\beta^*(-f)\pi_{\beta, \gamma}(f, g)x_{\beta+\gamma}^*(-f'g) \\
&= \pi_{\beta, \gamma}(f, g)\pi_{\beta, \gamma}(f', g).
\end{aligned}$$

We similarly obtain that

$$\begin{aligned}
\pi_{\beta, \gamma}(f, g + g') &= [x_\beta^*(f), x_\gamma^*(g + g')]x_{\beta+\gamma}^*(-f(g + g')) \\
&= x_\beta^*(f)x_\gamma^*(g)x_\gamma^*(g')x_\beta^*(-f)x_\gamma^*(-g')x_\gamma^*(-g)x_{\beta+\gamma}^*(-fg)x_{\beta+\gamma}^*(-fg') \\
&= x_\beta^*(f)x_\gamma^*(g')x_\beta^*(-f)[x_\beta^*(f), x_\gamma^*(g)]x_\gamma^*(-g')x_{\beta+\gamma}^*(-fg)x_{\beta+\gamma}^*(-fg') \\
&= \pi_{\beta, \gamma}(f, g')x_{\beta+\gamma}^*(fg')\pi_{\beta, \gamma}(f, g)x_\gamma^*(-g)x_{\beta+\gamma}^*(-fg)x_{\beta+\gamma}^*(-fg') \\
&= \pi_{\beta, \gamma}(f, g)\pi_{\beta, \gamma}(f, g').
\end{aligned}$$

Also, we see that

$$\begin{aligned}
\pi_{\beta, \gamma}(f, g) &= h_\beta^*(d)\pi_{\beta, \gamma}(f, g)h_\beta^*(d)^{-1} \\
&= h_\beta^*(d)[x_\beta^*(f), x_\gamma^*(g)]x_{\beta+\gamma}^*(-fg)h_\beta^*(d)^{-1} \\
&= [x_\beta^*(d^2f), x_\gamma^*(d^{-1}g)]x_{\beta+\gamma}^*(-dfg) \\
&= \pi_{\beta, \gamma}(d^2f, d^{-1}g)
\end{aligned}$$

and

$$\begin{aligned}
\pi_{\beta, \gamma}(f, g) &= h_{\beta+\gamma}^*(d)\pi_{\beta, \gamma}(f, g)h_{\beta+\gamma}^*(d)^{-1} \\
&= [x_\beta^*(df), x_\gamma^*(dg)]x_{\beta+\gamma}^*(-d^2fg) \\
&= \pi_{\beta, \gamma}(df, dg).
\end{aligned}$$

Therefore we obtain  $\pi_{\beta, \gamma}((d^3 - 1)f, g) = 1$ , in particular,  $\pi_{\beta, \gamma}(f, g) = 1$ .



Assume that  $i = l$ . Then, using the way similar to the case  $j = k$ , we see that

$$\begin{aligned}\pi_{\beta,\gamma}(f + f', g) &= \pi_{\beta,\gamma}(f, g)\pi_{\beta,\gamma}(f', g), \\ \pi_{\beta,\gamma}(f, g + g') &= \pi_{\beta,\gamma}(f, g)\pi_{\beta,\gamma}(f, g'), \\ \pi_{\beta,\gamma}(f, g) &= 1.\end{aligned}$$

This completes the proof of the lemma.  $\square$

Therefore, we obtain the following theorem (cf. [16, page 51]).

**Theorem 1.5.4.** If  $|Z(D)| \geq 5$  and  $|Z(D)| \neq 9$ , then  $\phi$  is a universal central extension of  $E(n, D_\tau)$ .

*Proof.* For any central extension  $\phi^* : E^* \rightarrow E(n, D_\tau)$ , we define  $x_\beta^*(f)$  for all  $\beta \in \Delta$  and  $f \in D_\tau$  as above. By Lemma 1.5.3, together with (ST1) and (ST2) (or (ST2)'), there exists a group homomorphism  $\hat{\phi}^* : St(n, D_\tau) \rightarrow E^*$  defined by  $\hat{\phi}^*(x_\beta(f)) = x_\beta^*(f)$  for  $\beta \in \Delta$  and  $f \in D_\tau$ . Thus we obtain  $\phi = \phi^* \circ \hat{\phi}^*$ .  $\square$

*Remark 1.5.5.* If the cardinality of  $Z(D)$  is less than 5 or is equal to 9, then there exist counterexamples (see [16, page 49]).

*Remark 1.5.6.* We remark that there exists a division ring  $D$  such that  $Z(D) = F$  for any field  $F$ . Indeed, let  $F$  be any field, and  $x_i$ ,  $i \in \mathbb{Z}$  be countable indeterminates. We put

$$K = F(x_i)_{i \in \mathbb{Z}} = \left\{ \frac{f(x_{i_1}, \dots, x_{i_r})}{g(x_{j_1}, \dots, x_{j_s})} \mid f, g \in F[x_i]_{i \in \mathbb{Z}} \right\}.$$

We define an automorphism  $\tau$  of  $K$  to be  $\tau(x_i) = x_{i+1}$ . Furthermore let  $t$  be a new indeterminate, and put

$$D = K((t)) = \left\{ \sum_{-\infty \leq k < \infty} a_k t^k \mid a_k \in K \right\}.$$

Then,  $D$  can be viewed as a division ring whose product is given by

$$\left( \sum_k a_k t^k \right) \left( \sum_l b_l t^l \right) = \sum_m \left( \sum_{k+l=m} a_k \tau^k(b_l) \right) t^m.$$

## 1.6 $K_1$ -groups

As a subgroup of  $GL(n, D_\tau)$ , we put  $H = \{\text{diag}(u_1, \dots, u_n) \mid u_i \in D_\tau^\times\}$ . Using the euclidean division, we see that  $GL(n, D_\tau) = \langle E(n, D_\tau), H \rangle$  since  $D_\tau$  is a euclidean ring (cf. [11], [15, Proposition 1.1.2]). If we set  $H_1 = \{\text{diag}(u, 1, \dots, 1) \mid u \in D_\tau^\times\} \cong D_\tau^\times$ , then we have

$$\begin{aligned}GL(n, D_\tau) &= \langle E(n, D_\tau), H \rangle \\ &= \langle E(n, D_\tau), H_1 \rangle \\ &\supset E(n, D_\tau).\end{aligned}$$

Define our  $K_1$ -group by  $K_1(n, D_\tau) = GL(n, D_\tau)/E(n, D_\tau)$  (cf. [8, page 25]). Then, we have

$$K_1(n, D_\tau) = GL(n, D_\tau)/E(n, D_\tau) \cong H_1/(E(n, D_\tau) \cap H_1).$$

In the following, we discuss the subgroup  $E(n, D_\tau) \cap H_1$ .

Since  $E(n, D_\tau)$  has the Bruhat decomposition (by Theorem 1.2.1), we consider  $BwB \cap H_1$  for  $w \in W_a$ , where  $B = B^+$ . By Proposition 1.1.2 and Lemma 1.2.2 (4), there exist  $\dot{w} \in W$  and  $h = \text{diag}(t^{m_1}, \dots, t^{m_n})$  such that  $w = \dot{w}h$ , where  $m_i \in \mathbb{Z}$  with  $m_1 + \dots + m_n = 0$ . Then we see that

$$BwB \cap H_1 = UT_0wT_0U \cap H_1 = U\dot{w}hT_0U \cap H_1.$$

If we set

$$D_\tau^{\geq 0} = D[t] \text{ and } D_\tau^{> 0} = D[t]t,$$

then we obtain

$$U \subset \begin{pmatrix} 1 + D_\tau^{\geq 0} & D_\tau^{\geq 0} & \dots & D_\tau^{\geq 0} \\ D_\tau^{> 0} & 1 + D_\tau^{\geq 0} & & \vdots \\ \vdots & & \ddots & D_\tau^{\geq 0} \\ D_\tau^{> 0} & \dots & D_\tau^{> 0} & 1 + D_\tau^{> 0} \end{pmatrix}.$$

Suppose  $BwB \cap H_1 \neq \emptyset$ . Then

$$\begin{aligned} BwB \cap H_1 \neq \emptyset &\Rightarrow U\dot{w}hT_0U \cap H_1 \neq \emptyset \\ &\Rightarrow U \cap H_1 UT_0h^{-1}\dot{w}^{-1} \neq \emptyset. \end{aligned}$$

Therefore, we can take  $d \in H_1$  of degree  $m$ ,  $x_+ \in U$ , and  $h_0 = \text{diag}(u_1, \dots, u_n) \in T_0$  such that  $x = dx_+h_0h^{-1}\dot{w}^{-1} \in U$ . Then we see that

$$x \in \begin{pmatrix} t^{m-m_1}(1 + D_\tau^{\geq 0}) & t^{m-m_2}D_\tau^{\geq 0} & \dots & t^{m-m_n}D_\tau^{\geq 0} \\ t^{-m_1}D_\tau^{> 0} & t^{-m_2}(1 + D_\tau^{\geq 0}) & \dots & t^{-m_n}D_\tau^{\geq 0} \\ \vdots & \vdots & \ddots & \vdots \\ t^{-m_1}D_\tau^{> 0} & t^{-m_2}D_\tau^{> 0} & \dots & t^{-m_n}(1 + D_\tau^{> 0}) \end{pmatrix} \dot{w}^{-1}.$$

If  $m_i > 0$  for some  $2 \leq i \leq n$ , then  $x$  has an entry with a negative power of  $t$  in a diagonal part. This is a contradiction, hence  $m_i \leq 0$  for all  $2 \leq i \leq n$ . In particular, we obtain  $m_1 \geq 0$  by  $m_1 + \dots + m_n = 0$ . Suppose that  $\dot{w}^{-1}$  is a permutation sending  $j$ th column to the 1st one for some  $2 \leq j \leq n$ . Then  $2 \leq \dot{w}^{-1}(k) \leq n$  for all  $2 \leq k \leq n$  with  $k \neq j$ . If  $\dot{w}^{-1}(i_1) > \dot{w}^{-1}(i_2)$  for  $2 \leq i_1 < i_2 \leq n$  except  $j$ , then we see that  $t^{-m_{i_1}}D_\tau^{> 0} \cap D_\tau^{\geq 0} = \emptyset$ . This is a contradiction. Thus, we obtain  $\dot{w}^{-1}(k) = k$  for such  $k$ .

Hence, it suffices to check the two cases: When  $\dot{w}^{-1}$  is a transposition of the 1st column and the  $j$ th column, and when  $\dot{w}^{-1}$  is an identity. Suppose that  $\dot{w}^{-1}$  is the transposition  $(1, j)$ . Then we have  $t^{m-m_1}(1 + D_\tau^{\geq 0}) \cap D_\tau^{\geq 0} \neq \emptyset$ , which implies  $m - m_1 \geq 0$ . On the other hand, we see that  $t^{m-m_j}D_\tau^{\geq 0} \cap (1 + D_\tau^{> 0}) \neq \emptyset$ , which implies  $m - m_j \leq 0$ . We know that  $m_1 + m_j = 0$ , so that we obtain

$$0 \leq -m_j \leq m \leq m_j \leq 0,$$

hence,  $m = m_1 = m_j = 0$ . However, the 1st column cannot be going to the  $j$ th column from the fact  $(1 + D_\tau^{>0}) \cap D_\tau^{>0} = \emptyset$ . Therefore,  $\dot{w} = 1$ .

Therefore, we obtain

$$\begin{aligned} E(n, D_\tau) \cap H_1 &= B \cap H_1 = (U \rtimes T_0) \cap H_1 \\ &= T_0 \cap H_1 \\ &\subset T \cap H_1. \end{aligned}$$

Recall that  $h_\beta(s) = h_{ij}(s)$  in  $T$  for  $\beta = \epsilon_i - \epsilon_j$  and  $s \in D_\tau^\times$ . From the definition of  $T$  and the relation  $h_{ij}(s) = h_{1j}(s)h_{1i}(s^{-1})$  for  $2 \leq i \neq j \leq n$  and  $s \in D_\tau^\times$ , we deduce that  $T$  is generated by  $h_{1j}(s)$  for  $2 \leq j \leq n$  and  $s \in D_\tau^\times$ . Thus every element  $h$  in  $T \cap H_1$  can be written in the form

$$h = h_{1,l_1}(s_1)h_{1,l_2}(s_2) \cdots h_{1,l_k}(s_k)$$

for some  $2 \leq l_i \leq n$  and  $s_i \in D_\tau^\times$ . Here, we set  $I_j = \{1 \leq i \leq k \mid l_i = j\}$  for  $2 \leq j \leq n$ . Then we see that  $\{l_1, \dots, l_k\} = I_2 \sqcup I_3 \sqcup \cdots \sqcup I_n$ , and

$$h = \begin{pmatrix} s_1 s_2 \cdots s_k & 0 & \cdots & 0 \\ 0 & S_{I_2} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & S_{I_n} \end{pmatrix},$$

where  $S_{I_j} = s_{i_1}^{-1} s_{i_2}^{-1} \cdots s_{i_r}^{-1}$  with  $i_1 < i_2 < \cdots < i_r$ ,  $i_1, \dots, i_r \in I_j$  for all  $2 \leq j \leq n$ . Then we have  $S_{I_j} = 1$  for all  $2 \leq j \leq n$  since  $h \in H_1$ . Therefore, if we put  $s = s_1 s_2 \cdots s_k$ , then we can rewrite  $s$  as

$$\begin{aligned} s &= s_1 \cdots s_k = s_1 \cdots s_k S_{I_2} S_{I_3} \cdots S_{I_n} \\ &= s_1 \cdots s_k s_{i_1}^{-1} s_{i_2}^{-1} \cdots s_{i_k}^{-1}, \end{aligned}$$

where  $\{i_1, \dots, i_k\} = \{2, \dots, n\}$ . If  $i_r = 1$ , then putting  $v_1 = s_2 \cdots s_k s_{i_1}^{-1} \cdots s_{i_{r-1}}^{-1}$ , we have

$$\begin{aligned} s &= [s_1, v_1] v_1 s_1 s_{i_r}^{-1} \cdots s_{i_k}^{-1} \\ &= [s_1, v_1] v_1 s_{i_{r+1}}^{-1} \cdots s_{i_k}^{-1} \\ &= [s_1, v_1] s_2 \cdots s_k s_{i_1}^{-1} \cdots s_{i_{r-1}}^{-1} s_{i_{r+1}}^{-1} \cdots s_{i_k}^{-1}. \end{aligned}$$

If  $i_{r'} = 2$ , then putting  $v_2 = s_3 \cdots s_k s_{i_1}^{-1} \cdots s_{i_{r'-1}}^{-1}$ , we also have

$$s = [s_1, v_1][s_2, v_2] s_3 \cdots s_k s_{i_1}^{-1} \cdots s_{i_{r'-1}}^{-1} s_{i_{r'+1}}^{-1} \cdots s_{i_{r-1}}^{-1} s_{i_{r+1}}^{-1} \cdots s_{i_k}^{-1}.$$

Repeating this computation, we obtain

$$s = [s_1, v_1][s_2, v_2] \cdots [s_k, v_k].$$

Thus we obtain

$$T \cap H_1 \subset \begin{pmatrix} [D_\tau^\times, D_\tau^\times] & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{pmatrix}.$$

On the other hand, we see that

$$\begin{pmatrix} [u, v] & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{pmatrix} = h_{12}(u)h_{12}(v)h_{12}(u^{-1}v^{-1})$$

for  $u, v \in D_\tau^\times$ . Since  $\deg([u, v]) = 0$ , we have

$$\begin{pmatrix} [D_\tau^\times, D_\tau^\times] & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{pmatrix} \subset T_0 \cap H_1.$$

These deduce that

$$E(n, D_\tau) \cap H_1 = \begin{pmatrix} [D_\tau^\times, D_\tau^\times] & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{pmatrix}.$$

Therefore we obtain the following theorem.

**Theorem 1.6.1.** Let  $n \geq 2$ . It holds that

$$K_1(n, D_\tau) \cong D_\tau^\times / [D_\tau^\times, D_\tau^\times].$$

## 2 Matsumoto type theorem for linear groups over $D_\tau$

### 2.1 Symplectic $K_2$ -groups

In this subsection, we consider the case of  $n = 2$ . Recall that  $D_\tau = D[t, t^{-1}]$  denotes the ring of non-commutative Laurent polynomials (see Section 1). Let  $P$  be the group generated by  $c(u, v)$ ,  $u, v \in D_\tau^\times$ , with the defining relations that for  $u, v, w, x, y \in D_\tau^\times$  and  $s \in D^\times$  with  $s \neq 1$ ,

- (P1)  $c(u, v)c(vu, w) = c(u, vw)c(v, w)$ ,
- (P2)  $c(u, v) = c(uvu, u^{-1})$ ,
- (P3)  $c(x, y)c(u, v)c(x, y)^{-1} = c([x, y]u, v)c(v, [x, y])$ ,
- (P4)  $c(s, v) = c(s, v(1 - s))$
- (P5)  $c(u, v) = c(u, -vu)$ .

We can show the following proposition by applying (P5) to (P1).

**Lemma 2.1.1.** In  $P$ , it holds that for  $u, v \in D_\tau^\times$ ,

$$(P5)' \quad c(u, v) = c(-uv, v).$$

Because  $[u, v] \in [D_\tau^\times, D_\tau^\times]$ ,  $u, v \in D_\tau^\times$ , satisfy the same relations as (P1)–(P5) (with  $c(u, v)$  replaced by  $[u, v]$ ), there exists a (unique) surjective group homomorphism  $\varphi : P \twoheadrightarrow [D_\tau^\times, D_\tau^\times]$  which sends  $c(u, v)$  to  $[u, v]$  for  $u, v \in D_\tau^\times$ . Set  $L = \text{Ker } \varphi$ ; note that

$$L = \{c(u_1, v_1)^{p_1} c(u_2, v_2)^{p_2} \cdots c(u_r, v_r)^{p_r} \mid r \geq 0, p_i = \pm 1, u_i, v_i \in D_\tau^\times, [u_1, v_1]^{p_1} [u_2, v_2]^{p_2} \cdots [u_r, v_r]^{p_r} = 1\}. \quad (\#)$$

We can show the following lemma in the same manner as [14, Lemma 2.2].

**Proposition 2.1.2.** The following exact is a central extension of  $[D_\tau^\times, D_\tau^\times]$  by  $L$ :

$$1 \longrightarrow L \longrightarrow P \xrightarrow{\varphi} [D_\tau^\times, D_\tau^\times] \longrightarrow 1.$$

By comparing (P1), (P2), (P3), (P4), (P5) with (T4), (T3), (T6), (T7), (T8), respectively, we see that there exists a (unique) surjective group homomorphism  $\zeta : P \rightarrow K_2(2, D_\tau)$  which maps  $c(u, v)$  to  $\hat{c}(u, v)$  for  $u, v \in D_\tau^\times$ . By Proposition 1.4.4 and (#), the restriction of this  $\zeta$  to  $L \subset P$  is a surjective group homomorphism from  $L$  onto  $K_2(2, D_\tau)$ .

**Theorem 2.1.3.** The group homomorphism  $\zeta : L \rightarrow K_2(2, D_\tau)$  is an isomorphism of groups.

We proceed the proof of this theorem along with [7], [14] and [15].

Let  $\tilde{H}$  be the group generated by  $\tilde{h}(u)$  for  $u \in D_\tau^\times$  and  $z(l)$  for  $l \in P$  with the defining relations that

$$\begin{aligned} (H1) \quad & \tilde{h}(u)\tilde{h}(v) = \tilde{h}(uvu)\tilde{h}(u^{-1}), \\ (H2) \quad & \tilde{h}(u)\tilde{h}(v) = z(c(u, v))\tilde{h}(vu), \\ (H3) \quad & z(l_1)z(l_2) = z(l_1l_2), \\ (H4) \quad & \tilde{h}(u)z(l) = z(c(u, \varphi_0(l)))z(l)\tilde{h}(u) \end{aligned}$$

for  $u, v \in D_\tau^\times$  and  $l, l_1, l_2 \in P$ . We deduce by (H3) that  $\{z(l) \mid l \in P\}$  is a subgroup of  $\tilde{H}$ . Moreover, we see by a way similar to [7, CHAPITRE II] and [10, Proposition 2] that  $L$  is isomorphic to  $\{z(l) \mid l \in L\}$ . We identify  $\{z(l) \mid l \in L\}$  with  $L$ , and write  $z(l)$  simply by  $l$  for  $l \in L$ .

We can easily check the next proposition.

**Proposition 2.1.4.** The same equalities as those in Lemma 1.4.5 with  $\hat{h}_{12}(u)$  replaced by  $\tilde{h}(u)$ , and with  $\hat{c}(u, v)$  replaced by  $z(c(u, v))$  hold in  $\tilde{H}$ .

**Proposition 2.1.5.** There exists a (unique) surjective group homomorphism  $\pi : \tilde{H} \rightarrow T$  which sends  $\tilde{h}(u)$  to  $h_{12}(u)$  for  $u \in D_\tau^\times$ , and  $z(c(v, w))$  to  $h_{12}(v)h_{12}(w)h_{12}(vw)^{-1}$  for  $v, w \in D_\tau^\times$ . The kernel  $\text{Ker } \pi$  is identical to  $L$ . Moreover,

$$1 \longrightarrow L \longrightarrow \tilde{H} \xrightarrow{\pi} T \longrightarrow 1$$

is a central extension of  $T$  by  $L$ .

*Proof.* The first assertion is obvious by  $h_{21}(u) = h_{12}(u^{-1}) \in T$  and Proposition 2.1.4. Let us show that  $\text{Ker } \pi = L$ . The inclusion  $\supset$  is obvious, so that we show the reverse inclusion  $\subset$ . Using (H2), (H3), and (H4) repeatedly, we deduce that every element  $h$  in  $\tilde{H}$  can be written in the form  $h = \xi \tilde{h}(s)$  with some  $s \in D_\tau^\times$  and  $\xi = z(c(u_1, v_1)^{p_1} c(u_2, v_2)^{p_2} \cdots c(u_r, v_r)^{p_r}) \in \{z(l) \mid l \in P\}$  with  $u_i, v_i \in D_\tau^\times$  and  $p_i \in \{\pm 1\}$ . If  $h \in \text{Ker } \pi$ , then

$$\begin{aligned} \pi(h) &= \pi(\xi)\pi(\tilde{h}(s)) \\ &= \begin{pmatrix} [u_1, v_1]^{p_1} [u_2, v_2]^{p_2} \cdots [u_r, v_r]^{p_r} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} = I. \end{aligned}$$

Therefore, we get  $s = 1$  and  $[u_1, v_1]^{p_1} [u_2, v_2]^{p_2} \cdots [u_r, v_r]^{p_r} = 1$ , which implies that  $\tilde{h}(s) = \tilde{h}(1) = 1$  by (T5)' for  $\tilde{H}$ , and  $\xi \in L$  by (#). Thus we have  $\text{Ker } \pi \subset L$ , and hence  $\text{Ker } \pi = L$ . By induction on  $r$ , we see that

$$h\tilde{h}(u)h^{-1} = \xi\tilde{h}(u)\xi^{-1} = \tilde{h}(\pi(\xi)u)\tilde{h}(\pi(\xi))^{-1}.$$

If  $h \in \text{Ker } \pi = L$ , then  $\pi(h) = \pi(\xi) = 1$ . Therefore  $L$  is central in  $\tilde{H}$ .  $\square$

By Propositions 2.1.2 and 2.1.5, we get the commutative diagram;

$$\begin{array}{ccccc} L & \longrightarrow & P & \xrightarrow{\varphi} & [D_\tau^\times, D_\tau^\times] \\ \parallel & & \downarrow z & & \downarrow d \\ L & \longrightarrow & \tilde{H} & \xrightarrow{\pi} & T, \end{array} \quad (\text{CD})$$

where  $d$  is the embedding of groups defined by  $d([u, v]) = \text{diag}([u, v], 1)$  for  $u, v \in D_\tau^\times$ , and  $z : P \rightarrow \tilde{H}$ ,  $l \mapsto z(l)$ . This implies that  $P$  is isomorphic to  $\{z(l) \mid l \in P\}$ . We identify  $\{z(l) \mid l \in P\}$  with  $P$ , and write  $z(l)$  simply by  $l$  for  $l \in P$ .

**Lemma 2.1.6.** If  $\tilde{h} \in \tilde{H}$  is such that  $\pi(\tilde{h}) = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}$  with some  $u_1, u_2 \in D_\tau^\times$ , then for  $u \in D_\tau^\times$ ,

$$\tilde{h}\tilde{h}(u)\tilde{h}^{-1} = \tilde{h}(u_1 u u_2^{-1})\tilde{h}(u_1 u_2^{-1})^{-1}.$$

*Proof.* In a way similar to the proof of Proposition 2.1.5, an element  $\tilde{h} \in \tilde{H}$  can be written in the form  $\tilde{h} = \tilde{h}(v)\xi$  with some  $v \in D_\tau^\times$  and  $\xi \in P \subset \tilde{H}$  by (H2), (H3), and (H4). If we put  $\varphi(\xi) = s \in [D_\tau^\times, D_\tau^\times]$ , then  $\pi(\tilde{h}) = \text{diag}(vs, v^{-1})$ , and

$$\begin{aligned} \tilde{h}\tilde{h}(u)\tilde{h}^{-1} &= \tilde{h}(v)\tilde{h}(su)\tilde{h}(s)^{-1}\tilde{h}(v)^{-1} \\ &= \tilde{h}(vsuv)\tilde{h}(vsv)^{-1} \end{aligned}$$

by (T1) and (T5).  $\square$

Next, we construct an extension of the monomial subgroup  $N$ , which is “compatible” with the extension  $(\tilde{H}, \pi)$  of  $T$  in Proposition 2.1.5 (see Proposition 2.1.9 below). For this, we give a presentation of  $N$ , and then define an action of  $N$  on  $\tilde{H}$ . The next lemma and proposition follow from [10, Proposition 3] and [15, Proposition 5.8].

**Lemma 2.1.7.** The subgroup  $N$  of  $E(2, D_\tau)$  is the group generated by  $w_{12}(u)$  for  $u \in D_\tau^\times$  with the defining relations that for  $u, v \in D_\tau^\times$ ,

$$\begin{aligned} \text{(N1)} \quad & w_{12}(u)^{-1} = w_{12}(-u), \\ \text{(N2)} \quad & w_{12}(1)h_{12}(u)w_{12}(1)^{-1} = h_{12}(u^{-1}), \\ \text{(N3)} \quad & h_{12}(u)h_{12}(v) = h_{12}(uvu)h_{12}(u^{-1}). \end{aligned}$$

**Proposition 2.1.8.** There exists an action of  $N$  on  $\tilde{H}$  defined by

$$w_{12}(u) \cdot \tilde{h}(v) = \tilde{h}(uv^{-1}u)\tilde{h}(u^2)^{-1}$$

for  $u, v \in D_\tau^\times$ ; we denote  $w_{12}(u) \cdot \tilde{h}(v)$  also by  $w_{12}(u)\tilde{h}(v)w_{12}(u)^{-1}$  for convenience.

Let  $\tilde{W} = \langle \vartheta \rangle \cong \mathbb{Z}$  be the cyclic group of infinite order. We put  $\tilde{T} = \langle \vartheta^2 \rangle \subset \tilde{W}$ , and  $N^* = \tilde{W} \ltimes \tilde{H}$ , where  $\tilde{W}$  acts on  $\tilde{H}$  by

$$\vartheta \cdot \tilde{h}(u) = w_{12}(-1) \cdot \tilde{h}(u) \quad \text{for } u \in D_\tau^\times.$$

There exists a group homomorphism  $\eta : \tilde{T} \rightarrow \tilde{H}$  defined by  $\eta(\vartheta^2) = \tilde{h}(-1)$ . Let  $J$  be the normal subgroup of  $N^* = \tilde{W} \ltimes \tilde{H}$  generated by  $\vartheta^{2n}\eta(\vartheta^{2n})^{-1}$  for  $n \in \mathbb{Z}$ , and let  $\tilde{N} = N^*/J$  be the quotient group. Let  $w_1 = \vartheta J$  be the coset in  $\tilde{N}$  containing  $\vartheta \in \tilde{W} \subset N^*$ . Let  $\tilde{\psi} : N^* \twoheadrightarrow N^*/J = \tilde{N}$  be the canonical homomorphism. We deduce that the restriction of  $\tilde{\psi}$  to  $\tilde{H} \subset N^*$  is injective; we regard  $\tilde{H}$  as a subgroup of  $\tilde{N}$ . If we set  $\tilde{w}(u) = \tilde{h}(u)\vartheta^{-1}J \in \tilde{N}$  for  $u \in D_\tau^\times$ , then we have, in  $\tilde{N} = N^*/J$ ,

$$\begin{aligned} \tilde{w}(-1) &= \tilde{h}(-1)\vartheta^{-1}J \\ &= \tilde{h}(-1)\tilde{h}(-1)^{-1}\vartheta J && \text{(by } \tilde{h}(-1) = \vartheta^2) \\ &= \vartheta J, \\ \tilde{w}(u)\tilde{w}(-u) &= \tilde{h}(u)\vartheta^{-1}\tilde{h}(-u)\vartheta^{-1}J \\ &= \tilde{h}(u)(\vartheta^{-1} \cdot \tilde{h}(-u))\vartheta^{-2}J \\ &= \tilde{h}(u)\tilde{h}(-u^{-1})\tilde{h}(-1)^{-1}J && \text{(by Proposition 2.1.5)} \\ &= c(u, -u^{-1})J && \text{(by the definition of } c(\cdot, \cdot)) \\ &= c(1, u^{-1})J && \text{(by (P5))} \\ &= J. \end{aligned}$$

Therefore,  $\tilde{w}(-1) = \vartheta$ ,  $\tilde{w}(u)^{-1} = \tilde{w}(-u)$ , and  $\tilde{h}(u) = \tilde{w}(u)\tilde{w}(-1)$  in  $\tilde{N}$ . Notice that there exists a group homomorphism  $\psi^*$  from  $N^* = \tilde{W} \ltimes \tilde{H}$  to  $N$  such that  $\psi^*(\vartheta) = w_{12}(-1)$  and  $\psi^*(\tilde{h}) = \pi(\tilde{h})$  for  $\tilde{h} \in \tilde{H}$  since it can easily be checked that the relations in  $\tilde{W}$  hold also in  $N$ . Since  $\psi^*(\vartheta^2\eta(\vartheta^2)^{-1}) = w_{12}(-1)^2h_{12}(-1)^{-1} = 1$ , we see that  $J \subset \text{Ker } \psi^*$ . Let  $\psi : \tilde{N} \rightarrow N$  be the induced group homomorphism,

which sends  $\tilde{w}(u)$  to  $w_{12}(u)$ . Then we see that  $\psi$  is surjective since  $w_{21}(u) = w_{12}(-u^{-1}) \in N$ , and that  $\psi$  is a central extension of  $N$  by  $L$  since  $\text{Ker } \psi \subset \tilde{H}$ . Summarizing these, we obtain the following proposition.

**Proposition 2.1.9.** The kernel  $\text{Ker } \psi$  of the group homomorphism  $\psi : \tilde{N} \rightarrow N$  is contained in the center of  $\tilde{N}$ , and is isomorphic to  $L$ . Namely,

$$1 \longrightarrow L \longrightarrow \tilde{N} \xrightarrow{\psi} N \longrightarrow 1$$

is a central extension of  $N$  by  $L$ . Moreover, the restriction of  $\psi$  to  $\tilde{H}$  coincides with the group homomorphism  $\pi : \tilde{H} \rightarrow T \subset N$  defined in Proposition 2.1.5.

Recall that  $w_1 = \vartheta J \in \tilde{N}$ .

**Lemma 2.1.10.** If  $h \in \tilde{H} \subset \tilde{N}$  is such that  $\psi(h) = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \in T$  with some  $u_1, u_2 \in D_\tau^\times$ , then

$$hw_1h^{-1} = \tilde{h}(u_1u_2^{-1})w_1.$$

*Proof.* We first show that

$$(L1) \quad w_1c(u, v)w_1^{-1} = \tilde{h}([u, v])^{-1}c(u, v),$$

$$(L2) \quad w_1c(u, v)w_1^{-1} = \tilde{h}([v, u])c(u, v).$$

We see by (T1) and (T1)' that

$$\begin{aligned} & \tilde{h}([u, v])w_1c(u, v)w_1^{-1} \\ &= \tilde{h}(uvu^{-1}v^{-1})\tilde{h}(u^{-1})\tilde{h}(v^{-1})\tilde{h}(u^{-1}v^{-1})^{-1} && \text{(by Proposition 2.1.8)} \\ &= \tilde{h}(u)\tilde{h}(v)\tilde{h}(u^{-1}v^{-1}u^{-1}v^{-1})\tilde{h}(u^{-1}v^{-1})^{-1} && \text{(by (T1)')} \\ &= c(u, v)\tilde{h}(vu)\tilde{h}(u^{-1}v^{-1}u^{-1}v^{-1})\tilde{h}(u^{-1}v^{-1})^{-1} \\ &= c(u, v). && \text{(by (T1))} \end{aligned}$$

In a way similar to (L1), we get (L2). These equalities and (T5) imply that

$$(L3) \quad \tilde{h}(s)^{-1} = \tilde{h}(s^{-1}) \quad \text{for } s \in [D_\tau^\times, D_\tau^\times],$$

$$(L4) \quad w_1\xi w_1^{-1} = \tilde{h}(\psi(\xi)^{-1})\xi \quad \text{for } \xi \in P.$$

As in the proof of Lemma 2.1.6, we write  $h \in \tilde{H}$  as  $h = \tilde{h}(v)\xi$  with some  $v \in D_\tau^\times$  and  $\xi \in P$ . If we set  $\psi(\xi) = \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}$  with  $s \in [D_\tau^\times, D_\tau^\times]$ , then  $\psi(h) = \begin{pmatrix} vs & 0 \\ 0 & v^{-1} \end{pmatrix}$ . Thus we compute

$$\begin{aligned} \tilde{h}w_1\tilde{h}^{-1} &= \tilde{h}(v)\xi w_1\xi^{-1}\tilde{h}(v)^{-1} \\ &= \tilde{h}(v)\tilde{h}(s^{-1})^{-1}w_1\tilde{h}(v)^{-1} && \text{(by (L4))} \\ &= \tilde{h}(v)\tilde{h}(s^{-1})^{-1}(w_1 \cdot \tilde{h}(v)^{-1})w_1 \\ &= \tilde{h}(v)\tilde{h}(s)\tilde{h}(u^{-1})^{-1}w_1 && \text{(by (L3) and Proposition 2.1.8)} \\ &= \tilde{h}(vsv)w_1. && \text{(by (T1))} \end{aligned}$$

□



We know the following lemma from [15, Lemma 5.10].

**Lemma 2.1.11.** Every matrix  $e \in E(2, D_\tau)$  can be written as  $e = uvv$  with some  $u, v \in U$  and  $w \in N$ . Moreover, the monomial matrix part  $w$  is uniquely determined by  $e$ ; we define  $\rho : E(2, D_\tau) \rightarrow N$  by  $\rho(e) = \rho(uvw) = w$ .

We remark that each  $w \in N$  is either of the following forms:

$$w = \underbrace{\begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}}_{\text{diagonal}} \quad \text{or} \quad \underbrace{\begin{pmatrix} 0 & u_2 \\ u_1 & 0 \end{pmatrix}}_{\text{anti-diagonal}} \quad (\spadesuit)$$

with some  $u_1, u_2 \in D_\tau^\times$ . Then we set  $d = d_w = \deg(u_1^{-1}u_2)$ . For  $\dot{\beta} = (\beta, m) \in \Delta_a$  with  $\beta = \epsilon_i - \epsilon_j \in \Delta$ ,  $m \in \mathbb{Z}$ , and  $f \in D$ , we have the following:

Case 1. If  $w$  is diagonal, then

$$w^{\pm 1} x_{\dot{\beta}}(f) w^{\mp 1} = \begin{cases} x_{(\beta, m \mp d)}(\tau^{m \mp d}(t^{\pm d} \tau^{-m}(u_1^{\pm 1} f) u_2^{\mp 1})) & \text{if } \beta \in \Delta^+, \\ x_{(\beta, m \mp d)}(\tau^{-m \pm d}(u_2^{\pm 1} \tau^m(f u_1^{\mp 1}) t^{\pm d})) & \text{if } \beta \in \Delta^-. \end{cases}$$

Case 2. If  $w$  is anti-diagonal, then

$$w^{\pm 1} x_{\dot{\beta}}(f) w^{\mp 1} = \begin{cases} x_{(-\beta, m-d)}(\tau^{-m+d}(u_{\pm}^{\pm 1} f \tau^m(u_{\mp}^{\mp 1}) t^d)) & \text{if } \beta \in \Delta^+, \\ x_{(-\beta, m-d)}(\tau^{m-d}(t^d \tau^{-m}(u_{\mp}^{\pm 1}) f u_{\pm}^{\mp 1})) & \text{if } \beta \in \Delta^-, \end{cases}$$

where  $u_+ = u_1$  and  $u_- = u_2$ .

In what follows, if  $w x_{\dot{\beta}}(f) w^{-1} = x_{\dot{\gamma}}(g)$  for suitable elements  $f, g \in D$ ,  $\dot{\beta}, \dot{\gamma} \in \Delta_a$ , and  $w \in N$ , then we write  $\dot{\gamma} = w(\dot{\beta})$ . Also, we know from Theorem 1.2.1 and Proposition 1.2.3 that every element  $e \in E(2, D_\tau)$  can be written in the form  $e = y x_{\dot{a}}(-f) w x_{\dot{b}}(g) z$  with  $w = \rho(e) \in N$  and some  $f, g \in D$ ,  $\dot{a}, \dot{b} \in \Pi_a$ ,  $y \in U'_{\dot{a}}$ ,  $z \in U'_{\dot{b}}$ .

**Lemma 2.1.12.** For  $e \in E(2, D_\tau)$  let  $\rho(e) = w$  be as in Lemma 2.1.11, and set  $e = y x_{\dot{a}}(-f) w x_{\dot{b}}(g) z$  with  $w = \rho(e)$  and  $f, g \in D$ ,  $\dot{a}, \dot{b} \in \Pi_a$ ,  $y \in U'_{\dot{a}}$ ,  $z \in U'_{\dot{b}}$ . Then the following hold.

Case 1 (for  $w_{\dot{a}}(1)e$ ).

If  $f = 0$  or  $w^{-1}(\dot{a}) \in \Delta_a^+$ , then  $\rho(w_{\dot{a}}(1)e) = w_{\dot{a}}(1)w$ .

If  $f \neq 0$  and  $w^{-1}(\dot{a}) \notin \Delta_a^+$ , then  $\rho(w_{\dot{a}}(1)e) = h_{\dot{a}}(1)h_{\dot{a}}(f)^{-1}w$ .

Case 2 (for  $ew_{\dot{b}}(-1)$ ).

If  $g = 0$  or  $w(\dot{b}) \in \Delta_a^+$ , then  $\rho(ew_{\dot{b}}(-1)) = w w_{\dot{b}}(-1)$ .

If  $g \neq 0$  and  $w(\dot{b}) \notin \Delta_a^+$ , then  $\rho(ew_{\dot{b}}(-1)) = w h_{\dot{b}}(g) h_{\dot{b}}(1)^{-1}$ .

*Proof.* It suffices to calculate

$$w_{\dot{a}}(1) x_{\dot{a}}(-f) w \quad \text{and} \quad w x_{\dot{b}}(g) w_{\dot{b}}(-1).$$

If  $f = 0$  or  $w^{-1}(\dot{a}) \in \Delta_a^+$ , then

$$w_{\dot{a}}(1)x_{\dot{a}}(-f)w = \underbrace{w_{\dot{a}}(1)w}_N \underbrace{x_{w^{-1}(\dot{a})}(k_1)}_U,$$

where

$$k_1 = \begin{cases} -u_1^{-1}fu_2t^{-d} & \text{if } \dot{a} = \alpha_1 \text{ and } w \text{ is diagonal,} \\ -t^du_2^{-1}fu_1 & \text{if } \dot{a} = \alpha_1 \text{ and } w \text{ is not diagonal,} \\ -t^{-d}\tau^{-1}(u_2^{-1})fu_1 & \text{if } \dot{a} = \alpha_0 \text{ and } w \text{ is diagonal,} \\ -u_1^{-1}\tau(fu_2)t^d & \text{if } \dot{a} = \alpha_0 \text{ and } w \text{ is not diagonal.} \end{cases}$$

If  $f \neq 0$  and  $w^{-1}(\dot{a}) \notin \Delta_a^+$ , then

$$w_{\dot{a}}(1)x_{\dot{a}}(-f)w = \underbrace{x_{\dot{a}}(f^{-1})}_U \underbrace{h_{\dot{a}}(1)h_{\dot{a}}(f)^{-1}w}_N \underbrace{x_{w^{-1}(-\dot{a})}(k_2)}_U,$$

where

$$k_2 = \begin{cases} -t^{-d}u_2^{-1}f^{-1}u_1 & \text{if } \dot{a} = \alpha_1 \text{ and } w \text{ is diagonal,} \\ -u_1^{-1}f^{-1}u_2t^d & \text{if } \dot{a} = \alpha_1 \text{ and } w \text{ is not diagonal,} \\ -u_1^{-1}f^{-1}\tau^{-1}(u_2)t^{-d} & \text{if } \dot{a} = \alpha_0 \text{ and } w \text{ is diagonal,} \\ -t^d\tau(u_2^{-1}f^{-1})u_1 & \text{if } \dot{a} = \alpha_0 \text{ and } w \text{ is not diagonal.} \end{cases}$$

If  $g = 0$  or  $w(\dot{b}) \in \Delta_a^+$ , then

$$wx_{\dot{b}}(g)w_{\dot{b}}(-1) = \underbrace{x_{w(\dot{b})}(k_3)}_U \underbrace{ww_{\dot{b}}(-1)}_N,$$

where

$$k_3 = \begin{cases} u_1gu_2^{-1}t^d & \text{if } \dot{b} = \alpha_1 \text{ and } w \text{ is diagonal,} \\ t^du_1gu_2^{-1} & \text{if } \dot{b} = \alpha_1 \text{ and } w \text{ is not diagonal,} \\ t^d\tau^{-1}(u_2)gu_1^{-1} & \text{if } \dot{b} = \alpha_0 \text{ and } w \text{ is diagonal,} \\ u_2\tau(gu_1^{-1})t^d & \text{if } \dot{b} = \alpha_0 \text{ and } w \text{ is not diagonal.} \end{cases}$$

If  $g \neq 0$  and  $w(\dot{b}) \notin \Delta_a^+$ , then

$$wx_{\dot{b}}(g)w_{\dot{b}}(-1) = \underbrace{x_{w(-\dot{b})}(k_4)}_U \underbrace{wh_{\dot{b}}(g)h_{\dot{b}}(1)^{-1}}_N \underbrace{x_{\dot{b}}(-g^{-1})}_U,$$

where

$$k_4 = \begin{cases} t^du_2g^{-1}u_1^{-1} & \text{if } \dot{b} = \alpha_1 \text{ and } w \text{ is diagonal,} \\ u_2g^{-1}u_1^{-1}t^d & \text{if } \dot{b} = \alpha_1 \text{ and } w \text{ is not diagonal,} \\ u_1g^{-1}\tau^{-1}(u_2^{-1})t^d & \text{if } \dot{b} = \alpha_0 \text{ and } w \text{ is diagonal,} \\ t^d\tau(u_1g^{-1})u_2^{-1} & \text{if } \dot{b} = \alpha_0 \text{ and } w \text{ is not diagonal.} \end{cases}$$

□

We put  $X = \{(e, \tilde{w}) \in E(2, D_\tau) \times \tilde{N} \mid \rho(e) = \psi(\tilde{w})\}$  (which is a set), and define permutations  $\lambda(h)$ ,  $\mu(u)$ ,  $\nu_{\dot{a}}$  (resp.  $\lambda(h)^*$ ,  $\mu(u)^*$ ,  $\nu_{\dot{a}}^*$ ) on  $X$  for  $h \in \tilde{H}$ ,  $u \in U$ , and  $\dot{a}, \dot{b} \in \Pi_a$  as follows (see Lemma 2.1.12):

$$\begin{aligned} \lambda(h)(e, \tilde{w}) &= (\psi(h)e, h\tilde{w}), \\ (e, \tilde{w})\lambda(h)^* &= (e\psi(h), \tilde{w}h), \\ \mu(u)(e, \tilde{w}) &= (ue, \tilde{w}), \\ (e, \tilde{w})\mu(u)^* &= (eu, \tilde{w}), \\ \nu_{\dot{a}}(e, \tilde{w}) &= \begin{cases} (w_{\dot{a}}(1)e, \tilde{w}_{\dot{a}}\tilde{w}) & \text{if } \rho(w_{\dot{a}}(1)e) = w_{\dot{a}}(1)w, \\ (w_{\dot{a}}(1)e, \tilde{h}_{\dot{a}}(f)^{-1}\tilde{w}) & \text{if } \rho(w_{\dot{a}}(1)e) = h_{\dot{a}}(1)h_{\dot{a}}(f)^{-1}w, \end{cases} \\ (e, \tilde{w})\nu_{\dot{b}}^* &= \begin{cases} (ew_{\dot{b}}(-1), \tilde{w}\tilde{w}_{\dot{b}}^{-1}) & \text{if } \rho(ew_{\dot{b}}(-1)) = ww_{\dot{b}}(-1), \\ (ew_{\dot{b}}(-1), \tilde{w}\tilde{h}_{\dot{b}}(g)) & \text{if } \rho(ew_{\dot{b}}(-1)) = wh_{\dot{b}}(g)h_{\dot{b}}(1)^{-1}, \end{cases} \end{aligned}$$

where

$$\tilde{w}_{\dot{a}} = \begin{cases} w_1 & \text{if } \dot{a} = \dot{\alpha}_1, \\ \tilde{h}(-t^{-1})w_1 & \text{if } \dot{a} = \dot{\alpha}_0, \end{cases} \quad \tilde{h}_{\dot{a}}(f) = \begin{cases} \tilde{h}(f) & \text{if } \dot{a} = \dot{\alpha}_1, \\ \tilde{h}(-f^{-1}t^{-1})\tilde{h}(-t^{-1})^{-1} & \text{if } \dot{a} = \dot{\alpha}_0. \end{cases}$$

*Remark 2.1.13.* For the actions of  $\lambda(h)$  (and  $\lambda(h)^*$ ), it is necessary to show that  $\rho(\psi(h)e) = \psi(h)w$ . For this, we need to take suitable  $\dot{a} \in \Pi_a$ ,  $y \in U'_a$  such that  $e = (yx_{\dot{a}}(-f))wv \in UNU$ , where  $yx_{\dot{a}}(-f) \in U$ .

Let  $G$  (resp.  $G^*$ ) be the group of permutations on  $X$  generated by  $\lambda(h)$ ,  $\mu(u)$ ,  $\nu_{\dot{a}}$  (resp.  $\lambda(h)^*$ ,  $\mu(u)^*$ ,  $\nu_{\dot{a}}^*$ ) for  $h \in \tilde{H}$ ,  $u \in U$ , and  $\dot{a} \in \Pi_a$ .

**Lemma 2.1.14.** For all  $(e, \tilde{w}) \in X$ ,  $g \in G$ , and  $g^* \in G^*$ , it holds that

$$(g(e, \tilde{w}))g^* = g((e, \tilde{w})g^*).$$

*Proof.* It suffices to show this equality for the generators of  $G$  and  $G^*$ . We can easily verify it, except for the case that  $g = \nu_{\dot{a}}$ ,  $g^* = \nu_{\dot{b}}^*$  for  $\dot{a}, \dot{b} \in \Pi_a$ . It can be easily checked that the first component of  $(\nu_{\dot{a}}(e, \tilde{w}))\nu_{\dot{b}}^*$  is equal to the one of  $\nu_{\dot{a}}((e, \tilde{w})\nu_{\dot{b}}^*)$ . Let us show that the second components of them are equal. We write  $e = yx_{\dot{a}}(-f)wx_{\dot{b}}(g)z$  with some  $f, g \in D$ ,  $w \in N$ ,  $y \in U'_a$ ,  $z \in U'_b$ , and  $\dot{a}, \dot{b} \in \Pi_a$ , as in Lemma 2.1.12. Recall that  $w$  is either of the forms in  $(\spadesuit)$ , that is,

$$w = \underbrace{\begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}}_{\text{diagonal}} \quad \text{or} \quad w = \underbrace{\begin{pmatrix} 0 & u_2 \\ u_1 & 0 \end{pmatrix}}_{\text{anti-diagonal}}$$

with some  $u_1, u_2 \in D_\tau^\times$ , and that  $d = d_w = \deg(u_1^{-1}u_2)$ . We denote the second component of  $(\nu_{\dot{a}}(e, \tilde{w}))\nu_{\dot{b}}^*$  (resp.  $\nu_{\dot{a}}((e, \tilde{w})\nu_{\dot{b}}^*)$ ) by  $(\nu_{\dot{a}}(\tilde{w}))\nu_{\dot{b}}^*$  (resp.  $\nu_{\dot{a}}((\tilde{w})\nu_{\dot{b}}^*)$ ) for convenience. We note that the following relation holds for all  $u \in D_\tau^\times$ :

$$(\star) \quad w_1\tilde{h}(u)w_1^{-1} = \tilde{h}(u^{-1}) = \tilde{h}(-u)^{-1}\tilde{h}(-1).$$

Case 1 Assume that  $w(\dot{b}) \neq \pm\dot{a}$ .

If we put  $\Gamma(\sigma_{\dot{c}}) = \{\dot{d} \in \Delta_a^+ \mid \sigma_{\dot{c}}(\dot{d}) \in \Delta_a^-\}$  for  $\dot{c} \in \Pi_a$ , then  $\Gamma(\sigma_{\dot{c}}) = \{\dot{c}\}$ . Therefore we obtain

$$(\nu_{\dot{a}}(\tilde{w}))\nu_b^* = \nu_{\dot{a}}((\tilde{w})\nu_b^*)$$

$$= \begin{cases} \tilde{w}_{\dot{a}}\tilde{w}\tilde{w}_b^{-1} & \text{if } "w^{-1}(\dot{a}) \in \Delta_a^+ \text{ or } f = 0" \text{ and } "w(\dot{b}) \in \Delta_a^+ \text{ or } g = 0", \\ \tilde{w}_{\dot{a}}\tilde{w}\tilde{h}_b(g) & \text{if } "w^{-1}(\dot{a}) \in \Delta_a^+ \text{ or } f = 0" \text{ and } "w(\dot{b}) \notin \Delta_a^+", \\ \tilde{h}_{\dot{a}}(f)^{-1}\tilde{w}\tilde{w}_b^{-1} & \text{if } "w^{-1}(\dot{a}) \notin \Delta_a^+" \text{ and } "w(\dot{b}) \in \Delta_a^+ \text{ or } g = 0", \\ \tilde{h}_{\dot{a}}(f)^{-1}\tilde{w}\tilde{h}_b(g) & \text{if } "w^{-1}(\dot{a}) \notin \Delta_a^+" \text{ and } "w(\dot{b}) \notin \Delta_a^+". \end{cases}$$

Case 2 Assume that  $w(\dot{b}) = \dot{a}$ .

We see that  $\dot{a} = \dot{b}$ ,  $d = 0$ , and  $w$  is diagonal. We know from [14, Lemma 4.7] that the claim holds when  $\dot{a} = \dot{\alpha}_1$ . It suffices to show the claim when  $\dot{a} = \dot{\alpha}_0$ . If  $f = 0$  or  $g = 0$ , then the claim follows from Lemmas 2.1.6 and 2.1.10. If  $f \neq 0$  and  $g \neq 0$ , then

$$w^{-1}x_{\dot{\alpha}_0}(-f)w = x_{w^{-1}(\dot{\alpha}_0)}(-\tau^{-1}(u_2^{-1})fu_1),$$

$$wx_{\dot{\alpha}_0}(g)w^{-1} = x_{w(\dot{\alpha}_0)}(\tau^{-1}(u_2)gu_1^{-1}).$$

We put  $s = -f + \tau^{-1}(u_2)gu_1^{-1}$  and  $s' = -\tau^{-1}(u_2^{-1})fu_1 + g$ . If  $s = 0$ , then  $s' = 0$ , and the claim is obvious. Assume that  $s \neq 0$ . We have to show that

$$\tilde{h}_{\dot{\alpha}_0}(-s)^{-1}\tilde{w}\tilde{w}_{\dot{\alpha}_0}^{-1} = \tilde{w}_{\dot{\alpha}_0}\tilde{w}\tilde{h}_{\dot{\alpha}_0}(s'),$$

which is equivalent to

$$\tilde{h}(s^{-1}t^{-1})\tilde{w}w_1 = w_1\tilde{w}\tilde{h}(-u_1^{-1}s^{-1}t^{-1}u_2).$$

We compute

$$\begin{aligned} w_1\tilde{w}\tilde{h}(-u_1^{-1}s^{-1}t^{-1}u_2) &= w_1\tilde{h}(-s^{-1}t^{-1})\tilde{h}(u_1u_2^{-1})^{-1}\tilde{w} && \text{(by Lemma 2.1.6)} \\ &= \tilde{h}(s^{-1}t^{-1})^{-1}\tilde{h}(-u_1u_2^{-1})w_1\tilde{w} && \text{(by } (\star) \text{)} \\ &= \tilde{h}(s^{-1}t^{-1})^{-1}\tilde{h}(-u_1u_2^{-1})\tilde{h}(u_1u_2^{-1})^{-1}\tilde{w}w_1 && \text{(by Lemma 2.1.10)} \\ &= \tilde{h}(-s^{-1}t^{-1})^{-1}\tilde{h}(-u_1u_2^{-1})\tilde{h}(-u_1u_2^{-1})^{-1}\tilde{w}w_1^{-1} && \text{(by } (\star) \text{)} \\ &= \tilde{h}(-s^{-1}t^{-1})^{-1}\tilde{w}w_1^{-1} \\ &= \tilde{h}(s^{-1}t^{-1})\tilde{w}w_1. && \text{(by } (\star) \text{)} \end{aligned}$$

Case 3 Assume that  $w(\dot{b}) = -\dot{a}$ .

We see that  $\dot{a} = \dot{b}$ ,  $d = -2$ , and  $w$  is not diagonal. We know from [14, Lemma 4.7] that the claim holds. It suffices to show the claim when  $\dot{a} = \dot{\alpha}_0$ . If  $f = 0$  or  $g = 0$ , then the claim follows from Lemmas 2.1.6 and 2.1.10. If  $f \neq 0$  and  $g \neq 0$ , then

$$w^{-1}x_{\dot{\alpha}_0}(-f)w = x_{w^{-1}(\dot{\alpha}_0)}(-\tau^{-1}(u_2^{-1})fu_1),$$

$$wx_{\dot{\alpha}_0}(g)w^{-1} = x_{w(\dot{\alpha}_0)}(\tau^{-1}(u_2)gu_1^{-1}).$$

Put  $s = -t^{-1}u_2^{-1}f^{-1}t^{-1}u_1 + g$  and  $s' = f - t^{-1}u_1g^{-1}t^{-1}u_2^{-1}$ . If  $s = 0$ , then  $s' = 0$ , and the claim follows from Case 2. Assume that  $s \neq 0$ . We have to show that

$$\tilde{h}_{\alpha_0}(f)^{-1}\tilde{w}\tilde{h}_{\alpha_0}(s) = \tilde{h}_{\alpha_0}(s')^{-1}\tilde{w}\tilde{h}_{\alpha_0}(g),$$

which is equivalent to

$$\tilde{h}(-f^{-1}t^{-1})^{-1}\tilde{w}\tilde{h}(-s^{-1}t^{-1}) = \tilde{h}(-s'^{-1}t^{-1})^{-1}\tilde{w}\tilde{h}(-g^{-1}t^{-1}).$$

We see that

$$\begin{aligned} \tilde{h}(-f^{-1}t^{-1})^{-1}\tilde{w}\tilde{h}(-s^{-1}t^{-1}) &= \tilde{h}(-s'^{-1}t^{-1})^{-1}\tilde{w}\tilde{h}(-g^{-1}t^{-1}) \\ \iff \tilde{w}\tilde{h}(-s^{-1})\tilde{h}(-g^{-1}t^{-1})^{-1}\tilde{w}^{-1} &= \tilde{h}(-f'^{-1}t^{-1})\tilde{h}(-s'^{-1}t^{-1})^{-1} \\ \iff (w_1\tilde{w})\tilde{h}(-s^{-1})\tilde{h}(-g^{-1}t^{-1})^{-1}(w_1\tilde{w})^{-1} &= w_1\tilde{h}(-f'^{-1}t^{-1})\tilde{h}(-s'^{-1}t^{-1})^{-1}w_1^{-1} \\ \iff \tilde{h}(u_1s^{-1}t^{-1}u_2^{-1})\tilde{h}(u_1g^{-1}t^{-1}u_2^{-1})^{-1} &= \tilde{h}(f^{-1}t^{-1})^{-1}\tilde{h}(s'^{-1}t^{-1}) \\ &\quad \text{(by Lemma 2.6 and } (\star)) \\ \iff \tilde{h}(f^{-1}t^{-1})\tilde{h}(u_1s^{-1}t^{-1}u_2^{-1}) &= \tilde{h}(s'^{-1}t^{-1})\tilde{h}(u_1g^{-1}t^{-1}u_2^{-1}) \\ \iff \tilde{h}(tf)\tilde{h}(u_2tsu_1^{-1}) &= \tilde{h}(ts')\tilde{h}(u_2tgu_1^{-1}) \quad \text{(by } (\star)) \end{aligned}$$

If we put  $x = u_2tgu_1^{-1}$  and  $y = tf$ , then the last equality is equivalent to

$$c(y, x - y^{-1}) = c(y - x^{-1}, x).$$

Since  $\deg(xy) = 0$ , we obtain

$$\begin{aligned} c(y, x - y^{-1}) &= c(y, (xy - 1)y^{-1}) \\ &= c(y(xy - 1), y^{-1}) \quad \text{(by (P2))} \\ &= c((y - x^{-1})xy, y^{-1}) \\ &= c(-(y - x^{-1})x, y^{-1}) \quad \text{(by (P5)')} \\ &= c(1 - yx, y^{-1}) \\ &= c(1 - yx, x) \quad \text{(by (P4))} \\ &= c(-(y - x^{-1})x, x) \\ &= c(y - x^{-1}, x). \quad \text{(by (P5)')} \end{aligned}$$

Thus we have proved Lemma 2.1.14.  $\square$

**Lemma 2.1.15.** The actions of  $G$  and  $G^*$  are simply transitive on  $X$ .

*Proof.* We give a proof only for  $G$  because the one for  $G^*$  is similar. We first show the transitivity. Let  $(e, \tilde{w}), (e', \tilde{w}') \in X$ . Since  $E(2, D_\tau)$  is generated by  $U$  and  $w_{12}(1)$ , there exists  $g_0 \in \langle \mu(u), \nu_{\dot{a}} \mid u \in U, \dot{a} \in \Pi_a \rangle \subset G$  such that  $g_0(e, \tilde{w}) = (e', \tilde{w}^*)$  for some  $\tilde{w}^*$  in  $N$ . Since both  $(e', \tilde{w}')$  and  $(e', \tilde{w}^*)$  are element of  $X$ , we see that  $\psi(\tilde{w}') = \rho(e') = \psi(\tilde{w}^*)$ . Since  $\text{Ker } \psi = L$  (see Proposition 2.1.9), it follows that  $\tilde{w}^* = l\tilde{w}$  for some  $l \in L$ . Then we have  $(e', \tilde{w}^*) = \lambda(l)(e', \tilde{w}')$ . Thus there exists  $g \in G$  such that  $g(e, \tilde{w}) = (e', \tilde{w}')$ .

Let  $g_1, g_2 \in G$ , and assume that  $g_1x = g_2x$  for some  $x \in X$ . Then we have  $g_1(xg^*) = g_2(xg^*)$  for any  $g^* \in G^*$  by Lemma 2.1.14. This implies that  $g_1x' = g_2x'$  for every  $x' \in X$  by the transitivity of  $G^*$ , which yields  $g_1 = g_2$ .  $\square$

**Theorem 2.1.16.** There exists a surjective group homomorphism  $\Psi : G \rightarrow E(2, D_\tau)$  which sends  $\lambda(h)$  to  $\psi(h)$  for  $h \in \tilde{H}$ ,  $\mu(u)$  to  $u$  for  $U$ , and  $\nu_{\dot{\alpha}}$  to  $w_{\dot{\alpha}}(1)$  for  $\dot{\alpha} \in \Pi_a$ . Moreover, the following exact sequence is a central extension of  $E(2, D_\tau)$ :

$$1 \longrightarrow L \longrightarrow G \xrightarrow{\Psi} E(2, D_\tau) \longrightarrow 1.$$

*Proof.* Recall that  $E(2, D_\tau)$  has a Bruhat decomposition;  $E(2, D_\tau) = UNU$ . The subgroup of  $G$  generated by  $\mu(u)$  for  $u \in U$  is isomorphic to  $U$ , and the subgroup generated by  $\lambda(h)$  and  $\nu_{\dot{\alpha}}$  for  $h \in \tilde{H}$ ,  $\dot{\alpha} \in \Pi$  is mapped onto  $N$  by Proposition 2.1.9 and Lemma 2.1.15. Therefore the first assertion is obvious. We show that  $\text{Ker } \Psi \cong L$ . Let  $(e, \tilde{w}) \in X$ . If  $g \in \text{Ker } \Psi$ , then  $g(e, \tilde{w}) = (e, \tilde{w}')$  for some  $\tilde{w}' \in \tilde{N}$ . Since  $\psi(\tilde{w}) = \rho(e) = \psi(\tilde{w}')$ , we have  $\tilde{w}' = l\tilde{w}$  for some  $l \in L$ , that is,  $g(e, \tilde{w}) = \lambda(l)(e, \tilde{w})$ . Thus we obtain  $g = \lambda(l)$  by Lemma 2.1.15. We see from Lemma 2.1.15 that  $\lambda(l)(e, \tilde{w}) = (e, \tilde{w}^*)$  for some  $\tilde{w}^* \in \tilde{N}$ . Therefore  $\lambda(l) \in \text{Ker } \Psi$ , and hence  $\text{Ker } \Psi \cong L$ . The centrality of  $\text{Ker } \Psi$  is easily checked by direct calculation: for  $(e, \tilde{w}) \in X$  and  $g \in G$ , if  $g(e, \tilde{w}) = (e', \tilde{w}')$ , then we see that  $\lambda(l)g(e, \tilde{w}) = (e', l\tilde{w}') = (e', \tilde{w}'l) = g\lambda(l)(e, \tilde{w})$ .  $\square$

**PROOF OF THEOREM 2.1.3.** By Theorems 1.4.1 and 2.1.16, we get the following diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & L & \longrightarrow & G & \xrightarrow{\Psi} & E(2, D_\tau) \longrightarrow 1 \quad (\text{exact}) \\ & & \downarrow \zeta & & \uparrow p & & \parallel \\ 1 & \longrightarrow & K_2(2, D_\tau) & \longrightarrow & St(2, D_\tau) & \xrightarrow{\phi} & E(2, D_\tau) \longrightarrow 1 \quad (\text{exact}) \end{array}$$

In a way similar to [13, Proposition 4.9], we see that there exists a unique homomorphism  $p : St(2, D_\tau) \rightarrow G$ , so that we find that  $p(K_2(2, D_\tau)) \subset L$  and  $p(\hat{c}(u, v)) = c(u, v)$  for all  $u, v \in D_\tau^\times$ . On the other hand, we know that  $\zeta$  sends  $c(u, v)$  to  $\hat{c}(u, v)$  for  $u, v \in D_\tau^\times$ . If we consider the generators of  $L$  and  $K_2(2, D_\tau)$ , then we obtain  $L \cong K_2(2, D_\tau)$ .  $\square$

**Corollary 2.1.17.** We have the following exact sequence:

$$1 \longrightarrow K_2(2, D_\tau) \longrightarrow P \xrightarrow{\varphi} D_\tau^\times \longrightarrow K_1(2, D_\tau) \longrightarrow 1.$$

## 2.2 Non-symplectic $K_2$ -groups

In this subsection, we assume that  $n \geq 3$ . Let  $Q$  be the group generated by  $c(u, v)$ ,  $u, v \in D_\tau^\times$ , with the defining relations that for  $u, v, w \in D_\tau^\times$  and  $s \in D^\times$  with  $s \neq 1$ ,

$$\begin{aligned} (\text{Q1}) \quad & c(uv, w) = c({}^u v, {}^u w) c(u, w), \\ (\text{Q2}) \quad & c(u, vw) = c(u, v) c({}^v u, {}^v w), \\ (\text{Q3}) \quad & c(s, 1-s) = 1, \end{aligned}$$

where  ${}^u v = uvu^{-1}$ , and  ${}^x c(u, v) = c({}^x u, {}^x v)$  for  $x, u, v \in D_\tau^\times$ . Because  $[u, v] \in [D_\tau^\times, D_\tau^\times]$ ,  $u, v \in D_\tau^\times$ , satisfy the same relations as (Q1)–(Q3) (with  $c(u, v)$  replaced by  $[u, v]$ ), there exists a (unique) surjective group homomorphism  $\varphi_0 : Q \rightarrow [D_\tau^\times, D_\tau^\times]$  which sends  $c(u, v)$  to  $[u, v]$  for  $u, v \in D_\tau^\times$ . Set  $L_0 = \text{Ker } \varphi_0$ ; note that

$$L_0 = \{c(u_1, v_1)^{p_1} c(u_2, v_2)^{p_2} \cdots c(u_r, v_r)^{p_r} \mid r \geq 0, p_i = \pm 1, u_i, v_i \in D_\tau^\times, [u_1, v_1]^{p_1} [u_2, v_2]^{p_2} \cdots [u_r, v_r]^{p_r} = 1\}. \quad (\#\#)$$

**Proposition 2.2.1.** The following is a central extension of  $[D_\tau^\times, D_\tau^\times]$  by  $L_0$ :

$$1 \longrightarrow L_0 \longrightarrow Q \xrightarrow{\varphi_0} [D_\tau^\times, D_\tau^\times] \longrightarrow 1.$$

*Proof.* First, we prove the following equality:

$$(Q4) \quad c(u, v)c(x, y) = {}^{[u, v]} c(x, y)c(u, v).$$

Let  $a, b, u, v \in D_\tau^\times$ . Using (Q1) and (Q2), we have

$$\begin{aligned} c(ua, vb) &= c({}^u a, {}^u vb)c(u, vb) \\ &= c({}^u a, {}^u b)c({}^{uv} a, {}^{uv} b)c(u, v)c({}^v u, {}^v b). \end{aligned}$$

Also, we have

$$\begin{aligned} c(ua, vb) &= c(ua, v)c({}^v ua, {}^v b) \\ &= c({}^u a, {}^u v)c(u, v)c({}^{vu} a, {}^{vu} b)c({}^v u, {}^v b). \end{aligned}$$

Therefore, we get  $c(u, v)c({}^{vu} a, {}^{vu} b) = c({}^{uv} a, {}^{uv} b)c(u, v)$ . Putting  $x = {}^{vu} a$  and  $y = {}^{vu} b$ , respectively, we see that  ${}^{uv} a = {}^{[u, v]} x$ ,  ${}^{uv} b = {}^{[u, v]} y$ . Hence (Q4) holds.

Applying (Q4) twice, we obtain that

$$c(u_2, v_2)c(u_1, v_1)c(x, y)c(u_1, v_1)^{-1}c(u_2, v_2)^{-1} = {}^{[u_2, v_2][u_1, v_1]} c(x, y).$$

Therefore, if we take  $\xi = c(u_1, v_1)^{p_1} c(u_2, v_2)^{p_2} \cdots c(u_r, v_r)^{p_r} \in Q$  with  $u_i, v_i \in D_\tau^\times$  and  $p_i \in \{\pm 1\}$ , then we see by induction on  $r$  that

$$\xi c(u, v) \xi^{-1} = {}^{\varphi_0(\xi)} c(u, v)$$

for all  $u, v \in D_\tau^\times$ . Thus  $\xi$  is central if  $\varphi_0(\xi) = 1$ .  $\square$

By comparing (Q1), (Q2), (Q3) with (TT6), (TT7), (TT3), respectively, we see that there exists a (unique) surjective group homomorphism  $\zeta_0 : Q \rightarrow K_2(n, D_\tau)$  which maps  $c(u, v)$  to  $\hat{c}(u, v)$  for  $u, v \in D_\tau^\times$ . By Proposition 2.2.1 and  $(\#\#)$ , the restriction of  $\zeta_0$  to  $L_0 \subset Q$  is a surjective group homomorphism from  $L_0$  onto  $K_2(n, D_\tau)$ .

**Theorem 2.2.2.** The group homomorphism  $\zeta_0 : L_0 \rightarrow K_2(n, D_\tau)$  is an isomorphism of groups.

Let  $\tilde{H}_0$  be the group generated by  $\tilde{h}_{ij}(u)$ ,  $u \in D_\tau^\times$ ,  $1 \leq i \neq j \leq n$ , and  $z(q)$ ,  $q \in Q$ , with the defining relations that for  $1 \leq i \neq j \leq n$ ,  $1 \leq k \neq l \leq n$ ,  $u, v \in D_\tau^\times$ , and  $q, q' \in Q$ ,

$$\begin{aligned}
(\text{HH1}) \quad & \tilde{h}_{ij}(u)\tilde{h}_{ji}(u) = 1, \\
(\text{HH2}) \quad & \tilde{h}_{ij}(u)\tilde{h}_{ki}(u)\tilde{h}_{jk}(u) = 1, \\
(\text{HH3}) \quad & \tilde{h}_{ij}(u)\tilde{h}_{ik}(v)\tilde{h}_{ij}(u)^{-1} = \tilde{h}_{ik}(uv)\tilde{h}_{ik}(u)^{-1} \quad (k \neq j), \\
(\text{HH4}) \quad & \tilde{h}_{ij}(u)\tilde{h}_{kj}(v)\tilde{h}_{ij}(u)^{-1} = \tilde{h}_{kj}(vu)\tilde{h}_{kj}(u)^{-1} \quad (k \neq i), \\
(\text{HH5}) \quad & [\tilde{h}_{ij}(u), \tilde{h}_{kl}(v)] = 1 \quad (k \neq i, j \text{ and } l \neq i, j), \\
(\text{HH6}) \quad & z(q)z(q') = z(qq'), \\
(\text{HH7}) \quad & \tilde{h}_{ij}(u)\tilde{h}_{ij}(v)\tilde{h}_{ij}(vu)^{-1} = z(c(u, v)), \\
(\text{HH8}) \quad & \tilde{h}_{ij}(u)\tilde{h}_{ik}(v)\tilde{h}_{ij}(u)^{-1}\tilde{h}_{ik}(v)^{-1} = z(c(u, v)), \\
(\text{HH9}) \quad & \tilde{h}_{ij}(u)z(c(u, v)) = z({}^x c(u, v))\tilde{h}_{ij}(u).
\end{aligned}$$

We deduce by (HH6) and (HH9) that  $\{z(q) \mid q \in Q\}$  is a normal subgroup of  $\tilde{H}_0$ . Moreover, we see by a way similar to [7, CHAPITRE II] and [10, Proposition 2] that  $L_0$  is isomorphic to  $\{z(l) \mid l \in L_0\}$ . We identify  $\{z(l) \mid l \in L_0\}$  with  $L_0$ , and write  $z(l)$  simply by  $l$  for  $l \in L_0$ .

We can easily check the following proposition.

**Proposition 2.2.3.** The same equalities as those in Lemmas 1.4.5 and 1.4.6 with  $\hat{h}_{ij}(u)$  replaced by  $\tilde{h}_{ij}(u)$ , and with  $\hat{c}_{ij}(u, v)$  replaced by  $z(c(u, v))$  hold in  $\tilde{H}_0$ .

**Proposition 2.2.4.** There exists a (unique) surjective group homomorphism  $\pi_0 : \tilde{H}_0 \twoheadrightarrow T$  which sends  $\tilde{h}_{ij}(u)$  to  $h_{ij}(u)$  for  $u \in D_\tau^\times$ , and  $z(c(v, w))$  to  $h_{12}(v)h_{12}(w)h_{12}(vw)^{-1}$  for  $v, w \in D_\tau^\times$ . The kernel  $\text{Ker } \pi_0$  of  $\pi_0$  is identical to  $L_0$ . Moreover,

$$1 \longrightarrow L_0 \longrightarrow \tilde{H}_0 \xrightarrow{\pi_0} T \longrightarrow 1$$

is a central extension of  $T$  by  $L_0$ .

*Proof.* We first note that the following relations hold by (HH7) and (HH8):

$$\begin{aligned}
\tilde{h}_{ij}(u)\tilde{h}_{ij}(v) &\equiv \tilde{h}_{ij}(vu) \quad \text{mod } z(Q), \\
\tilde{h}_{ij}(u)\tilde{h}_{ik}(v) &\equiv \tilde{h}_{ik}(v)\tilde{h}_{ij}(u) \quad \text{mod } z(Q),
\end{aligned}$$

where  $z(Q) = \{z(q) \mid q \in Q\}$ . Therefore, if we take  $\tilde{h} \in \text{Ker } \pi_0$ , then  $\tilde{h}$  is expressed as

$$\tilde{h} \equiv \tilde{h}_{12}(u_1)\tilde{h}_{13}(u_2) \cdots \tilde{h}_{1n}(u_{n-1}) \quad \text{mod } z(Q)$$

with some  $u_1, \dots, u_{n-1} \in D_\tau^\times$ . Applying  $\pi_0$  to  $\tilde{h}$ , we have

$$\pi(\tilde{h}) = \text{diag}(u_1 \cdots u_{n-1}s, u_1^{-1}, \dots, u_{n-1}^{-1})$$



for some  $s \in [D_\tau^\times, D_\tau^\times]$ , but this implies that  $u_i = 1$  for all  $1 \leq i \leq n-1$ . Since  $\tilde{h}_{ij}(1) = 1$  for all  $1 \leq i \neq j \leq n$  by (HH3), we obtain  $\tilde{h} \equiv 1 \pmod{z(Q)}$ . Thus  $\tilde{h} \in \text{Ker } \pi$  is of the form

$$\tilde{h} = z(c(u_1, v_1)c(u_2, v_2) \cdots c(u_r, v_r)) \in z(Q)$$

with some  $r \geq 0$ ,  $u_i, v_i \in D_\tau^\times$ . Therefore we see that  $\tilde{h} \in \text{Ker } \pi_0 \subset L_0$ . It is obvious that  $L_0 \subset \text{Ker } \pi_0$ .  $\square$

By Propositions 2.2.1 and 2.2.4, we get the following commutative diagram:

$$\begin{array}{ccccc} L_0 & \longrightarrow & Q & \xrightarrow{\varphi_0} & [D_\tau^\times, D_\tau^\times] \\ \parallel & & \downarrow z & & \downarrow d \\ L_0 & \longrightarrow & \tilde{H}_0 & \xrightarrow{\pi_0} & T, \end{array} \quad (\text{CD2})$$

where  $d$  is the embedding of groups defined by  $d([u, v]) = \text{diag}([u, v], 1, \dots, 1)$ . This implies that  $Q$  is isomorphic to  $\{z(q) \mid q \in Q\}$ . We identify  $\{z(q) \mid q \in Q\}$  with  $Q$ , and write  $z(q)$  simply by  $q$  for  $q \in Q$ .

Next, we construct the extension of the monomial subgroup  $N$ , which is “compatible” with the extension  $(\tilde{H}_0, \pi_0)$  of  $T$  in Proposition 2.2.4 (see Proposition 2.2.7 below). For this, we give a presentation of  $N$ , and then define an action of  $N$  on  $\tilde{H}_0$ . The next lemma and proposition follow from [10, Proposition 3] and [15, Proposition 5.8].

**Lemma 2.2.5.** The subgroup  $N$  of  $E(n, D_\tau)$  is the group generated by  $w_\alpha(u)$  for  $u \in D_\tau^\times$  and  $\alpha \in \Pi$  with the defining relations that for  $\alpha = \epsilon_i - \epsilon_j, \beta = \epsilon_k - \epsilon_l \in \Pi$ , and  $u, v \in D_\tau^\times$ ,

- (N1)  $w_\alpha(u)^{-1} = w_\alpha(-u)$ ,
- (N2)  $w_\alpha(1)h_\beta(u)w_\alpha(1)^{-1} = h_\beta(u)h_\alpha(u^{-\langle \alpha, \beta \rangle})$ ,
- (N3)  $h_\alpha(u)h_\alpha(v) = h_\alpha(uvu)h_\alpha(u^{-1})$ ,
- (N4)  $h_\alpha(u)h_\beta(v)h_\alpha(u)^{-1} = \begin{cases} h_\beta(u^{-1}v)h_\beta(u) & \text{if } j = k, \\ h_\beta(vu^{-1})h_\beta(u) & \text{if } i = l, \\ h_\beta(v) & \text{if } \alpha + \beta \notin \Delta, \end{cases}$
- (N5)  $w_\alpha(1)w_\beta(1)w_\alpha(1) = w_\beta(1)w_\alpha(1)w_\beta(1) \quad (\langle \alpha, \beta \rangle = -1)$ ,
- (N6)  $w_\alpha(1)w_\beta(1) = w_\beta(1)w_\alpha(1) \quad (\langle \alpha, \beta \rangle = 0)$ .

**Proposition 2.2.6.** There exists an action of  $N$  on  $\tilde{H}_0$  defined by

$$w_{ij}(u) \cdot \tilde{h}_{kl}(v) = \begin{cases} \tilde{h}_{kl}(v) & \text{if } i \neq k, l \text{ and } j \neq k, l, \\ \tilde{h}_{ji}(u^{-1}vu^{-1})\tilde{h}_{ji}(u^{-2})^{-1} & \text{if } i = k \text{ and } j = l, \\ \tilde{h}_{ij}(uvu)\tilde{h}_{ji}(u^2)^{-1} & \text{if } i = l \text{ and } j = k, \\ \tilde{h}_{jl}(u^{-1}v)\tilde{h}_{jl}(u^{-1})^{-1} & \text{if } i = k \text{ and } j \neq l, \\ \tilde{h}_{kj}(vu)\tilde{h}_{kj}(u)^{-1} & \text{if } i = l \text{ and } j \neq k, \\ \tilde{h}_{il}(uv)\tilde{h}_{il}(u)^{-1} & \text{if } i \neq l \text{ and } j = k, \\ \tilde{h}_{ki}(vu^{-1})\tilde{h}_{ki}(u^{-1})^{-1} & \text{if } i \neq k \text{ and } j = l \end{cases}$$

for  $1 \leq i \neq j \leq n$ ,  $1 \leq k \neq l \leq n$ , and  $u, v \in D_\tau^\times$ . We denote  $w_{ij}(u) \cdot \tilde{h}_{kl}(v)$  also by  $w_{ij}(u)\tilde{h}_{kl}(v)w_{ij}(u)^{-1}$  for convenience.

Let  $\tilde{W}_0$  be the group generated by  $\tilde{w}_\alpha$  for all  $\alpha \in \Pi$  with the following defining relations that

$$\begin{aligned} \text{(W1)} \quad & \tilde{h}_\alpha \tilde{w}_\beta \tilde{h}_\alpha^{-1} = \tilde{w}_\beta^\delta, \\ \text{(W2)} \quad & \tilde{w}_\alpha \tilde{w}_\gamma \tilde{w}_\alpha = \tilde{w}_\gamma \tilde{w}_\alpha \tilde{w}_\beta \quad \text{if } \langle \gamma, \alpha \rangle = -1, \\ \text{(W3)} \quad & \tilde{w}_\alpha \tilde{w}_\gamma = \tilde{w}_\gamma \tilde{w}_\alpha \quad \text{if } \langle \gamma, \alpha \rangle = 0, \end{aligned}$$

where  $\alpha \neq \pm\gamma$ ,  $\delta = (-1)^{\langle \alpha, \beta \rangle}$ , and  $\tilde{h}_\alpha = \tilde{w}_\alpha^2$ . Put  $\tilde{T} = \langle \tilde{h}_\alpha \mid \alpha \in \Pi \rangle$  and  $\tilde{N}_0^* = \tilde{H}_0 \rtimes \tilde{W}_0$ , where  $\tilde{W}_0$  acts on  $\tilde{H}_0$  by  $\tilde{w}_\alpha \cdot \tilde{h} = w_\alpha(-1) \cdot \tilde{h}$  for  $\alpha \in \Pi$  and  $\tilde{h} \in \tilde{H}_0$ . Then we see by Proposition 2.2.6 that  $\tilde{T}$  is the group generated by  $\tilde{h}_\alpha$  for all  $\alpha \in \Pi$  with the defining relation: (T)  $\tilde{h}_\alpha \tilde{h}_\beta \tilde{h}_\alpha^{-1} = \tilde{h}_\beta^\delta$ . Then there is the canonical homomorphism  $\eta : \tilde{T} \rightarrow \tilde{H}_0$  defined by  $\eta(\tilde{h}_\alpha) = \tilde{h}_\alpha(-1)$ . Let  $J_0$  be the subgroup of  $\tilde{N}_0^*$  generated by  $\tilde{t}\eta(\tilde{t})^{-1}$  for all  $\tilde{t} \in \tilde{T}$ , and let  $\tilde{N}_0 = \tilde{N}_0^*/J_0$  be the quotient group. Let  $\tilde{\psi}_0 : \tilde{N}_0^* \rightarrow \tilde{N}_0$  be the canonical homomorphism. We deduce that the restriction of  $\tilde{\psi}_0$  to  $\tilde{H}_0$  is injective; we regard  $\tilde{H}_0$  as a subgroup of  $\tilde{N}_0$ . If we set  $\tilde{w}_\alpha(u) = \tilde{h}_\alpha(u)\tilde{w}_\alpha^{-1}J_0 \in \tilde{N}_0$  for  $u \in D_\tau^\times$ , then we see, in exactly the same way as in the case of  $n = 2$  (see Subsection 2.1), that  $\tilde{w}_\alpha(-1) = \tilde{w}_\alpha J_0$ ,  $\tilde{w}_\alpha(u)^{-1} = \tilde{w}_\alpha(-u)$ , and  $\tilde{h}_\alpha(u) = \tilde{w}_\alpha(u)\tilde{w}_\alpha(-1)$  in  $\tilde{N}$ . Notice that there exists a group homomorphism  $\psi_0^*$  from  $\tilde{N}_0^*$  to  $N$  such that  $\psi_0^*(\tilde{w}_\alpha) = w_\alpha(-1)$  for  $\alpha \in \Pi$  and  $\psi_0^*(\tilde{h}) = \pi_0(\tilde{h})$  for  $\tilde{h} \in \tilde{H}_0$  since it can easily be checked that the relations in  $\tilde{W}_0$  hold in  $N$ . We see that  $J_0 \subset \text{Ker } \psi_0^*$ , and hence  $\psi_0^*$  induces a homomorphism  $\psi_0 : \tilde{N}_0 \rightarrow N$  which sends  $\tilde{w}_\alpha(u)$  to  $w_\alpha(u)$  for  $u \in D_\tau^\times$  and  $\alpha \in \Pi$ . It can be easily checked that  $\psi_0$  is surjective by (R6), and that  $\psi_0$  is a central extension of  $N$  by  $L$  as in the case of  $n = 2$  (see Subsection 2.1). Thus, we obtain the following proposition.

**Proposition 2.2.7.** The kernel  $\text{Ker } \psi_0$  of the group homomorphism  $\psi_0 : \tilde{N}_0 \rightarrow N$  is contained in the center of  $\tilde{N}_0$ , and is isomorphic to  $L_0$ . Namely,

$$1 \longrightarrow L_0 \longrightarrow \tilde{N}_0 \xrightarrow{\psi_0} N \longrightarrow 1$$

is a central extension of  $N$  by  $L_0$ . Moreover, the restriction of  $\psi_0$  to  $\tilde{H}_0$  coincides with the group homomorphism  $\pi_0 : \tilde{H}_0 \rightarrow T \subset N$  defined in Proposition 2.2.4.

**Lemma 2.2.8.** Every matrix  $e \in E(n, D_\tau)$  can be written as  $e = u w v$  with some  $u, v \in U$  and  $w \in N$ . Moreover, the monomial matrix part  $w$  is uniquely determined by  $e$ ; we define  $\rho : E(n, D_\tau) \rightarrow N$  by  $\rho(e) = \rho(u w v) = w$ .

For each  $w \in N$ , there exists a permutation  $\sigma$  of  $\{1, \dots, n\}$  such that  $w = P_\sigma \text{diag}(u_1, \dots, u_n)$  with suitable  $u_i \in D_\tau^\times$ , where  $P_\sigma$  is the permutation matrix corresponding to  $\sigma$ . In what follows, we write  $\dot{\beta} = (ij, m)$  for  $\dot{\beta} = (\beta, m) \in \Delta_a$  with  $\beta = \epsilon_i - \epsilon_j \in \Delta$ . Then, we see by (R3) and (R4) that for  $\dot{\beta} = (\beta, m)$  with  $\beta = \epsilon_i - \epsilon_j$ ,

$$\begin{aligned} w^{\pm 1} x_{\dot{\beta}}(f) w^{\mp 1} &= w^{\pm 1} x_{(ij, m)}(f) w^{\mp 1} \\ &= \begin{cases} x_{(\sigma(ij), m \mp d)}(\tau^{m \mp d} (t^{\pm d} \tau^{-m} (u_i^{\pm 1} f) u_j^{\mp 1})) & \text{if } \beta \in \Delta^+, \\ x_{(\sigma^{-1}(ij), m \mp d')}(\tau^{-m \pm d'} (u_{\sigma^{-1}(i)}^{\pm 1} \tau^m (f u_{\sigma^{-1}(j)}^{\mp 1}) t^{\pm d'})) & \text{if } \beta \in \Delta^-, \end{cases} \end{aligned}$$

where  $d = \deg(u_i u_j^{-1})$  and  $d' = \deg(u_{\sigma^{-1}(i)} u_{\sigma^{-1}(j)}^{-1})$ . In what follows, if  $w x_{\dot{\beta}}(f) w^{-1} = x_{\dot{\gamma}}(g)$  for suitable elements  $f, g \in D$  and  $\dot{\beta}, \dot{\gamma} \in \Delta_a$ , then we denote  $\dot{\gamma}$  by  $w(\dot{\beta})$ . We see from Theorem 1.2.1 and Proposition 1.2.3 that every element  $e \in E(n, D_\tau)$  can be written in the form  $e = y x_{\dot{a}}(-f) w x_{\dot{b}}(g) z$  with some  $f, g \in D$ ,  $\dot{a}, \dot{b} \in \Pi_a$ ,  $y \in U'_a$ , and  $z \in U'_b$ . Using the equality above and Lemma 2.1.12, we get the following lemma.

**Lemma 2.2.9.** For  $e \in E(n, D_\tau)$  let  $\rho(e) = w$  be as in Lemma 2.2.8, and set  $e = y x_{\dot{a}}(-f) w x_{\dot{b}}(g) z$  for  $f, g \in D$ ,  $\dot{a}, \dot{b} \in \Pi_a$ ,  $y \in Y_a$  and  $z \in Y_b$ . Then the following holds:

Case 1 (for  $w_{\dot{a}}(1)e$ ).

If  $f = 0$  or  $w^{-1}(\dot{a}) \in \Delta_a^+$ , then  $\rho(w_{\dot{a}}(1)e) = w_{\dot{a}}(1)w$ .

If  $f \neq 0$  and  $w^{-1}(\dot{a}) \notin \Delta_a^+$ , then  $\rho(w_{\dot{a}}(1)e) = h_{\dot{a}}(1)h_{\dot{a}}(f)^{-1}w$ .

Case 2 (for  $ew_{\dot{b}}(-1)$ ).

If  $g = 0$  or  $w(\dot{b}) \in \Delta_a^+$ , then  $\rho(ew_{\dot{b}}(-1)) = ww_{\dot{b}}(-1)$ .

If  $g \neq 0$  and  $w(\dot{b}) \notin \Delta_a^+$ , then  $\rho(ew_{\dot{b}}(-1)) = wh_{\dot{b}}(g)h_{\dot{b}}(1)^{-1}$ .

We put  $X_0 = \{(e, \tilde{w}) \in E(n, D_\tau) \times \tilde{N}_0 \mid \rho(e) = \psi_0(\tilde{w})\}$  and define permutations  $\lambda(h)$ ,  $\mu(u)$ ,  $\nu_{\dot{a}}$  (resp.  $\lambda(h)^*$ ,  $\mu(u)^*$ ,  $\nu_{\dot{a}}^*$ ) for  $\tilde{h} \in \tilde{H}_0$ ,  $u \in U$ , and  $\dot{a} \in \Pi_a$  as follows (see Subsection 2.1):

$$\begin{aligned} \lambda(h)(e, \tilde{w}) &= (\psi_0(h)e, h\tilde{w}), \\ (e, \tilde{w})\lambda(h)^* &= (e\psi_0(h), \tilde{w}h), \\ \mu(u)(e, \tilde{w}) &= (ue, \tilde{w}), \\ (e, \tilde{w})\mu(u)^* &= (eu, \tilde{w}), \\ \nu_{\dot{a}}(e, \tilde{w}) &= \begin{cases} (w_{\dot{a}}(1)e, \tilde{w}_{\dot{a}}\tilde{w}) & \text{if } \rho(w_{\dot{a}}(1)e) = w_{\dot{a}}(1)w, \\ (w_{\dot{a}}(1)e, \tilde{h}_{\dot{a}}(f)^{-1}\tilde{w}) & \text{if } \rho(w_{\dot{a}}(1)e) = h_{\dot{a}}(1)h_{\dot{a}}(f)^{-1}w, \end{cases} \\ (e, \tilde{w})\nu_{\dot{b}}^* &= \begin{cases} (ew_{\dot{b}}(-1), \tilde{w}\tilde{w}_{\dot{b}}^{-1}) & \text{if } \rho(ew_{\dot{b}}(-1)) = ww_{\dot{b}}(-1), \\ (ew_{\dot{b}}(-1), \tilde{w}\tilde{h}_{\dot{b}}(g)) & \text{if } \rho(ew_{\dot{b}}(-1)) = wh_{\dot{b}}(g)h_{\dot{b}}(1)^{-1}, \end{cases} \end{aligned}$$

where

$$\begin{aligned}\tilde{w}_{\dot{a}} &= \begin{cases} \tilde{w}_{\alpha}(1) & \text{if } \dot{a} = (\alpha, 0) \in \Pi, \\ \tilde{h}_{-\theta}(-t^{-1})\tilde{w}_{-\theta}(1) & \text{if } \dot{a} = \dot{\alpha}_0, \end{cases} \\ \tilde{h}_{\dot{a}}(f) &= \begin{cases} \tilde{h}_{\alpha}(f) & \text{if } \dot{a} = (\alpha, 0) \in \Pi, \\ \tilde{h}_{-\theta}(-f^{-1}t^{-1})\tilde{h}_{-\theta}(-t^{-1})^{-1} & \text{if } \dot{a} = \dot{\alpha}_0. \end{cases}\end{aligned}$$

From [13, Lemma 4.6], Lemmas 2.1.6 and 2.1.10, we get the following relations.

**Lemma 2.2.10.** Let  $v \in D_{\tau}^{\times}$ , and  $\tilde{w} \in \tilde{N}_0$ . If  $\psi_0(\tilde{w}) = P_{\sigma} \text{diag}(u_1, \dots, u_n)$  with some  $u_i \in D_{\tau}^{\times}$  and permutation  $\sigma$  on  $\{1, \dots, n\}$ , then we obtain

$$\begin{aligned}(1) \quad & \tilde{w}\tilde{h}_{ij}(v)\tilde{w}^{-1} = \tilde{h}_{\sigma(ij)}(u_i v u_j^{-1})\tilde{h}_{ij}(u_i u_j^{-1}), \\ (2) \quad & \tilde{w}\tilde{w}_{ij}(1)\tilde{w}^{-1} = \tilde{h}_{\sigma(ij)}(u_i u_j^{-1})\tilde{w}_{\sigma(ij)}(1).\end{aligned}$$

Let  $G_0$  (resp.  $G_0^*$ ) be the group of permutations on  $X_0$  generated by  $\lambda(h)$ ,  $\mu(u)$ ,  $\nu_{\dot{a}}$  (resp.  $\lambda(h)^*$ ,  $\mu(u)^*$ ,  $\nu_{\dot{a}}^*$ ) for  $h \in \tilde{H}_0$ ,  $u \in U$ , and  $\dot{a} \in \Pi_a$ . We obtain the following as Lemmas 2.1.14, 2.1.15, and Theorem 2.1.16, respectively.

**Lemma 2.2.11.** For all  $(e, \tilde{w}) \in X_0$ ,  $g \in G_0$  and  $g^* \in G_0^*$ , it holds that

$$(g(e, \tilde{w}))g^* = g((e, \tilde{w})g^*).$$

**Lemma 2.2.12.** The actions of  $G_0$  and  $G_0^*$  are simply transitive on  $X_0$ .

**Theorem 2.2.13.** There exists a surjective group homomorphism  $\Psi_0 : G_0 \rightarrow E(n, D_{\tau})$  which sends  $\lambda(h)$  to  $\psi_0(h)$  for  $h \in \tilde{H}_0$ ,  $\mu(u)$  to  $u$  for  $U$ , and  $\nu_{\dot{\alpha}}$  to  $w_{\dot{\alpha}}(1)$  for  $\dot{\alpha} \in \Pi_a$ . Moreover, the homomorphism  $\Psi_0$  satisfies that the following exact sequence is a central extension of  $E(n, D_{\tau})$ :

$$1 \longrightarrow L_0 \longrightarrow G_0 \xrightarrow{\Psi_0} E(n, D_{\tau}) \longrightarrow 1.$$

*PROOF OF THEOREM 2.2.2.* By Theorems 1.4.1 and 2.2.13, we get the following diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & L_0 & \longrightarrow & G_0 & \xrightarrow{\Psi_0} & E(n, D_{\tau}) \longrightarrow 1 \quad (\text{exact}) \\ & & \downarrow \zeta_0 & & \uparrow p_0 & & \parallel \\ 1 & \longrightarrow & K_2(n, D_{\tau}) & \longrightarrow & St(n, D_{\tau}) & \xrightarrow{\phi} & E(n, D_{\tau}) \longrightarrow 1 \quad (\text{exact}) \end{array}$$

In a way similar to [13, Proposition 4.9], we see that there exists a unique homomorphism  $p_0 : St(n, D_{\tau}) \rightarrow G_0$ , so that we find that  $p_0(K_2(n, D_{\tau})) \subset L_0$  and  $p_0(\hat{c}(u, v)) = c(u, v)$  for all  $u, v \in D_{\tau}^{\times}$ . On the other hand, we know  $\zeta_0$  sends  $c(u, v)$  to  $\hat{c}(u, v)$  for  $u, v \in D_{\tau}^{\times}$ . If we consider the generators of  $L_0$  and  $K_2(n, D_{\tau})$ , then we obtain  $L_0 \cong K_2(n, D_{\tau})$ . □

**Corollary 2.2.14.** We have the following exact sequence:

$$1 \longrightarrow K_2(n, D_\tau) \longrightarrow Q \xrightarrow{\varphi_0} D_\tau^\times \longrightarrow K_1(n, D_\tau) \longrightarrow 1.$$

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