Combinatorics related to Weyl groups, Young diagrams, and some special weights

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1 Introduction.

This thesis consists of three parts.

- Part I (Sections 2 – 8 and Appendix A). Strong minuscule elements in the finite Weyl groups.
- Part II (Sections 9 – 13). Game positions of Multiple Hook Removing Game.
- Part III (Sections 14 – 18 and Appendix B). Infinite pre-dominant integral weights and generalizations of MacMahon's identity.

1.1 Introduction to Part I : Strong minuscule elements in the finite Weyl groups.

The notion of (dominant) minuscule elements in a Weyl group was introduced by D. Peterson in order to study the number of reduced expressions for an element in the Weyl group; for the definition of (dominant) minuscule elements, see Definition 3.1 below. In this part, we study the following special class of dominant minuscule elements in the Weyl group for a finite-dimensional simple Lie algebra $\mathfrak{g}$; a dominant minuscule element w in the Weyl group of $\mathfrak{g}$ is called a strong minuscule element if there exists a unique dominant integral weight Λ (which we denote by Λ_w) such that w is Λ-minuscule. We denote by $\mathcal{SM}$ the set of strong minuscule elements. The following (Theorems 1.1, 1.2, and 1.3) are the main results in this part, which are stated also in Section 5 (for $\mathfrak{g}$ of classical type) and Appendix A (for $\mathfrak{g}$ of exceptional type); we prove all of these statements in the case that $\mathfrak{g}$ is of exceptional type by using computer programs due to Kawai and Tada [4]. Let $\{\alpha_i\}_{i\in I}$ be the set of simple roots for $\mathfrak{g}$; in this part, all simple roots in the simply-laced cases are treated as short roots.

Theorem 1.1. It holds that

$$\mathcal{SM} = \bigsqcup_{i\in K} \mathcal{SM}_i,$$

where $K := \{i \in I \mid \alpha_i \text{ is short}\}$, and $\mathcal{SM}_i = \{w \in \mathcal{SM} \mid \Lambda_w = \Lambda_i\}$ for $i \in K$.

In order to prove Theorem 1.1, we introduce a special element $v_i \in W$ for each $i \in K$ in Definition 6.3 (for $\mathfrak{g}$ of classical type) and Table 1 in Appendix A (for $\mathfrak{g}$ of exceptional type). Using this element, we show (in Lemma 6.4) that if $w \in \mathcal{SM}$, and $w = s_{i_1} \cdots s_{i_r}$ is a reduced expression of w, then $w \in \mathcal{SM}_{i_r}$. In fact, Theorem 1.1 follows immediately from this fact.

Now, fix $i \in K$. Let $J_i := \{s_j\}_{j\in I} \setminus \{s_i\}$, where $s_j \in W$ is the simple reflection in the simple root α_j for $j \in I$, and W_{J_i} the parabolic subgroup of W generated by J_i. Let W^{J_i} be the set of minimal-length coset representatives for cosets in W / W_{J_i}. In Proposition 7.3, we show that $\mathcal{SM}_i \subset W^{J_i}$ for $i \in K$. Then, by using the description of W^{J_i} due to Stumbo [21], we prove the following theorem.

Theorem 1.2. Assume that $\mathfrak{g}$ is of classical type, and let $i \in K$.

(i) If $\mathfrak{g}$ is of type A_n, then $\#\mathcal{SM}_i = \binom{n-1}{i-1}$ for $1 \le i \le n$.
(ii) If $\mathfrak{g}$ is of type B_n, then $\#\mathcal{SM}_1 = 2^{n-1}$.
(iii) If $\mathfrak{g}$ is of type C_n, then $\#\mathcal{SM}_i = \binom{n-1}{i-2}$ for $2 \le i \le n-1$, and $\#\mathcal{SM}_n = n$.
(iv) If $\mathfrak{g}$ is of type D_n, then $\#\mathcal{SM}_1 = \#\mathcal{SM}_2 = 2^{n-2} - 1$, $\#\mathcal{SM}_i = \binom{n-2}{i-3}$ for $3 \le i \le n-1$, and $\#\mathcal{SM}_n = n-1$.

In the case that $\mathfrak{g}$ is of exceptional type, $\#\mathcal{SM}_i$ is given by Table 2 in Appendix A, which is obtained by use of computer.

Furthermore, we describe $\mathcal{SM}_i$ in terms of a Bruhat interval for $i \in K$ such that Λ_i is a minuscule weight. Let w_0 (resp., $w_{J_i,0}$) be the longest element of W (resp., of W_{J_i}), and set $w_0^{J_i} := w_0 w_{J_i,0}$. For $x, y \in W^{J_i}$, we set $[x, y]^{J_i} := \{w \in W^{J_i} \mid x \le w \le y\}$, where $\le$ is the Bruhat order. Recall that v_i, $i \in K$, is defined in Definition 6.3 and Table 1 in Appendix A.

Theorem 1.3. Let $i \in K$ be such that Λ_i is a minuscule weight.

(i') If $\mathfrak{g}$ is of type A_n, then $\mathcal{SM}_i = [v_i, w_0^{J_i}]^{J_i}$ for $1 \le i \le n$.
(ii') If $\mathfrak{g}$ is of type B_n, then $\mathcal{SM}_1 = [v_1, w_0^{J_1}]^{J_1}$.
(iii') If $\mathfrak{g}$ is of type C_n, then $\mathcal{SM}_n = [v_n, w_0^{J_n}]^{J_n} \setminus \{w_0^{J_n}\}$.
(iv') If $\mathfrak{g}$ is of type D_n, then $\mathcal{SM}_1 = [v_1, w_0^{J_1}]^{J_1}$, $\mathcal{SM}_2 = [v_2, w_0^{J_2}]^{J_2}$, and $\mathcal{SM}_n = [v_n, w_0^{J_n}]^{J_n} \setminus \{w_0^{J_n}\}$.

Also in the case that $\mathfrak{g}$ is of exceptional type, $\mathcal{SM}_i$ is identical to $[v_i, w_0^{J_i}]^{J_i}$ for each $i \in K$ such that Λ_i is a minuscule weight (see Appendix A).

As an application of Theorems 1.2 and 1.3 (and the theory of Lakshmibai-Seshadri paths due to Littelmann [6]), we obtain the following dimension formula for a Demazure module. We set $\overline{v_i} := w_0 v_i w_{J_i,0}$.

Corollary 1.4. Let $i \in K$ be such that Λ_i is a minuscule weight. It hold that

$$\dim E_{\overline{v_i}}(\Lambda_i) = \begin{cases} \binom{n-1}{i-1} & (1 \le i \le n \text{ in type } \mathrm{A}_n), \\ 2^{n-1} & (i = 1 \text{ in type } \mathrm{B}_n), \\ n+1 & (i = n \text{ in type } \mathrm{C}_n), \\ 2^{n-2} - 1 & (i = 1, 2 \text{ in type } \mathrm{D}_n), \\ n & (i = n \text{ in type } \mathrm{D}_n), \\ 16 & (i = 1, 5 \text{ in type } \mathrm{E}_6), \\ 43 & (i = 6 \text{ in type } \mathrm{E}_7), \end{cases}$$

where $E_{\overline{v_i}}(\Lambda_i) := U(\mathfrak{n}_+) L(\Lambda_i)_{\overline{v_i}(\Lambda_i)}$ is the Demazure module of lowest weight $\overline{v_i}(\Lambda_i)$ in the finite-dimensional irreducible $\mathfrak{g}$-module $L(\Lambda_i)$ of highest weight Λ_i.

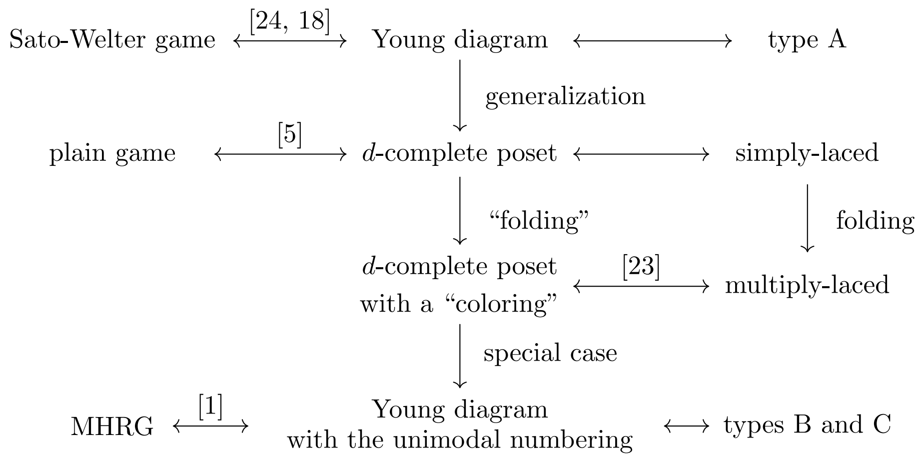
Part I is organized as follows. In Section 2, we fix our notation for Lie algebras. In Section 3, we recall the definition of minuscule elements. In Section 4, we introduce the notion of strong minuscule elements, which is the main object in this part, and prove its basic property (Proposition 4.2). In Section 5, we state our main results (Theorems 1.1, 1.2, and 1.3 above) in this part in the case that $\mathfrak{g}$ is of classical type. In Section 6, we introduce the special element v_i, and then prove Theorem 1.1 above. In Section 7, we prove Theorem 1.2 above by using the description of the set W^{J_i} of minimal-length coset representatives due to Stumbo. In Section 8, we prove Theorem 1.3 above. Then, as an application of our results, we give a dimension formula for certain Demazure modules (Corollary 1.4 above). In Appendix A, by use of computer, we prove statements similar to Theorems 1.1, 1.2, 1.3, and Corollary 1.4 in the case that $\mathfrak{g}$ is of exceptional type.

1.2 Introduction to Part II : Game positions of Multiple Hook Removing Game.

The Sato-Welter game is an impartial game studied by Welter [24] and Sato [18], independently. This game is played in terms of Young diagrams. The rule is given as follows:

(i) The starting position is a Young diagram Y.
(ii) Assume that a Young diagram Y' appears as a game position. A player chooses a box $(i, j) \in Y'$, and moves game position from Y' to $Y'\langle i, j\rangle$, where $Y'\langle i, j\rangle$ is the Young diagram which is obtained by removing the hook at (i, j) from Y' and filling the gap between two diagrams (see Figure 1 below).
(iii) The (unique) ending position is the empty Young diagram $\emptyset$. The winner is the player who makes $\emptyset$ after his/her operation (ii).

Kawanaka [5] introduced the notion of a plain game, as a generalization of the Sato-Welter game. A plain game is played in terms of d-complete posets which was introduced and classified by Proctor [15, 16], and can be thought of as a generalization of Young diagrams. It is known that d-complete posets are closely related to not only the combinatorial game theory, but also the representation theory and the algebraic geometry associated with simply-laced finite-dimensional simple Lie algebras. For example, the weight system of a minuscule representation (which is identical to the Weyl group orbit of a minuscule fundamental weight) for a simply-laced finite-dimensional simple Lie algebra can be described in terms of a d-complete poset. Applying the "folding" technique to this fact for the simply-laced case, Tada [23] described the Weyl group orbits of some fundamental weights for multiply-laced finite-dimensional simple Lie algebras in terms of d-complete posets with "colorings".



Based on [23], Abuku and Tada [1] introduced a new impartial game, named Multiple Hook Removing Game (MHRG for short). MHRG is played in terms of Young diagrams with the unimodal numbering; for the definition of unimodal numbering, see Section 10. Let us explain the rule of MHRG. We fix positive integers $m, n \in \mathbb{N}$ such that $m \leq n$. Let $Y_{m,n} := \{(i,j) \in \mathbb{N}^2 \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$ be the rectangular Young diagram of size $m \times n$. We denote by $\mathcal{F}(Y_{m,n})$ the set of all Young diagrams contained in the rectangular Young diagram $Y_{m,n}$. For a game position G of an impartial game, we denote by $\mathcal{O}(G)$ the set of all options of G. The rule of MHRG is given as follows:

(1) All game positions are some Young diagrams contained in $\mathcal{F}(Y_{m,n})$ with the unimodal numbering. The starting position is the rectangular Young diagram $Y_{m,n}$.
(2) Assume that $Y \in \mathcal{F}(Y_{m,n})$ appears as a game position. If $Y \neq \emptyset$ (the empty Young diagram), then a player chooses a box $(i,j) \in Y$, and remove the hook at (i,j) in Y. We denote by $Y\langle i,j\rangle$ the resulting Young diagram. Then we know from [1] (see also Lemma 11.4 below) that $f := \#\{(i',j') \in Y\langle i,j\rangle \mid \mathcal{H}_{Y\langle i,j\rangle}(i',j') = \mathcal{H}_Y(i,j) \text{ (as multisets)}\} \leq 1$,where $\mathcal{H}_Y(i,j)$ (resp., $\mathcal{H}_{Y\langle i,j\rangle}(i',j')$) is the numbering multiset for the hook at $(i,j) \in Y$ (resp., $(i',j') \in Y\langle i,j\rangle$); see Section 10. If $f = 0$, then a player moves Y to $Y\langle i,j\rangle \in \mathcal{O}(Y)$. If $f = 1$, then a player moves Y to $(Y\langle i,j\rangle)\langle i',j'\rangle \in \mathcal{O}(Y)$, where $(i',j') \in Y\langle i,j\rangle$ is the unique element such that $\mathcal{H}_{Y\langle i,j\rangle}(i',j') = \mathcal{H}_Y(i,j)$.
(3) The (unique) ending position is the empty Young diagram $\emptyset$. The winner is the player who makes $\emptyset$ after his/her operation (2).

In general, not all Young diagrams in $\mathcal{F}(Y_{m,n})$ appear as game positions of MHRG (see Example 11.3). The goal of this paper is to give a characterization of the set of all game positions in MHRG. Let us explain our results more precisely. Let $\binom{[1,m+n]}{m}$ denote the set of all subsets of $[1, m+n] := \{x \in \mathbb{N} \mid 1 \leq x \leq m+n\}$ having m elements. Then there exists a bijection I from $\mathcal{F}(Y_{m,n})$ onto $\binom{[1,m+n]}{m}$ (see Subsection 9.1 below). Let Y^D denote the dual Young diagram of Y in $Y_{m,n}$ (see Subsection 9.1). We set $c := (m+n-1+\chi)\,/\,2$, where $\chi = 0$ (resp., $\chi = 1$) if $m+n$ is odd (resp., even). For $Y \in \mathcal{F}(Y_{m,n})$, we set $I_R(Y) := I(Y) \cap [c+1-\chi, m+n]$. We denote by $\mathcal{S}(Y_{m,n})$ the set of all those Young diagrams in $\mathcal{F}(Y_{m,n})$ which appear as game positions of MHRG (with $Y_{m,n}$ the starting position).

Theorem 1.5 (= Theorem 12.1)**.** Let $Y \in \mathcal{F}(Y_{m,n})$, and $\lambda = (\lambda_1, \dots, \lambda_m)$ the partition corresponding to Y. The following (I), (II), (III), and (IV) are equivalent.
(I) $Y \in \mathcal{S}(Y_{m,n})$. (II) $Y^D \in \mathcal{S}(Y_{m,n})$. (III) $I_R(Y) \cap I_R(Y^D) = \emptyset$.
(IV) $\lambda_i + \lambda_j \neq n - m + i + j - 1$ for all $1 \leq i \leq j \leq m$.

Theorem 1.6 (= Theorem 13.1)**.** Let $t \in \mathbb{N}_0$ and $m, n \in \mathbb{N}$ such that $t \leq m \leq n$. For a Young diagram Y having at most t rows, $Y \in \mathcal{S}(Y_{m,n})$ if and only if $Y \in \mathcal{S}(Y_{t,n-m+t})$. Moreover, the Grundy value of Y as an element of $\mathcal{S}(Y_{m,n})$ is equal to the Grundy value of Y as an element of $\mathcal{S}(Y_{t,n-m+t})$.

In [22], Tada proves that there exists a bijection between the set of all game positions of MHRG and the set of Young diagrams, which corresponds to the Weyl group orbit of the weight in types B and C; as an application of Theorem 1.5, he also gives a description of the Weyl group orbit of the weight (in types B and C).

Part II is organized as follows. In Section 9, we fix our notation for Young diagrams, and recall some basic facts on the combinatorial game theory. In Section 10, we recall the definition of the unimodal numbering and the diagonal expression for Young diagrams. In Section 11, we recall the rule of MHRG, and a basic property (Lemma 11.4). In Sections 12 and 13, we prove Theorems 1.5 and 1.6 above, respectively.

1.3 Introduction to Part III : Infinite pre-dominant integral weights and generalizations of MacMahon's identity.

Let $\mathfrak{g}$ be a Kac-Moody algebra with $\{\alpha_i\}_{i\in I}$ the set of simple roots (possibly, the index set I is an infinite set); for simplicity of notation, we assume that $\mathfrak{g}$ is simply-laced, that is, the off-diagonal entries of the Cartan matrix of $\mathfrak{g}$ are all 0 or -1. An integral weight Λ of $\mathfrak{g}$ is said to be pre-dominant if $\langle \Lambda, \beta^\vee \rangle \geq -1$ for all positive real roots $\beta \in \Phi_+$, where $\beta^\vee$ is the coroot of β. For a pre-dominant integral weight Λ, we set $D(\Lambda) = \{\alpha \in \Phi_+ \mid \langle \Lambda, \alpha^\vee \rangle = -1\}$, and define a partial order $\leq$ on $D(\Lambda)$ by: for $\alpha, \beta \in D(\Lambda)$, $\alpha \geq \beta$ if $\alpha - \beta \in \sum_{i\in I} \mathbb{N}_0 \alpha_i$. Then the poset $(D(\Lambda), \leq)$ can be regarded as a generalization of Young diagrams in the sense of [13, Remark 6.11]. A pre-dominant integral weight Λ is said to be finite (resp., infinite) if $\#D(\Lambda) < \infty$ (resp., $\#D(\Lambda) = \infty$).

In [13], Nakada proved the following multivariable q-hook formula for the generalized Young diagram $D(\Lambda)$ for a finite pre-dominant integral weight Λ for a Kac-Moody algebra:

$$T(D(\Lambda), \leq) = \prod_{\beta \in D(\Lambda)} \frac{1}{1 - \mathbf{q}^{H_\Lambda(\beta)}}. \tag{1.1}$$

Here, $T(D(\Lambda), \leq)$ is the trace generating function of $(D(\Lambda), \leq)$ (see [13, Section 2]) corresponding to the coloring $c_\Lambda \colon D(\Lambda) \to I$ in [13, Definition 6.4]. Also, $H_\Lambda(\beta)$ is the hook at β in the diagram $D(\Lambda)$ which is defined as the intersection of $D(\Lambda)$ and the inversion set $\Phi(s_\beta)$ of the reflection s_β in β, and $\mathbf{q}^{H_\Lambda(\beta)} := \prod_{\gamma \in H_\Lambda(\beta)} q_{c_\Lambda(\gamma)}$. Taking the specialization $q_i \to q$, we obtain

$$U(D(\Lambda), \leq) := T(D(\Lambda), \leq)|_{q_i \to q} = \prod_{\beta \in D(\Lambda)} \frac{1}{1 - q^{\# H_\Lambda(\beta)}}. \tag{1.2}$$

Moreover, Nakada proved the same formula as (1.1) and (1.2) for a certain infinite pre-dominant integral weight in the case that $\mathfrak{g}$ is of type A_∞. In particular, in the case of type A_∞, by applying the formula (1.2) to the case of $\Lambda = -\Lambda_0$, we obtain MacMahon's identity ([7]):

$$U(D(-\Lambda_0), \leq^*) = \prod_{n=1}^{\infty} \left(\frac{1}{1 - q^n} \right)^n,$$

where $\leq^*$ is the dual order of $\leq$.

In Part III, based on comments in [10, 12], we treat a more general pre-dominant integral weight Λ_λ for A_∞ and D_∞ associated to a partition or strict partition λ; we see that $\Lambda_\lambda^c := -\Lambda_\lambda$ is an infinite pre-dominant integral weight. In Theorem 16.4 (resp., Theorem 16.6), we prove that there exists an order-preserving (resp., order-reversing) isomorphism from the diagram $D(\Lambda_\lambda)$ (resp., $D(\Lambda_\lambda^c)$) of a finite (resp., infinite) pre-dominant integral weight Λ_λ (resp., Λ_λ^c) onto the corresponding Young diagram Y_λ (resp., complementary Young diagram Y_λ^c; see Definition 15.1) which preserves the hooks in type A_∞. In

[25], Wildon found a "complementary" relation between the hook length sequence of a Young diagram Y and the hook length sequence of its dual Young diagram Y^D in a rectangular Young diagram of finite size. In Part III, we generalize this result to the case that Y^c is defined in the "rectangular Young diagram of infinite size" (see Example 15.2). Combining these results, we give another proof for a generalized MacMahon's identity for type A_∞ (Corollary 18.1) which was obtained in [17, Theorem 2.1]. Also, in type D_∞, we obtain a similar formula by the same argument as for type A_∞, with Young diagrams replaced by shifted Young diagrams (Corollary B.10); this formula is slightly different from [17, Theorem 3.1] since our formulation is motivated by the pre-dominant integral weights Λ_λ and Λ_λ^c, while the formulation in [17] is natural from the view point of the weight system of $\mathfrak{g}$.

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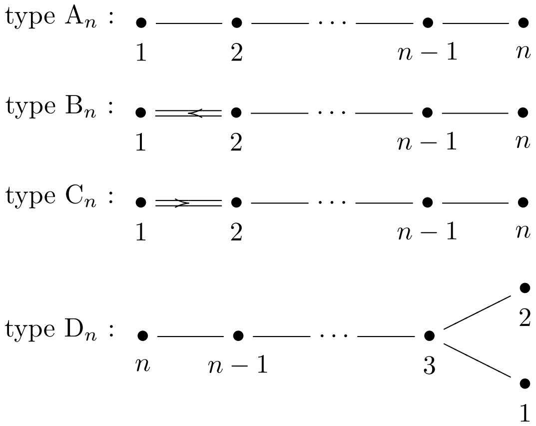
Part I

2 Preliminaries.

Let $\mathbb{N}_0$ denote the set of nonnegative integers. Throughout this part, except for Appendix A, $\mathfrak{g}$ is the finite-dimensional classical simple Lie algebra of type A_n, B_n, C_n, or D_n over $\mathbb{C}$; the Dynkin diagram for $\mathfrak{g}$ is as follows.



Let $(a_{ij})_{i,j\in I}$ be the Cartan matrix of $\mathfrak{g}$, where $I = \{1, 2, \dots, n\}$. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$, and set $\mathfrak{h}^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$. We denote by $\langle \cdot, \cdot \rangle \colon \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$ the standard pairing. Denote by $\Pi = \{\alpha_i \mid i \in I\}$ (resp., $\Pi^\vee = \{\alpha_i^\vee \mid i \in I\}$) the set of simple roots (resp., simple coroots); note that $\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij}$.

Let $P = \bigoplus_{i\in I} \mathbb{Z}\Lambda_i$ (resp., $P^+ = \sum_{i\in I} \mathbb{N}_0\Lambda_i$) be the set of integral weights (resp., dominant integral weights), where Λ_i is the fundamental weight for $i \in I$. We denote by $W = \langle s_i \mid i \in I \rangle \subset GL(\mathfrak{h}^*)$ the Weyl group of $\mathfrak{g}$, where s_i is the simple reflection in α_i, and denote by $\ell \colon W \to \mathbb{N}_0$ the length function on W. Denote by Φ (resp., Φ_+) the set of roots (resp., positive roots) for $\mathfrak{g}$. For $\beta \in \Phi$, $\beta^\vee$ denotes the coroot of β.

Let K be the subset of $I = \{1, 2, \dots, n\}$ given as follows:

$$K := \begin{cases} I & \text{if type } \mathrm{A}_n \text{ or } \mathrm{D}_n, \\ \{1\} & \text{if type } \mathrm{B}_n, \\ I \setminus \{1\} & \text{if type } \mathrm{C}_n. \end{cases} \tag{2.1}$$

Namely, the set K is identical to I if $\mathfrak{g}$ is of type A_n or D_n, and to $\{i \in I \mid \alpha_i \text{ is a short simple root}\}$ if $\mathfrak{g}$ is of type B_n or C_n. For $i \in I$, we set

$$\begin{gathered} \mathrm{adj}(i) := \{j \in I \mid a_{ij} \neq 0, 2\}, \quad \mathrm{adj}_s(i) := \{j \in \mathrm{adj}(i) \mid a_{ij} = -1\}, \\ \mathrm{adj}_\ell(i) := \mathrm{adj}(i) \setminus \mathrm{adj}_s(i) = \{j \in \mathrm{adj}(i) \mid a_{ij} = -2\}. \end{gathered}$$

3 Minuscule elements in the Weyl group.

Definition 3.1 (see, e.g., [15], [20])**.** Let $\Lambda \in P$. A Weyl group element $w \in W$ is said to be Λ-*minuscule* if there exists a reduced expression $w = s_{i_1} \cdots s_{i_r}$ such that

$$\langle s_{i_{p+1}} \cdots s_{i_r}(\Lambda), \alpha_{i_p}^\vee \rangle = 1 \text{ for all } 1 \le p \le r. \tag{3.1}$$

If $w \in W$ is Λ-minuscule for some integral weight $\Lambda \in P$ (resp., dominant integral weight $\Lambda \in P^+$), then we say that w is *minuscule* (resp., *dominant minuscule*). The set of minuscule (resp., dominant minuscule) elements in W is denoted by $\mathcal{M}$ (resp., $\mathcal{M}^+$).

Remark 3.2 ([20, Proposition 2.1])**.** Let $\Lambda \in P$, and $w \in W$. If condition (3.1) holds for some reduced expression of w, then it holds for every reduced expression of w. Hence the definition of a Λ-minuscule element is independent of the choice of a reduced expression.

4 Strong minuscule elements.

Definition 4.1. A dominant minuscule element $w \in W$ is said to be *strong minuscule* if there exists a unique dominant integral weight $\Lambda \in P^+$ (which we denote by Λ_w) such that w is Λ-minuscule. The set of strong minuscule elements in W is denoted by $\mathcal{SM}$.

The following is a basic property of strong minuscule elements.

Proposition 4.2. Let $w \in \mathcal{SM}$, and $w = s_{i_1} \cdots s_{i_r}$ a reduced expression of w. Then, $\#\{1 \le p \le r \mid i_p = i\} \ge 1$ for each $i \in I$. Namely, each of the simple reflections appears at least once in each reduced expression of w.

Proof. Suppose, for a contradiction, that s_j does not appear in the reduced expression $w = s_{i_1} \cdots s_{i_r}$ for some $j \in I$. In this case, since $s_{i_{p+1}} \cdots s_{i_r}(\Lambda_j) = \Lambda_j$ and $\langle \Lambda_j, \alpha_{i_p}^\vee \rangle = 0$ for all $1 \le p \le r$, we see that w is also $(\Lambda_w + \Lambda_j)$-minuscule. Because $\Lambda_w + \Lambda_j \in P^+$, this contradicts the assumption that $w \in \mathcal{SM}$. □

5 Main results.

Assume that $\mathfrak{g}$ is of classical type; for the case that $\mathfrak{g}$ is of exceptional type, see Appendix A below.

Theorem 5.1 (will be proved in Section 6)**.** It holds that

$$\mathcal{SM} = \bigsqcup_{i \in K} \mathcal{SM}_i,$$

where $\mathcal{SM}_i \coloneqq \{w \in \mathcal{SM} \mid \Lambda_w = \Lambda_i\}$ for $i \in K$.

Theorem 5.2 (will be proved in Section 7)**.** Let $i \in K$.

(i) If $\mathfrak{g}$ is of type A_n, then $\#\mathcal{SM}_i = \binom{n-1}{i-1}$ for $1 \le i \le n$.
(ii) If $\mathfrak{g}$ is of type B_n, then $\#\mathcal{SM}_1 = 2^{n-1}$.
(iii) If $\mathfrak{g}$ is of type C_n, then $\#\mathcal{SM}_i = \binom{n-1}{i-2}$ for $2 \le i \le n-1$, and $\#\mathcal{SM}_n = n$.
(iv) If $\mathfrak{g}$ is of type D_n, then $\#\mathcal{SM}_1 = \#\mathcal{SM}_2 = 2^{n-2} - 1$, $\#\mathcal{SM}_i = \binom{n-2}{i-3}$ for $3 \le i \le n-1$, and $\#\mathcal{SM}_n = n-1$.

Recall that the Weyl group W of $\mathfrak{g}$ is generated by $S \coloneqq \{s_1, \dots, s_n\}$. For $J \subset S$, let W_J be the parabolic subgroup of W generated by J. Let $W^J \cong W / W_J$ be the set of minimal-length coset representatives for cosets in W / W_J (see [2, Corollary 2.4.5]). We denote by $\le$ the Bruhat order on W (see, e.g., [2, Chapter 2]). For $u, w \in W$, we set $[u, w] \coloneqq \{v \in W \mid u \le v \le w\}$. For $u, w \in W^J$, we set $[u, w]^J \coloneqq [u, w] \cap W^J$. Denote by w_0 (resp., $w_{J,0}$) the longest element in W (resp., W_J); note that $w \le w_0$ (resp., $w \le w_{J,0}$) for all $w \in W$ (resp., $w \in W_J$). Define $w_0^J = w_0 w_{J,0} \in W^J$. Note that $w \le w_0^J$ for all $w \in W^J$ (see [2, Section 2.5]).

Theorem 5.3 (will be proved in Section 8)**.** Let $i \in K$ be such that Λ_i is a minuscule weight in the sense that $\langle \Lambda_i, \beta^\vee \rangle \in \{0, \pm 1\}$ for all $\beta \in \Phi$. Define $v_i \in W$ as Definition 6.3 below, and set $J_i \coloneqq S \setminus \{s_i\} \subset S$.

(i') If $\mathfrak{g}$ is of type A_n, then $\mathcal{SM}_i = [v_i, w_0^{J_i}]^{J_i}$ for $1 \le i \le n$.
(ii') If $\mathfrak{g}$ is of type B_n, then $\mathcal{SM}_1 = [v_1, w_0^{J_1}]^{J_1}$.
(iii') If $\mathfrak{g}$ is of type C_n, then $\mathcal{SM}_n = [v_n, w_0^{J_n}]^{J_n} \setminus \{w_0^{J_n}\}$.
(iv') If $\mathfrak{g}$ is of type D_n, then $\mathcal{SM}_1 = [v_1, w_0^{J_1}]^{J_1}$, $\mathcal{SM}_2 = [v_2, w_0^{J_2}]^{J_2}$, and $\mathcal{SM}_n = [v_n, w_0^{J_n}]^{J_n} \setminus \{w_0^{J_n}\}$.

6 Proof of Theorem 5.1.

Lemma 6.1 ([20, Proposition 2.5])**.** Let $w \in \mathcal{M}^+$, and fix a reduced expression $w = s_{i_1} \cdots s_{i_r}$ of w. Fix $i \in I$, and set $a \coloneqq \max\{1 \le p \le r \mid i_p = i\}$. Then,

$$\#\{a+1 \le p \le r \mid i_p \in \mathrm{adj}_s(i)\} \le 1, \tag{6.1}$$

$$\#\{a+1 \le p \le r \mid i_p \in \mathrm{adj}_\ell(i)\} = 0. \tag{6.2}$$

Remark 6.2. Let $w \in \mathcal{M}^+$, and $w = s_{i_1} \cdots s_{i_r}$ be a reduced expression of w. We claim that if $i_r \in I \setminus K$, then $w \notin \mathcal{SM}$. Indeed, suppose, for a contradiction, that $w \in \mathcal{SM}$. By Proposition 4.2, there exists $1 \le p \le r-1$ such that $i_r \in \mathrm{adj}_\ell(i_p)$. This contradicts (6.2).

Definition 6.3. Recall that K is as (2.1). For $i \in K$, we define $v_i \in W$ as follows (note that $\ell(v_i) = n$ in all cases).

(a) If $\mathfrak{g}$ is of type A_n, then $v_i \coloneqq s_n s_{n-1} \cdots s_{i+1} s_1 s_2 \cdots s_{i-1} s_i$ for $i \in K = I$.
(b) If $\mathfrak{g}$ is of type B_n, then $v_1 \coloneqq s_n s_{n-1} \cdots s_2 s_1$.
(c) If $\mathfrak{g}$ is of type C_n, then $v_i \coloneqq s_n s_{n-1} \cdots s_{i+1} s_1 s_2 \cdots s_{i-1} s_i$ for $i \in K = I \setminus \{1\}$.
(d) If $\mathfrak{g}$ is of type D_n, then $v_1 \coloneqq s_2 s_n s_{n-1} \cdots s_3 s_1, v_2 \coloneqq s_1 s_n s_{n-1} \cdots s_3 s_2$, and $v_i \coloneqq s_n s_{n-1} \cdots s_{i+1} s_1 s_2 s_3 \cdots s_{i-1} s_i$ for $i \in K \setminus \{1,2\} = I \setminus \{1,2\}$.

Lemma 6.4. Let $w \in \mathcal{M}$, and let $w = s_{i_1} \cdots s_{i_r}$ be a reduced expression of w. Set $k \coloneqq i_r \in I$. Then, w is a strong minuscule element if and only if $k \in K$ and there exists $u \in W$ such that $w = uv_k$ and $\ell(w) = \ell(u) + n$. Moreover, it holds that $\Lambda_w = \Lambda_k$ in this case.

Proof. We give a proof only for the cases of type A_n, B_n, or C_n; the proof for the case of type D_n is similar. Assume that $w \in \mathcal{SM}$; in particular, $w \in \mathcal{M}^+$. It follows from Remark 6.2 that $k \in K$. First, we show by (descending) induction on $1 \le p \le k$ (starting from $p = k$) that w has a reduced expression of the form

$$w = \cdots s_p s_{p+1} \cdots s_{k-1} s_k. \tag{6.3}$$

If $p = k$, then the assertion is obvious by assumption. Assume that $1 < p \le k$; by the induction hypothesis, we have a reduced expression for w of the form:

$$w = \cdots s_p s_{p+1} \cdots s_{k-1} s_k. \tag{6.4}$$

By Proposition 4.2, s_{p-1} appears in this reduced expression. Let us take the right-most one:

$$w = \cdots s_{p-1} \underbrace{\cdots}_{(*)} s_p s_{p+1} \cdots s_{k-1} s_k; \tag{6.5}$$

there is no s_{p-1} in $(*)$. Also, by (6.1), neither s_p nor s_{p-2} appears in $(*)$, which implies that every simple reflection in $(*)$ commutes with s_{p-1}. Hence, we get a reduced expression for w of the form:

$$w = \cdots s_{p-1} s_p s_{p+1} \cdots s_{k-1} s_k, \tag{6.6}$$

as desired. In particular, we obtain a reduced expression of the form

$$w = \cdots s_1 s_2 \cdots s_{k-1} s_k. \tag{6.7}$$

Similarly, we can show by induction on $k \le q \le n$ that w has a reduced expression of the form:

$$w = \cdots s_q \cdots s_{k+2} s_{k+1} s_1 s_2 \cdots s_{k-1} s_k.$$

In particular, we obtain a reduced expression of the form

$$w = \underbrace{\cdots}_{=: u} \underbrace{s_n s_{n-1} \cdots s_{k+2} s_{k+1} s_1 s_2 \cdots s_{k-1} s_k}_{= v_k}. \tag{6.8}$$

If we set $u \coloneqq wv_k^{-1}$, then we have $w = uv_k$ with $\ell(w) = \ell(u) + n$, as desired.

Conversely, assume that ($w \in \mathcal{M}$, and) there exists $u \in W$ such that $w = uv_k$ with $\ell(w) = \ell(u) + n$; note that w has a reduced expression of the form (6.8). Let $\Lambda \in P$ be such that w is Λ-minuscule, and write it Λ as: $\Lambda = \sum_{i=1}^{n} c_i \Lambda_i$ with $c_i \in \mathbb{Z}$. Since $\langle \Lambda, \alpha_k^\vee \rangle = 1$ by the assumption that w is Λ-minuscule (see also Remark 3.2), we get $c_k = 1$. Also, we see that $\langle \Lambda - \alpha_k, \alpha_{k-1}^\vee \rangle = 1$ and $k \in \mathrm{adj}_s(k-1)$, which implies that $c_{k-1} = 0$. Repeating this argument, we get $c_{k-1} = c_{k-2} = \cdots = c_1 = 0$. Similarly, we see that $\langle \Lambda - \alpha_k - \alpha_{k-1} - \cdots - \alpha_1, \alpha_{k+1}^\vee \rangle = 1$ and $k \in \mathrm{adj}_s(k+1)$, which implies that $c_{k+1} = 0$. Repeating this argument, we get $c_{k+2} = c_{k+3} = \cdots = c_n = 0$. Therefore, we conclude that $\Lambda = \Lambda_k \in P^+$; in paticular, w is dominant minuscule. Furthermore, the argument above shows the uniqueness of $\Lambda \in P^+$ such that w is Λ-minuscule. Thus we have proved Lemma 6.4. □

Theorem 5.1 follows immediately from Lemma 6.4 and the definition of strong minuscule elements.

7 Proof of Theorem 5.2.

For $j \in I$, we define $w_j \in W$ as follows.

(a') If $\mathfrak{g}$ is of type A_n, then $w_j \coloneqq s_1 s_2 \cdots s_{j-1} s_j$ for $j \in I$.
(b') If $\mathfrak{g}$ is of type B_n, then $w_j \coloneqq s_n s_{n-1} \cdots s_2 s_1 s_2 \cdots s_{j-1} s_j$ for $j \in I$.
(c') If $\mathfrak{g}$ is of type C_n, then $w_j \coloneqq s_n s_{n-1} \cdots s_2 s_1 s_2 \cdots s_{j-1} s_j$ for $j \in I$.
(d') If $\mathfrak{g}$ is of type D_n, then $w_1 \coloneqq s_n s_{n-1} \cdots s_4 s_3 s_1$, $w_2 \coloneqq s_n s_{n-1} \cdots s_4 s_3 s_2$, and $w_j \coloneqq s_n s_{n-1} \cdots s_3 s_1 s_2 s_3 \cdots s_{j-1} s_j$ for $j \in I \setminus \{1, 2\}$.

For $j \in I$ and $0 \le l \le \ell(w_j)$, define $w_j(l)$ to be the product of l simple reflections from the right in the expression of w_j above, except for the case that $\mathfrak{g}$ is of type D_n, $j \in I \setminus \{1, 2\}$, and $l = j-1$. When $\mathfrak{g}$ is of type D_n, and $j \in I \setminus \{1, 2\}$, the element $w_j(j-1)$ represents both $s_1 s_3 \cdots s_j$ and $s_2 s_3 \cdots s_j$; for example, the sentence "a proposition holds for $w_j(j-1)$" means that the proposition holds for both $s_1 s_3 \cdots s_j$ and $s_2 s_3 \cdots s_j$.

Proposition 7.1 ([21, Theorems 2 and 6])**.** Assume that $\mathfrak{g}$ is of type A_n, B_n, or C_n. For $i \in I$, it holds that

$$W^{J_i} = \{ w_n(l_n) w_{n-1}(l_{n-1}) \cdots w_i(l_i) \mid l_i, \ldots, l_{n-1}, l_n \text{ satisfy condition } (\#) \}, \tag{7.1}$$

where condition (#) is given by (A) (resp., (BC1), (BC2), and (BC3)) below if $\mathfrak{g}$ is of type A_n (resp., of type B_n or C_n).

(A) $0 \le l_n \le l_{n-1} \le \cdots \le l_i \le i$;
(BC1) $0 \le l_j \le j + i - 1$,
(BC2) $l_{j+1} \le l_j + 1$, and
(BC3) if $l_j \le j - 1$, then $l_{j+1} \le l_j$.

Moreover, for each element $w_n(l_n) w_{n-1}(l_{n-1}) \cdots w_i(l_i)$ of the right-hand side of (7.1), it holds that

$$\ell(w_n(l_n) w_{n-1}(l_{n-1}) \cdots w_i(l_i)) = \ell(w_n(l_n)) + \ell(w_{n-1}(l_{n-1})) + \cdots + \ell(w_i(l_i)).$$

Proposition 7.2 ([21, Theorem 4])**.** Assume that $\mathfrak{g}$ is of type D_n. For $i \in I \setminus \{1, 2\}$, it holds that

$$W^{J_i} = \{ w_n(l_n) w_{n-1}(l_{n-1}) \cdots w_i(l_i) \mid l_i, \ldots, l_{n-1}, l_n \text{ satisfy conditions (D1)–(D4)} \}, \tag{7.2}$$

where

(D1) $0 \le l_j \le j + i - 2$,
(D2) $l_{j+1} \le l_j + 1$,
(D3) if $l_j \le j - 2$, then $l_{j+1} \le l_j$, and
(D4) if $l_{j+1} = l_j + 1 = j$, then $w_j(l_j)$ and $w_{j+1}(l_{j+1})$ must be chosen in such a way that the one has s_1 as the left-most simple reflection, and the other has s_2.

Moreover, for each element $w_n(l_n)w_{n-1}(l_{n-1})\cdots w_i(l_i)$ of the right-hand side of (7.2), it holds that

$$\ell(w_n(l_n)w_{n-1}(l_{n-1})\cdots w_i(l_i)) = \ell(w_n(l_n)) + \ell(w_{n-1}(l_{n-1})) + \cdots + \ell(w_i(l_i)).$$

For $i = 1$, it holds that

$$\begin{aligned} W^{J_1} = \{ & w_{\frac{3+(-1)^h}{2}}(l_h) w_{\frac{3+(-1)^{h-1}}{2}}(l_{h-1}) \cdots w_2(l_4)w_1(l_3)w_2(l_2)w_1(l_1) \mid \\ & 0 \le h \le n-1,\, 1 \le l_h < l_{h-1} < \cdots < l_1 \le n-1 \}. \end{aligned} \tag{7.3}$$

For $i = 2$, W^{J_2} is given by the same formula as (7.3) with w_1 and w_2 interchanged. Moreover, the "length additivity" holds also for the cases that $i = 1$ and $i = 2$.

Proposition 7.3. For $i \in K$, the set $\mathcal{SM}_i = \{w \in \mathcal{SM} \mid \Lambda_w = \Lambda_i\}$ (see Theorem 5.1) is contained in W^{J_i}. If $\mathfrak{g}$ is of type A_n, B_n, or C_n, then it holds that

$$\mathcal{SM}_i = \{w_n(l_n)\cdots w_{i+1}(l_{i+1})w_i(l_i) \mid l_i, \ldots l_{n-1}, l_n \text{ satisfy condition } (\star)\}, \tag{7.4}$$

where condition $(\star)$ is given by (SA) (resp., (SB), (SC)) below if $\mathfrak{g}$ is of type A_n (resp., of type B_n, of type C_n).

(SA) Condition (A) in Proposition 7.1, and $l_i = i$, $l_n \neq 0$;
(SB) Conditions (BC1)–(BC3) (with $i = 1$) and $l_n \neq 0$;
(SC) If $2 \le i \le n-1$, then $i \le l_i \le 2i-2$ and $1 \le l_n \le \cdots \le l_{i+1} \le 2i - l_i - 1$. If $i = n$, then $n \le l_n \le 2n-1$.

Also, if $\mathfrak{g}$ is of type D_n, then it holds that

$$\begin{aligned} \mathcal{SM}_1 = \{ & w_{\frac{3+(-1)^h}{2}}(l_h) w_{\frac{3+(-1)^{h-1}}{2}}(l_{h-1}) \cdots w_2(l_4)w_1(l_3)w_2(l_2)w_1(l_1) \mid \\ & 2 \le h \le n-1,\, 1 \le l_h < l_{h-1} < \cdots < l_2 < l_1 = n-1 \}. \end{aligned} \tag{7.5}$$

For $i = 2$, $\mathcal{SM}_2$ is given by the same formula as (7.5) with w_1 and w_2 interchanged. Moreover, $\mathcal{SM}_i$, $3 \le i \le n-1$, and $\mathcal{SM}_n$ are given as follows:

$$\mathcal{SM}_i = \{w_n(l_n)\cdots w_i(l_i) \mid i \le l_i \le 2i-3\,,\, 1 \le l_n \le \cdots \le l_{i+1} \le 2i - l_i - 2\}, \tag{7.6}$$

$$\mathcal{SM}_n = \{w_n(l_n) \mid n \le l_n \le 2n-2\}. \tag{7.7}$$

The following lemma will be used in proof of Proposition 7.3.

Lemma 7.4. Let $w \in \mathcal{SM}_i$, and $w = s_{i_1}\cdots s_{i_r}$ be a reduced expression of w; recall that $i_r = i$. For each $1 \le p \le r-1$, we set $u_p \coloneqq \#\{p+1 \le a \le r \mid i_a \in \mathrm{adj}_s(i_p)\}$. Then,

$$u_p \in 2\mathbb{N}_0 \quad \text{if } \; i_p = i, \tag{7.8}$$

$$u_p \in 2\mathbb{N}_0 + 1 \quad \text{if } \; i_p \neq i. \tag{7.9}$$

Proof. By (3.1), we have $\langle \Lambda_i - \alpha_{i_r} - \cdots - \alpha_{i_{p+1}}, \alpha_{i_p}^\vee \rangle = 1$ for all $1 \le p \le r$, and hence $\delta_{i,i_p} - a_{i_p,i_r} - \cdots - a_{i_p,i_{p+1}} = 1$. Now, we set $t_p \coloneqq \#\{p+1 \le a \le r \mid i_a \in \mathrm{adj}_\ell(i_p)\}$ and $q_p \coloneqq \#\{p+1 \le a \le r \mid i_a = i_p\}$. If $i_p = i$, then $1 - u_p - 2t_p + 2q_p = 1$. Therefore, we obtain $u_p = 2(q_p - t_p) \in 2\mathbb{N}_0$. If $i_p \neq i$, then $-u_p - 2t_p + 2q_p = 1$. Hence we have $u_p = 2(q_p - t_p) - 1 \in 2\mathbb{N}_0 + 1$. □

Proof of Proposition 7.3. We give a proof only (7.6); the proofs for (7.4), (7.5), and (7.7) are similar and simpler. In order to show the inclusion $\subset$, let $w \in \mathcal{SM}_i$. By Lemma 6.4, in any reduced expression of w, the right-most generator is s_i. Hence, we have $w \in W^{J_i}$ by [2, Lemma 2.4.3]. By Proposition 7.2, we can write w as

$$w = w_n(l_n)\cdots w_i(l_i) \tag{7.10}$$

for some $l_i, \dots, l_{n-1}, l_n$ satisfying conditions (D1)–(D4). If $l_j = 0$ for some $i \le j \le n$, then $l_n = l_{n-1} = \cdots = l_{j+1} = 0$, which implies that s_n does not appear in (7.10). However, this contradicts Proposition 4.2. Thus we obtain $l_j \ge 1$ for all $i \le j \le n$. Let $w = s_{j_r} \cdots s_{j_1}$ be a reduced expression of w obtained by the product of reduced expressions of each $w_j(l_j)$ in (7.10). Suppose, for a contradiction, that $l_i = 2i-2$. Since $l_{i+1} \ge 1$ and $j_{2i-1} = i+1 \ne i$, this contradicts (7.9) because $u_{2i-1} = 2 \notin 2\mathbb{N}_0 + 1$. Hence we have $l_i \le 2i-3$. Next, let us show that $i \le l_i$. If $l_i \le i-1$, then we have $l_i = i-1$ and $l_{i+1} = i$ because both s_1 and s_2 appear in (7.10) by Proposition 4.2. Since $j_{2i-1} = 1 \ne i$ (or $j_{2i-1} = 2 \ne i$), this contradicts (7.9) because $u_{2i-1} = 2 \notin 2\mathbb{N}_0 + 1$. Therefore, we have $i \le l_i \le 2i-3$. Suppose, for a contradiction, that $l_{i+1} \ge 2i - l_i - 1$. Since $j_{2i-1} = l_i - i - 3$, it follows that $u_{2i-1} = 4 \notin 2\mathbb{N}_0 + 1$ (resp., $u_{2i-1} = 3 \notin 2\mathbb{N}_0$) if $i \le l_i \le 2i-4$ (resp., $l_i = 2i-3$). This contradicts (7.9) (resp., (7.8)). Hence we have $l_{i+1} \le 2i - l_i - 2$. Recall from (D2) that $l_{j+1} \le l_j + 1$ for all $i+1 \le j \le n-1$. Suppose, for a contradiction, that $l_{j+1} = l_j + 1$ for some $i+1 \le j \le n-1$. If we set $m \coloneqq \min\{i+1 \le j \le n-1 \mid l_{j+1} = l_j + 1\}$, then $l_m \le l_{m-1} \le \cdots \le l_{i+1}$. By direct computation, we obtain

$$u_M = \begin{cases} 2l_m & \text{if } l_m \le m-i+1, \\ 2(m-i)+1 & \text{if } l_m = m-i+1, \\ 2(m-i+1) & \text{if } l_m \ge m-i+1, \end{cases}$$

where $M \coloneqq l_{m+1} + l_m + \cdots + l_i$; remark that $l_m = m-i+1$ if and only if $j_M = i$. This contradicts (7.8) and (7.9). Therefore we obtain $1 \le l_n \le l_{n-1} \le \cdots \le l_{i+1} \le 2i - l_i - 2$, as desired. Thus we have shown the inclusion $\subset$.

Next, let us show the reverse inclusion $\supset$. Let $3 \le i \le n-1$, and let $w = w_n(l_n) \cdots w_i(l_i)$ with $i \le l_i \le 2i-3$ and $1 \le l_n \le \cdots \le l_{i+1} \le 2i - l_i - 2$. Set $k_i \coloneqq l_i - i + 2$; note that $2 \le k_i \le i-1$. Take $\varepsilon_i \in \mathfrak{h}^*$, $i \in I$, such that $\alpha_1 = \varepsilon_2 + \varepsilon_1$, $\alpha_j = \varepsilon_j - \varepsilon_{j-1}$ for $2 \le j \le n$, $\Lambda_1 = (\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n)/2$, $\Lambda_2 = (-\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n)/2$ and $\Lambda_j = \varepsilon_j + \varepsilon_{j+1} + \cdots + \varepsilon_n$ for $3 \le j \le n$. Then, we compute

$$\begin{aligned} w_i(l_i)\Lambda_i &= s_{k_i} \cdots s_3 s_1 s_2 s_3 \cdots s_{i-1} \underbrace{s_i(\varepsilon_n + \varepsilon_{n-1} + \cdots + \varepsilon_{i+1} + \varepsilon_i)}_{(\varepsilon_n + \cdots + \varepsilon_{i+1} + \varepsilon_i, \varepsilon_i - \varepsilon_{i-1}) = 1} \\ &= s_{k_i} \cdots s_3 s_1 s_2 s_3 \cdots \underbrace{s_{i-1}(\varepsilon_n + \varepsilon_{n-1} + \cdots + \varepsilon_{i+1} + \varepsilon_{i-1})}_{(\varepsilon_n + \cdots + \varepsilon_{i+1} + \varepsilon_{i-1}, \varepsilon_{i-1} - \varepsilon_{i-2}) = 1} \\ &= \cdots\cdots\cdots\cdots \\ &= s_{k_i} \cdots s_3 \underbrace{s_1(\varepsilon_n + \varepsilon_{n-1} + \cdots + \varepsilon_{i+1} + \varepsilon_1)}_{(\varepsilon_n + \cdots + \varepsilon_{i+1} + \varepsilon_1, \varepsilon_2 + \varepsilon_1) = 1} \\ &= s_{k_i} \cdots \underbrace{s_3(\varepsilon_n + \varepsilon_{n-1} + \cdots + \varepsilon_{i+1} - \varepsilon_2)}_{(\varepsilon_n + \cdots + \varepsilon_{i+1} - \varepsilon_2, \varepsilon_3 - \varepsilon_2) = 1} \\ &= \cdots\cdots\cdots\cdots \\ &= \underbrace{s_{k_i}(\varepsilon_n + \varepsilon_{n-1} + \cdots + \varepsilon_{i+1} - \varepsilon_{k_i - 1})}_{(\varepsilon_n + \cdots + \varepsilon_{i+1} - \varepsilon_{k_i-1}, \varepsilon_{k_i} - \varepsilon_{k_i-1}) = 1} \\ &= \varepsilon_n + \varepsilon_{n-1} + \cdots + \varepsilon_{i+1} - \varepsilon_{k_i}. \end{aligned}$$

Since $1 \le l_n \le \cdots \le l_{i+1} \le 2i - l_i - 2 \le i-2$, we can write $w_j(l_j)$ as $w_j(l_j) = s_{p_j} s_{p_j+1} \cdots s_{j-1} s_j$, where $p_j \coloneqq j - l_j + 1$ for $i+1 \le j \le n$; remark that $p_j \le j$ and $k_i + 1 < p_{i+1} < p_{i+2} < \cdots < p_n \le n$. We compute

$$\begin{aligned} w_{i+1}(l_{i+1})(\varepsilon_n + \cdots + \varepsilon_{i+1} - \varepsilon_{k_i}) &= s_{p_{i+1}} s_{p_{i+1}+1} \cdots s_i \underbrace{s_{i+1}(\varepsilon_n + \cdots + \varepsilon_{i+1} - \varepsilon_{k_i})}_{(\varepsilon_n + \cdots + \varepsilon_{i+1} - \varepsilon_{k_i}, \varepsilon_{i+1} - \varepsilon_i) = 1} \\ &= s_{p_{i+1}} s_{p_{i+1}+1} \cdots \underbrace{s_i(\varepsilon_n + \cdots + \varepsilon_{i+2} + \varepsilon_i - \varepsilon_{k_i})}_{(\varepsilon_n + \cdots + \varepsilon_{i+2} + \varepsilon_i - \varepsilon_{k_i}, \varepsilon_i - \varepsilon_{i-1}) = 1} \\ &= \cdots\cdots\cdots\cdots \end{aligned}$$

$$
\begin{aligned}
&= \underbrace{s_{p_{i+1}}(\varepsilon_n + \cdots + \varepsilon_{i+2} + \varepsilon_{p_{i+1}} - \varepsilon_{k_i})}_{(\varepsilon_n+\cdots+\varepsilon_{i+2}+\varepsilon_{p_{i+1}}-\varepsilon_{k_i},\varepsilon_{p_{i+1}}-\varepsilon_{p_{i+1}-1})=1} \\
&= \varepsilon_n + \cdots + \varepsilon_{i+2} + \varepsilon_{p_{i+1}-1} - \varepsilon_{k_i},
\end{aligned}
$$

which implies that $w_{i+1}(l_{j+1})w_i(l_i)$ is Λ_i-minuscule. Similarly, we see that for $i+1 \le j \le n-2$,

$$
\begin{aligned}
&w_{j+1}(l_{j+1})(\varepsilon_n + \cdots + \varepsilon_{j+1} + \varepsilon_{p_j-1} + \varepsilon_{p_{j-1}-1} + \cdots + \varepsilon_{p_{i+1}-1} - \varepsilon_{k_i}) \\
&\qquad = \varepsilon_n + \cdots + \varepsilon_{j+2} + \varepsilon_{p_{j+1}-1} + \varepsilon_{p_j-1} + \cdots + \varepsilon_{p_{i+1}-1} - \varepsilon_{k_i},
\end{aligned}
$$

and hence $w_{j+1}(l_{j+1}) \cdots w_{i+1}(l_{i+1})w_i(l_i)$ is Λ_i-minuscule. Then,

$$
\begin{aligned}
&w_n(l_n)(\varepsilon_n + \varepsilon_{p_{n-1}-1} + \cdots + \varepsilon_{p_{i+1}-1} - \varepsilon_{k_i}) \\
&= s_{p_n}s_{p_n+1}\cdots s_{n-1}\underbrace{s_n(\varepsilon_n + \varepsilon_{p_{n-1}-1} + \cdots + \varepsilon_{p_{i+1}-1} - \varepsilon_{k_i})}_{(\varepsilon_n+\varepsilon_{p_{n-1}-1}+\cdots+\varepsilon_{p_{i+1}-1}-\varepsilon_{k_i},\varepsilon_n-\varepsilon_{n-1})=1} \\
&= \cdots\cdots\cdots\cdots \\
&= \underbrace{s_{p_n}(\varepsilon_{p_n} + \varepsilon_{p_{n-1}-1} + \cdots + \varepsilon_{p_{i+1}-1} - \varepsilon_{k_i})}_{(\varepsilon_{p_n}+\varepsilon_{p_{n-1}-1}+\cdots+\varepsilon_{p_{i+1}-1}-\varepsilon_{k_i},\varepsilon_{p_n}-\varepsilon_{p_n-1})=1},
\end{aligned}
$$

which implies $w = w_n(l_n) \cdots w_{i+1}(l_{i+1})w_i(l_i)$ is Λ_i-minuscule.

Finally, let us show that $w = w_n(l_n) \cdots w_{i+1}(l_{i+1})w_i(l_i)$ is a strong minuscule element. In the expression $w = w_n(l_n) \cdots w_{i+1}(l_{i+1})w_i(l_i)$, we move the right-most s_j in each $w_j(l_j)$ to the right position, by using the commutation relation $s_ps_q = s_qs_p$ for $3 \le p, q \le n$ with $|p-q| \ge 2$, as follows:

$$
\begin{aligned}
&\overbrace{s_{n-l_n+1}\cdots s_{n-1}s_n}^{w_n(l_n)=}\overbrace{\underbrace{s_{(n-1)-l_{n-1}+1}\cdots s_{n-2}}_{\text{these commute with } s_n}s_{n-1}}^{w_{n-1}(l_{n-1})=}w_{n-2}(l_{n-2})\cdots w_i(l_i) \\
&= (w_n(l_n)s_n)(w_{n-1}(l_{n-1})s_{n-1})s_ns_{n-1}\overbrace{\underbrace{s_{(n-2)-l_{n-2}+1}\cdots s_{n-3}}_{\text{these commute with } s_ns_{n-1}}s_{n-2}}^{w_{n-2}(l_{n-2})=}\cdots w_i(l_i) \\
&= (w_n(l_n)s_n)(w_{n-1}(l_{n-1})s_{n-1})(w_{n-2}(l_{n-2})s_{n-2})s_ns_{n-1}s_{n-2}w_{n-3}(l_{n-3})\cdots w_i(l_i) \\
&= \cdots\cdots\cdots\cdots \\
&= \underbrace{(w_n(l_n)s_n)(w_{n-1}(l_{n-1})s_{n-1})\cdots(w_{i+1}(l_{i+1})s_{i+1})}_{=:u'}s_n\cdots s_{i+1}w_i(l_i) \\
&= u's_n\cdots s_{i+1}\underbrace{s_{k_i}\cdots s_3}_{\text{these commute with } s_n\cdots s_{i+1}}s_1s_2s_3\cdots s_i \\
&= \underbrace{u's_{k_i}\cdots s_3}_{=:u}\underbrace{s_n\cdots s_{i+1}s_1s_2s_3\cdots s_i}_{=v_i};
\end{aligned}
$$

remark that if $i = 3$, then $u = e$. Therefore it follows from Lemma 6.4 that $w = w_n(l_n) \cdots w_{i+1}(l_{i+1})w_i(l_i)$ is a strong minuscule element. This completes the proof of Proposition 7.3. □

Proof of Theorem 5.2. We give proofs only for the cases of type B_n and type C_n; the proofs for the other cases are similar or simpler. In this proof, we denote by $W(\mathrm{B}_n)$ the Weyl group of type B_n, and set $J_i^{(n)} := \{s_1, \ldots, s_n\} \setminus \{s_i\}$. In the case of type B_n, we see from Proposition 7.3 that $\mathcal{SM}_1 = \{w_n(l_n) \cdots w_1(l_1) \mid l_n, \ldots, l_1$ satisfy $l_n \neq 0$ and (BC1)–(BC3)$\}$. It is easy to see by Proposition 7.1 that

$$
\#\{w_n(l_n)\cdots w_1(l_1) \mid l_n, \ldots, l_1 \text{ satisfy } l_n = 0 \text{ and (BC1)–(BC3)}\} = \#W(\mathrm{B}_{n-1})^{J_1^{(n-1)}}.
$$

Therefore, we obtain

$$
\#\mathcal{SM}_1 = \#W(\mathrm{B}_n)^{J_1^{(n)}} - \#W(\mathrm{B}_{n-1})^{J_1^{(n-1)}} = \frac{\#W(\mathrm{B}_n)}{\#W(\mathrm{B}_n)_{J_1^{(n)}}} - \frac{\#W(\mathrm{B}_{n-1})}{\#W(\mathrm{B}_{n-1})_{J_1^{(n-1)}}}
$$

$$= \frac{n! \times 2^n}{n!} - \frac{(n-1)! \times 2^{n-1}}{(n-1)!} = 2^n - 2^{n-1} = 2^{n-1},$$

as desired.

In the case of type C_n with $2 \le i \le n-1$, we see from Proposition 7.3 that $\mathcal{SM}_i = \{w_n(l_n) \cdots w_i(l_i) \mid i \le l_i \le 2i-2\,,\ 1 \le l_n \le \cdots \le l_{i+1} \le 2i - l_i - 1\}$. Hence we have

$$\begin{aligned}\#\mathcal{SM}_i = \sum_{l_i=i}^{2i-2} \binom{n+i-2-l_i}{n-i} &= \binom{n-2}{n-i} + \binom{n-3}{n-i} + \cdots + \binom{n-i}{n-i} \\ &= \binom{n}{i-1} - \binom{n-1}{i-1} = \binom{n-1}{i-2};\end{aligned}$$

remark that $\binom{n}{r} = \sum_{k=r-1}^{n-1} \binom{k}{r-1}$ and $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$. □

8 Proof of Theorem 5.3.

Proof of Theorem 5.3. We give a proof only for the case of type A_n; the proofs for the other cases are similar or simpler. Let $w \in \mathcal{SM}_i$. By Proposition 7.3, we have $w \in W^{J_i}$, and hence $w \le w_0^{J_i}$. By Lemma 6.4, there exists $u \in W$ such that $w = uv_i$ with $\ell(w) = \ell(u) + \ell(v_i)$. Hence, by the subword property of the Bruhat order (see, e.g., [2, Theorem 2.2.2]), we have $v_i \le w$. Therefore, we conclude that $w \in [v_i, w_0^{J_i}]^{J_i}$.

Conversely, let $w \in [v_i, w_0^{J_i}]^{J_i} = [v_i, w_0^{J_i}] \cap W^{J_i}$. By Proposition 7.1, there exist $0 \le p_n \le \cdots \le p_i \le i$ such that $w = w_n(p_n) \cdots w_i(p_i)$. Since $v_i \le w$ by assumption, it follows from the subword property that both s_1 and s_n appear in any reduced expression for w. Observe that for $i < j \le n$, the element $w_j(p_j)$ does not have a reduced expression in which s_1 appears, and that the element $w_i(p_i)$ has a reduced expression in which s_1 appears if and only if $p_i = i$. Thus we conclude that $p_i = i$. Also, observe that for $i \le j < n$, the element $w_j(p_j)$ does not have a reduced expression in which s_n appears, and that the element $w_n(p_n)$ has a reduced expression in which s_n appears if and only if $p_n \ge 1$. Thus we conclude that $p_n \ge 1$. Therefore, by Proposition 7.3, we have $w \in \mathcal{SM}_i$, as desired. □

Remark 8.1. In general, $[v_i, w_0^{J_i}]^{J_i} \subsetneq [v_i, w_0^{J_i}]$. Indeed, in the Weyl group of type A_4, we see that $s_2v_3 = s_2s_1s_2s_4s_3 \in [v_3, w_0^{J_3}] \setminus [v_3, w_0^{J_3}]^{J_3}$; note that this element is not a minuscule element, and hence Lemma 6.4 is not valid for this element.

For $\Lambda \in P^+$, let $L(\Lambda)$ denote the finite-dimensional irreducible $\mathfrak{g}$-module of highest weight Λ, with $L(\Lambda) = \bigoplus_{\mu \in P} L(\Lambda)_\mu$ the weight space decomposition; recall that $\dim L(\Lambda)_{\tau(\Lambda)} = 1$ for all $\tau \in W$. Denote by $\mathfrak{n}_+$ the subalgebra of $\mathfrak{g}$ generated by the root spaces corresponding to positive roots. For $\tau \in W$, we denote by $E_\tau(\Lambda)$ the $\mathfrak{n}_+$-submodule of $L(\Lambda)$ generated by $L(\Lambda)_{\tau(\Lambda)}$, which we call the *Demazure module* of lowest weight $\tau(\Lambda)$.

Remark 8.2. For $i \in I$, we assume that $\Lambda = \Lambda_i$ is a minuscule weight. In this case, the dimension of the Demazure module $E_\tau(\Lambda)$ for $\tau \in W^{J_i}$ is equal to $[e, \tau]^{J_i}$ (this fact follows from, for example, the theory of Lakshmibai-Seshadri paths; see [6, Theorem 5.2]).

Let and fix $i \in I$. For $\tau \in W^{J_i}$, we set $\bar{\tau} := w_0 \tau w_{J_i,0}$, where $w_{J_i,0} \in W_{J_i}$ is the longest element of W_{J_i}. Then we see by [2, Proposition 2.5.4] that $\bar{\tau} \in W^{J_i}$, and that the map $\bar{\cdot}\colon W^{J_i} \to W^{J_i}$, $\tau \mapsto \bar{\tau}$, is an order-reversing involution on W^{J_i}.

Corollary 8.3. Let $i \in K$ be such that Λ_i is a minuscule weight. It hold that

(1) If $\mathfrak{g}$ is of type A_n, then $\dim E_{\overline{v_i}}(\Lambda_i) = \binom{n-1}{i-1}$ for each $i \in I$.
(2) If $\mathfrak{g}$ is of type B_n, then $\dim E_{\overline{v_1}}(\Lambda_1) = 2^{n-1}$.
(3) If $\mathfrak{g}$ is of type C_n, then $\dim E_{\overline{v_n}}(\Lambda_n) = n+1$.
(4) If $\mathfrak{g}$ is of type D_n, then $\dim E_{\overline{v_1}}(\Lambda_1) = \dim E_{\overline{v_2}}(\Lambda_2) = 2^{n-2} - 1$, and $\dim E_{\overline{v_n}}(\Lambda_n) = n$.

Proof. We see that

$$\overline{[v_i, w_0^{J_i}]^{J_i}} = [\overline{w_0^{J_i}}, \overline{v_i}]^{J_i} = [w_0 w_0^{J_i} w_{J_i,0}, \overline{v_i}]^{J_i} = [w_0^2, \overline{v_i}]^{J_i} = [e, \overline{v_i}]^{J_i}.$$

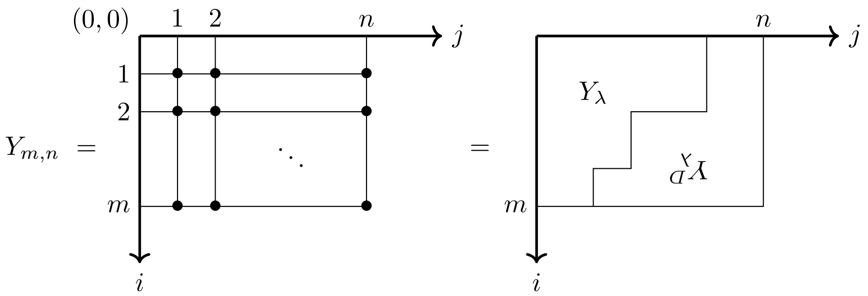
Hence, $\#[v_i, w_0^{J_i}]^{J_i} = \#[e, \overline{v_i}]^{J_i}$. Because we have $\#[v_i, w_0^{J_i}]^{J_i} = \#\mathcal{SM}_i$ or $\#[v_i, w_0^{J_i}]^{J_i} = \#\mathcal{SM}_i + 1$ by Theorems 5.2 and 5.3, we conclude by using Remark 8.2 that $\dim E_{\overline{v_i}}(\Lambda_i) = \#[e, \overline{v_i}]^{J_i} = \#[v_i, w_0^{J_i}]^{J_i} = \#\mathcal{SM}_i$ or $\#\mathcal{SM}_i + 1$, as desired. □

Part II

9 Preliminaries.

9.1 Young diagrams.

Let $\mathbb{N}$ denote the set of positive intgers. For $a, b \in \mathbb{Z}$, we set $[a, b] := \{x \in \mathbb{Z} \mid a \leq x \leq b\}$. Throughout this part, we fix $m, n \in \mathbb{N}$ such that $m \leq n$. For a positive integer $x \in \mathbb{N}$, we set $\overline{x} := m + n + 1 - x$. Let $\mathcal{Y}_m(m+n)$ be the set of partitions $\lambda = (\lambda_1, \dots, \lambda_m)$ of length at most m such that $n \geq \lambda_1 \geq \cdots \geq \lambda_m \geq 0$. We can identify $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathcal{Y}_m(m+n)$ with the *Young diagram* $Y_\lambda := \{(i, j) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N} \mid 1 \leq i \leq m, 1 \leq j \leq \lambda_i\}$ of shape λ; if $\lambda = (0, 0, \dots, 0) \in \mathcal{Y}_m(m+n)$, then we denote Y_λ by $\emptyset$, and call it the *empty Young diagram*. We identify $(i, j) \in Y_\lambda$ with the square in $\mathbb{R}^2$ whose vertices are $(i-1, j-1), (i-1, j), (i, j-1)$, and (i, j); elements in Y_λ are called *boxes* in Y_λ. Let $Y_{m,n} := \{(i, j) \in \mathbb{N}^2 \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ be the rectangular Young diagram of size $m \times n$, which corresponds to $(n, n, \dots, n) \in \mathcal{Y}_m(m+n)$. Set $\mathcal{F}(Y_{m,n}) := \{Y_\lambda \mid \lambda \in \mathcal{Y}_m(m+n)\}$; notice that $\mathcal{F}(Y_{m,n})$ is identical to the set of all Young diagrams contained in the rectangular Young diagram $Y_{m,n}$. We set $\lambda^D := (n - \lambda_m, \dots, n - \lambda_1) \in \mathcal{Y}_m(m+n)$. The Young diagram $Y_\lambda^D := Y_{\lambda^D}$ is called the *dual Young diagram* of Y_λ (in $Y_{m,n}$).



Let $\binom{[1,m+n]}{m}$ denote the set of all subsets of $[1, m+n]$ having m elements. For $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathcal{Y}_m(m+n)$, we set $i'_t := \lambda_{m-t+1} + t$ for $1 \leq t \leq m$; observe that $I_\lambda := \{i'_1 < \cdots < i'_m\} \in \binom{[1,m+n]}{m}$. It is well-known that the map $\lambda \mapsto I_\lambda$ is a bijection from $\mathcal{Y}_m(m+n)$ onto $\binom{[1,m+n]}{m}$. By the composition of this bijection and the inverse of the bijection $\mathcal{Y}_m(m+n) \to \mathcal{F}(Y_{m,n})$, $\lambda \mapsto Y_\lambda$, we obtain a bijection I from $\mathcal{F}(Y_{m,n})$ onto $\binom{[1,m+n]}{m}$. Let $Y \in \mathcal{F}(Y_{m,n})$. For $(i, j) \in Y$, we set $H_Y(i, j) := \{(i, j)\} \cup \{(i, j') \in Y \mid j < j'\} \cup \{(i', j) \in Y \mid i < i'\}$, and call it the *hook at* (i, j) in Y. Also, for $(i, j) \in Y$, we set

$$
\begin{aligned}
Y\langle i, j\rangle := \{(i', j') \mid (i', j') \in Y, \text{ and } i' < i \text{ or } j' < j\} \\
\cup \{(i'-1, j'-1) \mid (i', j') \in Y,\ i' > i \text{ and } j' > j\}.
\end{aligned}
$$

The procedure which obtains $Y\langle i, j\rangle$ from Y is called *removing the hook* at (i, j) from Y (see Figure 1 below).

9.2 Combinatrial game theory.

For the general theory of combinatorial games, we refer the reader to [19, Chapters 1 and 2]. In this subsection, we fix an impartial game in normal play whose game positions are all short (in the sense of [19, pages 4 and 9]).

Definition 9.1. A game position of an impartial game is called an *$\mathcal{N}$-position* (resp., a *$\mathcal{P}$-position*) if the next player (resp., the previous player) has a winning strategy.

Definition 9.2. For a (proper) subset X of $\mathbb{N}_0$, we set $\operatorname{mex} X := \min(\mathbb{N}_0 \setminus X)$.

For a game position G of an impartial game, we denote by $\mathcal{O}(G)$ the set of all options of G.

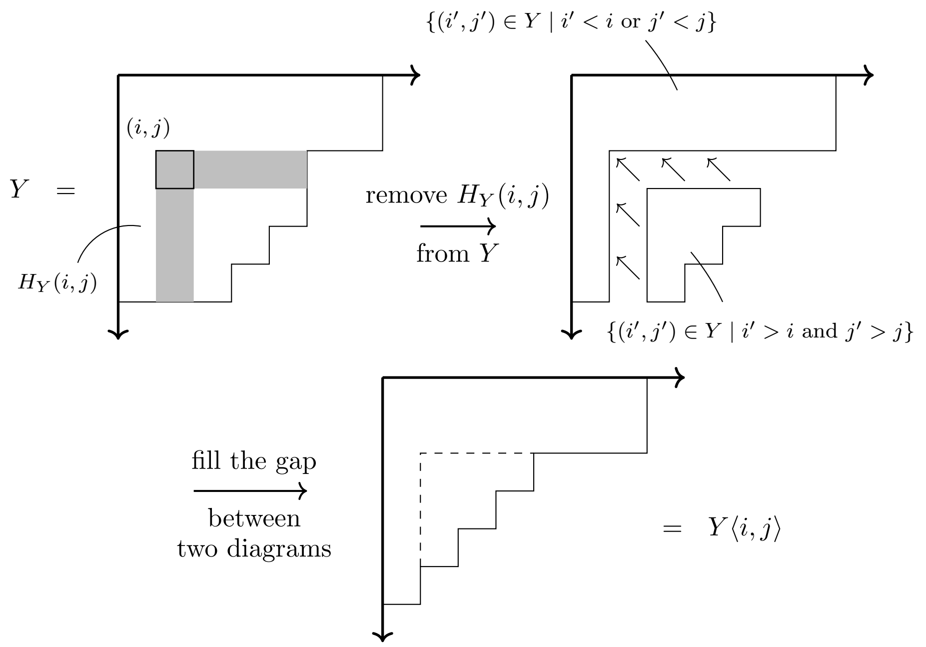


Figure 1. Removing the hook at (i, j) from Y.

Definition 9.3. Let G be a game position. The *Grundy value* $\mathcal{G}(G)$ of G is defined by

$$\mathcal{G}(G) := \begin{cases} 0 & \text{if } G \text{ is an ending position,} \\ \operatorname{mex}\{\mathcal{G}(P) \mid P \in \mathcal{O}(G)\} & \text{if } G \text{ is not an ending position.} \end{cases}$$

Recall from [19, page 6] that each game position of an impartial game is either an $\mathcal{N}$-position or a $\mathcal{P}$-position. The following result is well-known in the combinatorial game theory.

Theorem 9.4 ([19, Theorem 2.1])**.** A game position G is a $\mathcal{P}$-position if and only if $\mathcal{G}(G) = 0$.

10 Unimodal numbering on Young diagrams.

Let $Y \in \mathcal{F}(Y_{m,n})$. For each box $(i, j) \in Y$, we write $c\,(i, j) := \min\,(j - i + m, i - j + n)$ on it; we call this numbering on Y the *unimodal numbering* on Y.

Example 10.1. Assume that $m = 3$ and $n = 5$. The Young diagram $Y = Y_{(4,4,2)} \in \mathcal{F}(Y_{3,5})$ with the unimodal numbering is as follows:

3	4	3	2
2	3	4	3
1	2		

It can be easily checked that $c := (m + n - 1 + \chi)\,/\,2$ is the maximum number appearing in the moutainous numbering, where

$$\chi := \begin{cases} 1 & \text{if } m + n \in 2\mathbb{N}, \\ 0 & \text{if } m + n \in 2\mathbb{N} + 1. \end{cases}$$

We define $\mathbb{D}_{m,n} \subset \mathbb{N}_0^{m+n+1}$ by

$$\mathbb{D}_{m,n} := \{(a_1, a_2, a_3, \dots, a_{m+n-1}, a_{m+n}, a_{m+n+1}) \in \mathbb{N}_0^{m+n+1} \mid$$
$$a_1 = a_{m+n+1} = 0\,,\ 0 \leq a_k - a_{k-1} \leq 1 \text{ for } 2 \leq k \leq m+1,$$
$$0 \leq a_k - a_{k+1} \leq 1 \text{ for } m+1 \leq k \leq m+n\}.$$

For $Y \in \mathcal{F}(Y_{m,n})$, we set $d_k = d_k(Y) := \#\{(i,j) \in Y \mid j - i = -m - 1 + k\}$ for each $1 \leq k \leq m+n+1$; note that $d_1 = d_{m+n+1} = 0$. We know from [1, Proposition 3.6] that

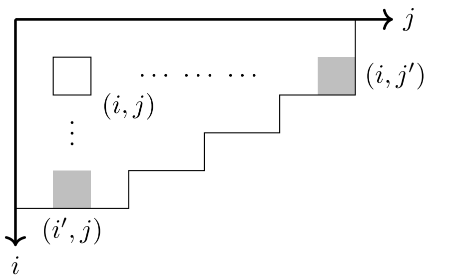
$$D_{m,n}(Y) := (d_1, d_2, d_3, \dots, d_{m+n-1}, d_{m+n}, d_{m+n+1})$$

is an element of $\mathbb{D}_{m,n}$. Thus we obtain the map $D_{m,n} \colon \mathcal{F}(Y_{m,n}) \to \mathbb{D}_{m,n}$, $Y \mapsto D_{m,n}(Y)$. An element $D_{m,n}(Y) \in \mathbb{D}_{m,n}$ is called the *diagonal expression* of Y. For simplicity of notation, we denote $D_{m,n}$ by D.

Example 10.2. Assume that $m = 3$ and $n = 5$. Let $\lambda = (4, 3, 1) \in \mathcal{Y}_3(8)$. Then we have $D_{3,5}(Y_\lambda) = (0, 1, 1, 2, 2, 1, 1, 0, 0) \in \mathbb{D}_{3,5}$.

Proposition 10.3 ([1, Proposition 3.6])**.** The map $D_{m,n} \colon \mathcal{F}(Y_{m,n}) \to \mathbb{D}_{m,n}$ is bijective.

Here we recall from [1, Subsection 3.3] the relation between "removing a hook" (see Figure 1) and the diagonal expression (see Example 10.5 below). For a subset S of $Y \in \mathcal{F}(Y_{m,n})$, we define $\mathcal{H}_Y(S)$ to be the multiset consisting of $c\,(i,j)$ for $(i,j) \in S$. The multiset $\mathcal{H}_Y(S)$ is called the *numbering multiset* for S. In particular, if $S = H_Y(i,j)$ for some $(i,j) \in Y$, then we denote $\mathcal{H}_Y(S)$ by $\mathcal{H}_Y(i,j)$. We deduce that $\mathcal{H}_Y(Y) = \mathcal{H}_Y(Y\langle i,j\rangle) \cup \mathcal{H}_Y(i,j)$ (the union of multisets). Now, let $Y \in \mathcal{F}(Y_{m,n})$, and fix $(i,j) \in Y$. Let i' (resp., j') be such that $(i',j) \in Y$ and $(i'+1, j) \notin Y$ (resp., $(i,j') \in Y$ and $(i, j'+1) \notin Y$).



Then we see that

$$\begin{aligned} &\#\{(x,y) \in Y \mid y - x = -m + k\} - \#\{(x,y) \in Y\langle i,j\rangle \mid y - x = -m + k\} \\ &= \begin{cases} 1 & \text{if } m + j - i' \leq k \leq m + j' - i, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore, if

$$D(Y) = (d_1, \dots, d_{m+j-i'}, d_{m+j-i'+1}, d_{m+j-i'+2}, \dots,$$
$$d_{m+j'-i}, d_{m+j'-i+1}, d_{m+j'-i+2}, \dots, d_{m+n+1}),$$

then

$$D(Y\langle i,j\rangle) = (d_1, \dots, d_{m+j-i'}, d_{m+j-i'+1} - 1, d_{m+j-i'+2} - 1 \dots,$$
$$d_{m+j'-i} - 1, d_{m+j'-i+1} - 1, d_{m+j'-i+2} \dots, d_{m+n+1}).$$

Thus, if we remove a hook from $Y \in \mathcal{F}(Y_{m,n})$, then 1 is subtracted from some consecutive entries in $D(Y)$; in the case above, the consecutive entries are $d_l, d_{l+1}, \dots, d_r$, with $l = m + j - i' + 1$ and $r = m + j' - i + 1$.

Definition 10.4. Let $\mathbf{a} = (a_1, a_2, \ldots, a_{m+n}, a_{m+n+1}) \in \mathbb{D}_{m,n}$; recall that $a_1 = a_{m+n+1} = 0$. For $2 \le l \le r \le m+n$, we write $\mathbf{a} \xrightarrow{l,r} \mathbf{a}'$ if $a_k \ge 1$ for all $l \le k \le r$, and $\mathbf{a}' = (a_1, a_2, \ldots, a_{l-1}, a_l - 1, a_{l+1} - 1, \ldots, a_{r-1} - 1, a_r - 1, a_{r+1}, \ldots, a_{m+n}, a_{m+n+1}) \in \mathbb{N}_0^{m+n+1}$.

Recall that the map $D = D_{m,n} \colon \mathcal{F}(Y_{m,n}) \to \mathbb{D}_{m,n}$ is bijective. Let $\mathbf{a}, \mathbf{a}' \in \mathbb{D}_{m,n}$, and set $Y := D^{-1}(\mathbf{a})$, $Y' := D^{-1}(\mathbf{a}')$. If $\mathbf{a} \xrightarrow{l,r} \mathbf{a}'$ for some $2 \le l \le r \le m+n$, then we write $Y \xrightarrow{l,r} Y'$.

Example 10.5. Keep the notation and setting in Example 10.2. It follows that $Y_\lambda\langle 2,1\rangle = \{(1,1),(1,2),(1,3),(1,4)\}$, and hence $D(Y_\lambda\langle 2,1\rangle) = (0,0,0,1,1,1,1,0,0)$. Thus we have $D(Y_\lambda) \xrightarrow{2,5} D(Y_\lambda\langle 2,1\rangle)$ (and hence $Y_\lambda \xrightarrow{2,5} Y_\lambda\langle 2,1\rangle$).

11 Multiple Hook Removing Game.

Abuku and Tada [1] introduced an impartial game, named Multiple Hook Removing Game (MHRG for short), whose rule is given as follows; recall that m and n are fixed positive integers such that $m \le n$:

(1) All game positions are some Young diagrams contained in $\mathcal{F}(Y_{m,n})$ with the unimodal numbering. The starting position is the rectangular Young diagram $Y_{m,n}$.

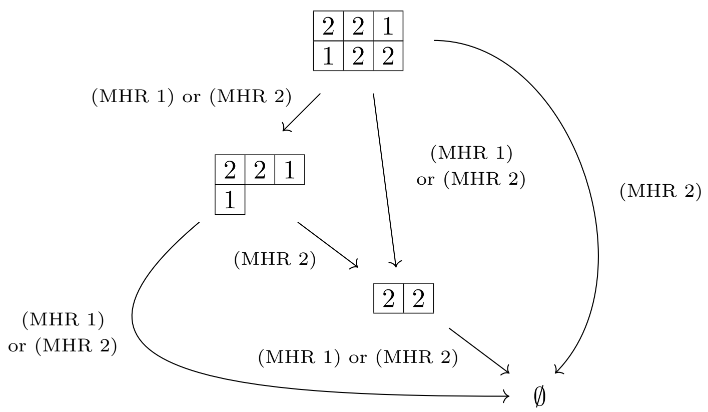
(2) Assume that $Y \in \mathcal{F}(Y_{m,n})$ appears as a game position. If $Y \ne \emptyset$ (the empty Young diagram), then a player chooses a box $(i,j) \in Y$, and remove the hook at (i,j) in Y; recall from Subsection 9.1 that the resulting Young diagram is $Y\langle i,j\rangle$. Then we know from [1, Lemma 3.15] (see also Lemma 11.4 below) that $f := \#\{(i',j') \in Y\langle i,j\rangle \mid \mathcal{H}_{Y\langle i,j\rangle}(i',j') = \mathcal{H}_Y(i,j) \text{ (as multisets)}\} \le 1$. If $f = 0$, then a player moves Y to $Y\langle i,j\rangle \in \mathcal{O}(Y)$; we call this case and this operation (MHR 1). If $f = 1$, then a player moves Y to $(Y\langle i,j\rangle)\langle i',j'\rangle \in \mathcal{O}(Y)$, where $(i',j') \in Y\langle i,j\rangle$ is the unique element such that $\mathcal{H}_{Y\langle i,j\rangle}(i',j') = \mathcal{H}_Y(i,j)$; we call this case and this operation (MHR 2).

(3) The (unique) ending position is the empty Young diagram $\emptyset$. The winner is the player who makes $\emptyset$ after his/her operation (2).

Definition 11.1. We denote by $\mathcal{S}(Y_{m,n})$ the set of all those Young diagrams in $\mathcal{F}(Y_{m,n})$ which appear as game positions of MHRG (with $Y_{m,n}$ the starting position); in general, $\mathcal{S}(Y_{m,n}) \subsetneq \mathcal{F}(Y_{m,n})$ as Example 11.3 below shows.

Definition 11.2. Let $Y \in \mathcal{S}(Y_{m,n})$, and $Y' \in \mathcal{O}(Y)$. If a player moves Y to Y' by operation (MHR 1) (resp., (MHR 2)), then we write $Y \xrightarrow{\text{(MHR 1)}} Y'$ (resp., $Y \xrightarrow{\text{(MHR 2)}} Y'$).

Example 11.3. Assume that $m = 2$ and $n = 3$. The elements of $\mathcal{S}(Y_{2,3})$ are



The following elements of $\mathcal{F}(Y_{2,3})$ are not contained in $\mathcal{S}(Y_{2,3})$:

$$\begin{array}{|c|c|c|}\hline 2&2&1\\\hline 1&2\\\cline{1-2}\end{array}\,,\quad \begin{array}{|c|c|}\hline 2&2\\\hline 1&2\\\hline\end{array}\,,\quad \begin{array}{|c|c|}\hline 2&2\\\hline 1\\\cline{1-1}\end{array}\,,\quad \begin{array}{|c|c|c|}\hline 2&2&1\\\hline\end{array}\,,\quad \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}\,,\quad \begin{array}{|c|}\hline 2\\\hline\end{array}\,.$$

Lemma 11.4 ([1, Lemma 3.15]). Let $Y \in \mathcal{F}(Y_{m,n})$, and $(i,j) \in Y$. Assume that there exists a box $(i',j') \in Y\langle i,j\rangle$ such that $\mathcal{H}_{Y\langle i,j\rangle}(i',j') = \mathcal{H}_Y(i,j)$ (as multisets). If $Y \xrightarrow{l,r} Y\langle i,j\rangle$, then $Y\langle i,j\rangle \xrightarrow{\overline{r-1},\overline{l-1}} (Y\langle i,j\rangle)\langle i',j'\rangle$. In particular, $\#\{(i',j') \in Y\langle i,j\rangle \mid \mathcal{H}_{Y\langle i,j\rangle}(i',j') = \mathcal{H}_Y(i,j) \text{ (as multisets)}\} \leq 1$.

Remark 11.5. In fact, the following holds (see [1, Lemma 3.15]), although we do not use these facts in this thesis.

(1) Keep the notation and setting in Lemma 11.4. There does not exist $(i'',j'') \in (Y\langle i,j\rangle)\langle i',j'\rangle$ such that $\mathcal{H}_{(Y\langle i,j\rangle)\langle i',j'\rangle}(i'',j'') = \mathcal{H}_Y(i,j)$.
(2) Let $(i,j),(k,l) \in Y$. Assume that $\mathcal{H}_Y(i,j) = \mathcal{H}_Y(k,l)$. If there exists a box $(i',j') \in Y\langle i,j\rangle$ such that $\mathcal{H}_{Y\langle i,j\rangle}(i',j') = \mathcal{H}_Y(i,j)$, then there exists a (unique) box $(k',l') \in Y\langle k,l\rangle$ such that $\mathcal{H}_{Y\langle k,l\rangle}(k',l') = \mathcal{H}_Y(i,j)$. Moreover, in this case, we have $(Y\langle i,j\rangle)\langle i',j'\rangle = (Y\langle k,l\rangle)\langle k',l'\rangle$.

12 Description of $\mathcal{S}(Y_{m,n})$.

Recall that $m,n \in \mathbb{N}$ are such that $m \leq n$, and that $c = \max\{c(i,j) \mid (i,j) \in Y_{m,n}\}$ is equal to $(m+n-1+\chi)/2$, where $\chi = 0$ (resp., $\chi = 1$) if $m+n$ is odd (resp., even). Also, we have a canonical bijection $I\colon \mathcal{F}(Y_{m,n}) \to \binom{[1,m+n]}{m}$ (see Subsection 9.1).

Let $Y \in \mathcal{F}(Y_{m,n})$. We set $I_R(Y) := I(Y) \cap [c+1-\chi, m+n]$; note that $\overline{c+1-\chi} = m+n+1-(c+1-\chi) = c+1 \geq c+1-\chi$.

Theorem 12.1. Let $Y \in \mathcal{F}(Y_{m,n})$, and $\lambda = (\lambda_1, \ldots, \lambda_m)$ the partition corresponding to Y, that is, $Y = Y_\lambda$. The following (I), (II), (III), and (IV) are equivalent.
(I) $Y \in \mathcal{S}(Y_{m,n})$. (II) $Y^D \in \mathcal{S}(Y_{m,n})$. (III) $I_R(Y) \cap I_R(Y^D) = \emptyset$.
(IV) $\lambda_i + \lambda_j \neq n - m + i + j - 1$ for all $1 \leq i,j \leq m$.

The rest of this section is devoted to a proof of Theorem 12.1. We can easily show the following lemma.

Lemma 12.2. (A) It holds that $I(Y^D) = \{\bar{i} = m+n+1-i \mid i \in I(Y)\} = \overline{I(Y)}$ for $Y \in \mathcal{F}(Y_{m,n})$.
(B) Let $Y \in \mathcal{F}(Y_{m,n})$, and let $l,r \in [2, m+n]$ such that $l \leq r$. Then, $l-1 \notin I(Y)$ and $r \in I(Y)$ if and only if there exists a (unique) box $(i,j) \in Y$ such that $Y \xrightarrow{l,r} Y\langle i,j\rangle$; in this case, $I(Y\langle i,j\rangle) = (I(Y) \setminus \{r\}) \cup \{l-1\}$ and $I(Y\langle i,j\rangle^D) = (I(Y^D) \setminus \{\bar{r}\}) \cup \{\overline{l-1}\}$.

Remark 12.3. Let $Y \in \mathcal{F}(Y_{m,n})$, and $(i,j) \in Y$. Let $2 \leq l \leq r \leq m+n$ be such that $Y \xrightarrow{l,r} Y\langle i,j\rangle$. By Lemmas 11.4 and 12.2 (B), it follows that $\bar{r} \notin I(Y\langle i,j\rangle)$ and $\overline{l-1} \in I(Y\langle i,j\rangle)$ if and only if there exists a (unique) box $(i',j') \in Y\langle i,j\rangle$ such that $Y\langle i,j\rangle \xrightarrow{\overline{r-1},\overline{l-1}} (Y\langle i,j\rangle)\langle i',j'\rangle$; in particular, in this case, it holds that $\mathcal{H}_{Y\langle i,j\rangle}(i',j') = \mathcal{H}_Y(i,j)$ (as multisets).

We first show (I) $\Rightarrow$ (III). Since $Y \in \mathcal{S}(Y_{m,n})$ by (I), there exists a sequence of game positions of the form

$$Y_{m,n} = Y_0 \xrightarrow{t_1} Y_1 \xrightarrow{t_2} Y_2 \xrightarrow{t_3} \cdots \xrightarrow{t_p} Y_p = Y,$$

where t_i is either (MHR 1) or (MHR 2) for each $1 \leq i \leq p$. For $1 \leq i \leq p$ such that t_i is (MHR 2), we see from Lemmas 11.4 and 12.2 (B) that $Y_{i-1} \xrightarrow{l_i,r_i} Y_i' \xrightarrow{\overline{r_i-1},\overline{l_i-1}} Y_i$ for some $2 \leq l_i \leq r_i \leq m+n$ with $l_i - 1 \notin I(Y_{i-1})$, $r_i \in I(Y_{i-1})$, and $Y_i' \in \mathcal{F}(Y_{m,n})$. Similarly, for $1 \leq i \leq p$ such that t_i is (MHR 1), there exists $2 \leq l_i \leq r_i \leq m+n$ with $l_i - 1 \notin I(Y_{i-1})$ and $r_i \in I(Y_{i-1})$ such that $Y_{i-1} \xrightarrow{l_i,r_i} Y_i$; we set $Y_i' := Y_i$ by convention. We show by induction on p that $I_R(Y_p) \cap I_R(Y_p^D) = \emptyset$. If $p = 0$, then it is obvious that $I_R(Y_{m,n}) \cap I_R(Y_{m,n}^D) = \emptyset$, since $I_R(Y_{m,n}) = \{n+1, n+2, \ldots, m+n\}$ and

$$I_R(Y_{m,n}^D) = I_R(\emptyset) = \begin{cases} \emptyset & \text{if } m < n, \\ \{m\} & \text{if } m = n. \end{cases}$$

Assume that $p > 0$; by the induction hypothesis,

$$I_R(Y_{p-1}) \cap I_R(Y_{p-1}^D) = \emptyset. \tag{12.1}$$

By Lemma 12.2 (B), we have

$$I_R(Y'_p) \setminus \{l_p - 1\} = I_R(Y_{p-1}) \setminus \{r_p\}, \tag{12.2}$$

$$I_R(Y'^D_p) \setminus \{\overline{l_p - 1}\} = I_R(Y_{p-1}^D) \setminus \{\overline{r_p}\}. \tag{12.3}$$

Lemma 12.4. It holds that $I_R(Y'_p) \cap I_R(Y'^D_p) \neq \emptyset$ if and only if $\overline{l_p - 1} \in I(Y_{p-1}) \setminus \{r_p\}$ or $l_p - 1 = \overline{l_p - 1}$; notice that $l_p - 1 = \overline{l_p - 1}$ if and only if $\chi = 0$ and $l_p - 1 = c + 1$.

Proof. Assume first that $l_p - 1 < c + 1 - \chi$; recall that $\overline{l_p - 1} > \overline{c + 1 - \chi} = c + 1 \geq c + 1 - \chi$. It follows from (12.2) and (12.3) that

$$I_R(Y'_p) = I_R(Y_{p-1}) \setminus \{r_p\}, \quad I_R(Y'^D_p) = (I_R(Y_{p-1}^D) \setminus \{\overline{r_p}\}) \cup \{\overline{l_p - 1}\}.$$

Because $I_R(Y_{p-1}) \cap I_R(Y_{p-1}^D) = \emptyset$ by the induction hypothesis, we see that $I_R(Y'_p) \cap I_R(Y'^D_p) \neq \emptyset$ if and only if $\overline{l_p - 1} \in I_R(Y_{p-1}) \setminus \{r_p\}$. Assume next that $l_p - 1 \geq c + 1 - \chi$. It follows from (12.2) and (12.3) that

$$I_R(Y'_p) = (I_R(Y_{p-1}) \setminus \{r_p\}) \cup \{l_p - 1\},$$

$$I_R(Y'^D_p) = \begin{cases} I_R(Y_{p-1}^D) \setminus \{\overline{r_p}\} & \text{if } \overline{l_p - 1} < c + 1 - \chi, \\ (I_R(Y_{p-1}^D) \setminus \{\overline{r_p}\}) \cup \{\overline{l_p - 1}\} & \text{if } \overline{l_p - 1} \geq c + 1 - \chi. \end{cases}$$

Here we note that $\overline{l_p - 1} \in I(Y_{p-1}) \setminus \{r_p\}$ if and only if $l_p - 1 \in I(Y_{p-1}^D) \setminus \{\overline{r_p}\}$ by Lemma 12.2 (A). If $\overline{l_p - 1} < c + 1 - \chi$ (resp., $\overline{l_p - 1} \geq c + 1 - \chi$), then it holds that $I_R(Y'_p) \cap I_R(Y'^D_p) \neq \emptyset$ if and only if $\overline{l_p - 1} \in I(Y_{p-1}) \setminus \{r_p\}$ (resp., $\overline{l_p - 1} \in I_R(Y_{p-1}) \setminus \{r_p\}$ or $l_p - 1 = \overline{l_p - 1}$). Thus we have proved the lemma. □

Proposition 12.5. (1) The operation t_p is (MHR 1) if and only if either of the following (a) or (b) holds.

(a) $\overline{l_p - 1} \notin I(Y_{p-1})$ and $l_p - 1 \neq \overline{l_p - 1}$.

(b) $l_p - 1 = \overline{r_p}$ (notice that $l_p - 1 \neq \overline{l_p - 1}$ also in this case since $l_p - 1 \neq r_p = \overline{l_p - 1}$).

(2) The operation t_p is (MHR 2) if and only if $\overline{l_p - 1} \in I(Y_{p-1}) \setminus \{r_p\}$ or $l_p - 1 = \overline{l_p - 1}$.

Proof. It suffices to show only part (2). We first show the "only if" part of (2). Assume that t_p is (MHR 2); recall that $Y_{p-1} \xrightarrow{l_p, r_p} Y'_p \xrightarrow{\overline{r_p - 1}, \overline{l_p - 1}} Y_p$. It follows from Lemma 12.2 (B) (applied to $Y = Y'_p$ and $Y\langle i, j \rangle = Y_p$) that $\overline{l_p - 1} \in I(Y'_p) = (I(Y_{p-1}) \setminus \{r_p\}) \cup \{l_p - 1\}$. Thus we have $\overline{l_p - 1} \in I(Y_{p-1}) \setminus \{r_p\}$ or $\overline{l_p - 1} = l_p - 1$. We next show the "if" part of (2); by Remark 12.3, and Lemmas 11.4 and 12.2 (B), it suffices to show that $\overline{r_p} \notin I(Y'_p)$ and $\overline{l_p - 1} \in I(Y'_p)$. Because $I(Y'_p) = (I(Y_{p-1}) \setminus \{r_p\}) \cup \{l_p - 1\}$, it is obvious from the assumption that $\overline{l_p - 1} \in I(Y'_p)$. Let us show that $\overline{r_p} \notin I(Y'_p)$. Suppose, for a contradiction, that $\overline{r_p} \in I(Y'_p)$. Since $I(Y'_p) = (I(Y_{p-1}) \setminus \{r_p\}) \cup \{l_p - 1\}$, and since $\overline{r_p} \neq l_p - 1$, we have $\overline{r_p} \in I(Y_{p-1}) \setminus \{r_p\} \subset I(Y_{p-1})$, and hence $r_p \in I(Y_{p-1}^D)$ by Lemma 12.2 (A). If $c + 1 - \chi \leq r_p$, then $r_p \in I_R(Y_{p-1}^D)$. Since $r_p \in I_R(Y_{p-1})$ by Lemma 12.2 (B) (applied to $Y_{p-1} \xrightarrow{l_p, r_p} Y'_p$), we get $r_p \in I_R(Y_{p-1}) \cap I_R(Y_{p-1}^D)$, which contradicts the induction hypothesis (12.1). If $c + 1 - \chi > r_p$, then $c + 1 - \chi \leq c + 1 = \overline{c + 1 - \chi} < \overline{r_p}$, which implies that $\overline{r_p} \in I_R(Y_{p-1})$. Since $r_p \in I(Y_{p-1})$, we have $\overline{r_p} \in I_R(Y_{p-1}^D)$ by Lemma 12.2 (A). Hence we get $\overline{r_p} \in I_R(Y_{p-1}) \cap I_R(Y_{p-1}^D)$, which contradicts the induction hypothesis (12.1). Therefore we obtain $\overline{r_p} \notin I(Y'_p)$, as desired. Thus we have proved the proposition. □

If t_p is (MHR 1) (recall that $Y'_p = Y_p$ and $Y^{D\prime}_p = Y^D_p$ in this case), then we see by Lemma 12.4 and Proposition 12.5 (1) that $I_R(Y_p) \cap I_R(Y^D_p) = \emptyset$. Assume that t_p is (MHR 2), or equivalently, $\overline{l_p - 1} \in I(Y_{p-1}) \setminus \{r_p\}$ or $l_p - 1 = \overline{l_p - 1}$ by Proposition 12.5 (2). Because $Y_{p-1} \xrightarrow{l_p, r_p} Y'_p \xrightarrow{\overline{r_p-1}, \overline{l_p-1}} Y_p$ in this case, it follows from Lemma 12.2 (B) that

$$I_R(Y_p) \setminus \{\overline{r_p}, l_p - 1\} = I_R(Y_{p-1}) \setminus \{r_p, \overline{l_p - 1}\}, \tag{12.4}$$

$$I_R(Y^D_p) \setminus \{r_p, \overline{l_p - 1}\} = I_R(Y^D_{p-1}) \setminus \{\overline{r_p}, l_p - 1\}. \tag{12.5}$$

Hence, by (12.4) and (12.5), together with the induction hypothesis (12.1), we obtain $I_R(Y_p) \cap I_R(Y^D_p) = \emptyset$. Thus we have proved (I) $\Rightarrow$ (III) in Theorem 12.1.

Conversely, we prove (III) $\Rightarrow$ (I), that is, $Y \in \mathcal{S}(Y_{m,n})$ if $I_R(Y) \cap I_R(Y^D) = \emptyset$. We show by (descending) induction on $\langle I(Y)\rangle := \sum_{i \in I(Y)} i$. It is obvious that $Y_{m,n} \in \mathcal{S}(Y_{m,n})$. Assume that $\langle I(Y)\rangle < \langle I(Y_{m,n})\rangle$. Since $I(Y_{m,n}) = [n+1, m+n]$, and $I(Y) \neq I(Y_{m,n})$ with $\#I(Y) = m$, there exists $r \notin I(Y)$ such that $n + 1 \leq r$. Also, there exists $l \leq r$ such that $l - 1 \in I(Y)$; note that $l - 1 < r$. Here we show that $\overline{l-1} \notin I(Y)$. Suppose, for a contradiction, that $\overline{l-1} \in I(Y)$. If $c + 1 - \chi \geq l - 1$, then $c + 1 - \chi \leq c + 1 = \overline{c + 1 - \chi} \leq \overline{l - 1}$, and hence $\overline{l-1} \in I_R(Y)$. By Lemma 12.2 (A) applied to $l - 1 \in I(Y)$, it follows that $\overline{l-1} \in I_R(Y^D)$. Thus we obtain $\overline{l-1} \in I_R(Y) \cap I_R(Y^D)$, which contradicts the assumption that $I_R(Y) \cap I_R(Y^D) = \emptyset$. If $c + 1 - \chi < l - 1$, then $l - 1 \in I_R(Y^D)$ because $\overline{l-1} \in I(Y)$. Since $l - 1 \in I_R(Y)$, we get $l - 1 \in I_R(Y) \cap I_R(Y^D)$, which contradicts the assumption that $I_R(Y) \cap I_R(Y^D) = \emptyset$. Therefore we obtain $\overline{l-1} \notin I(Y)$.

Proposition 12.6. Keep the setting above.

(1) If $\overline{r} \notin I(Y)$ or $\overline{r} = l - 1$, then there exists a (unique) Young diagram Y' such that $I(Y') = (I(Y) \setminus \{l-1\}) \cup \{r\}$ and $I(Y'^D) = (I(Y^D) \setminus \{\overline{l-1}\}) \cup \{\overline{r}\}$. Furthermore, $Y' \in \mathcal{S}(Y_{m,n})$, and $Y' \xrightarrow{\text{(MHR 1)}} Y$.
(2) If $\overline{r} \in I(Y)$ and $\overline{r} \neq l - 1$, then there exists a (unique) Young diagram Y'' such that $I(Y'') = (I(Y) \setminus \{\overline{r}, l-1\}) \cup \{r, \overline{l-1}\}$ and $I(Y''^D) = (I(Y^D) \setminus \{r, \overline{l-1}\}) \cup \{\overline{r}, l-1\}$. Furthermore, $Y'' \in \mathcal{S}(Y_{m,n})$, and $Y'' \xrightarrow{\text{(MHR 2)}} Y$.

Proof. (1) Recall that $l - 1 \in I(Y)$ and $r \notin I(Y)$, which implies that $(I(Y) \setminus \{l-1\}) \cup \{r\} \in \binom{[1,m+n]}{m}$. Since $I \colon \mathcal{F}(Y_{m,n}) \to \binom{[1,m+n]}{m}$ is a bijection, there exists unique $Y' \in \mathcal{F}(Y_{m,n})$ such that $I(Y') = (I(Y) \setminus \{l-1\}) \cup \{r\}$; note that $I(Y'^D) = (I(Y^D) \setminus \{\overline{l-1}\}) \cup \{\overline{r}\}$ by Lemma 12.2 (A). Then it follows from Lemma 12.2 (B) that $Y' \xrightarrow{l,r} Y$. Because $\overline{r} \notin I(Y)$ or $\overline{r} = l - 1$ by the assumption of (1), and $I_R(Y) \cap I_R(Y^D) = \emptyset$ by assumption, it can be easily verified that $I_R(Y') \cap I_R(Y'^D) = \emptyset$. Since $l - 1 < r$, we have $\langle I(Y')\rangle > \langle I(Y)\rangle$, and hence $Y' \in \mathcal{S}(Y_{m,n})$ by the induction hypothesis. Because $\overline{l-1} \notin I(Y)$, we see from Remark 12.3 that there does not exist a box $(i,j) \in Y$ such that $Y \xrightarrow{\overline{r-1}, \overline{l-1}} Y\langle i,j\rangle$. Thus we obtain $Y' \xrightarrow{\text{(MHR 1)}} Y$, as desired.
(2) Let Y' be as in the proof of part (1). Since $\overline{r} \in I(Y)$ and $\overline{r} \neq l - 1$ by the assumption of (2), and $\overline{l-1} \notin I(Y)$ as seen above,

$$(I(Y') \setminus \{\overline{r}\}) \cup \{\overline{l-1}\} = (I(Y) \setminus \{\overline{r}, l-1\}) \cup \{r, \overline{l-1}\} \in \binom{[1, m+n]}{m}.$$

Thus there exists $Y'' \in \mathcal{F}(Y_{m,n})$ such that $I(Y'') = (I(Y) \setminus \{\overline{r}, l-1\}) \cup \{r, \overline{l-1}\}$; note that $I(Y''^D) = (I(Y^D) \setminus \{r, \overline{l-1}\}) \cup \{\overline{r}, l-1\}$ by Lemma 12.2 (A). It follows from Lemma 12.2 (B) that $Y'' \xrightarrow{\overline{r-1}, \overline{l-1}} Y' \xrightarrow{l,r} Y$. Because $\overline{r} \in I(Y)$ and $\overline{r} \neq l - 1$ by the assumption of (2), and $I_R(Y) \cap I_R(Y^D) = \emptyset$ by assumption, it can be easily verified that $I_R(Y'') \cap I_R(Y''^D) = \emptyset$. Since $l - 1 < r$ and $\overline{l-1} > \overline{r}$, we have $\langle I(Y'')\rangle > \langle I(Y)\rangle$, and hence $Y'' \in \mathcal{S}(Y_{m,n})$ by the induction hypothesis. We see from Lemma 11.4 that $Y'' \xrightarrow{\text{(MHR 2)}} Y$, as desired. □

By Proposition 12.6, we obtain $Y \in \mathcal{S}(Y_{m,n})$. This completes the proof of (III) $\Rightarrow$ (I), and hence (I) $\Leftrightarrow$ (III). The equivalence (II) $\Leftrightarrow$ (III) follows from the equivalence (I) $\Leftrightarrow$ (III) since $I_R(Y^D) \cap I_R((Y^D)^D) = I_R(Y) \cap I_R(Y^D)$.

Finally, let us show the equivalence (III) $\Leftrightarrow$ (IV). Let $Y \in \mathcal{F}(Y_{m,n})$, and $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathcal{Y}_m(m+n)$ be such that $Y = Y_\lambda$. We first show (IV) $\Rightarrow$ (III). Obviously, if $I_R(Y) \cap I_R(Y^D) \neq \emptyset$, then $I(Y) \cap I(Y^D) \neq \emptyset$. It follows from Subsection 9.1 that

$$I(Y) = \{\lambda_p + m - p + 1 \mid 1 \leq p \leq m\},$$
$$I(Y^D) = \{n - \lambda_q + q \mid 1 \leq q \leq m\}.$$

Hence, $I(Y) \cap I(Y^D) \neq \emptyset$ if and only if $\lambda_i + m - i + 1 = n - \lambda_j + j$ (or equivalently, $\lambda_i + \lambda_j = n - m + i + j - 1$) for some $1 \leq i, j \leq m$. Thus we have shown (IV) $\Rightarrow$ (III).

We next show (III) $\Rightarrow$ (IV). Assume that $\lambda_i + \lambda_j = n - m + i + j - 1$ for some $1 \leq i, j \leq m$; we may assume that $i \leq j$. As seen above, we have $\lambda_i + m - i + 1 \in I(Y) \cap I(Y^D)$. Hence it suffices to show that if $\lambda_i + \lambda_j = n - m + i + j - 1$, then $\lambda_i + m - i + 1 \in [c + 1 - \chi, m + n]$. Indeed, suppose, for a contradiction, that $\lambda_i + m - i + 1 \notin [c + 1 - \chi, m + n]$. Then, $\lambda_i + m - i + 1 < c + 1 - \chi$ or $m + n < \lambda_i + m - i + 1$. Because $\lambda_i + m - i + 1 \leq n + m - i + 1 \leq n + m$, we get $\lambda_i + m - i + 1 < c + 1 - \chi$. Since $i \leq j$ (and hence $\lambda_i \geq \lambda_j$) and $\lambda_i < c - m - \chi + i$, we have $\lambda_i + \lambda_j \leq 2\lambda_i < (m + n - 1 + \chi) - 2m - 2\chi + 2i = n - m - \chi + 2i - 1 \leq n - m + i + j - 1 = \lambda_i + \lambda_j$, which is a contradiction. Therefore, we conclude that $\lambda_i + m - i + 1 \in [c + 1 - \chi, m + n]$. Thus we have shown (III) $\Rightarrow$ (IV), thereby completing the proof of (III) $\Leftrightarrow$ (IV).

13 Application.

Let $t \in \mathbb{N}_0$ and $m, n \in \mathbb{N}$ such that $t \leq m \leq n$. For $(\lambda_1, \dots, \lambda_t) \in \mathcal{Y}_t(t + n)$, we set

$$[\![\lambda_1, \dots, \lambda_t]\!] := (\lambda_1, \dots, \lambda_t, \lambda_{t+1}, \dots, \lambda_m) \in \mathcal{Y}_m(m + n),$$

with $\lambda_k := 0$ for $t + 1 \leq k \leq m$.

Theorem 13.1. Under the notation and setting above, $Y_{[\![\lambda_1,\dots,\lambda_t]\!]} \in \mathcal{S}(Y_{m,n})$ if and only if $Y_{(\lambda_1,\dots,\lambda_t)} \in \mathcal{S}(Y_{t,n-m+t})$. Moreover, the Grundy value of $Y_{[\![\lambda_1,\dots,\lambda_t]\!]} \in \mathcal{S}(Y_{m,n})$ is equal to the Grundy value of $Y_{(\lambda_1,\dots,\lambda_t)} \in \mathcal{S}(Y_{t,n-m+t})$.

Proof. Since $\lambda_k = 0$ for $t + 1 \leq k \leq m$, it follows from Theorem 12.1 that $Y_{[\![\lambda_1,\dots,\lambda_t]\!]} \in \mathcal{S}(Y_{m,n})$ if and only if $\lambda_i + \lambda_j \neq n - m + i + j - 1$ for all $1 \leq i \leq j \leq t$ and

$$\lambda_s \neq n - m + s + k - 1 \text{ for all } 1 \leq s \leq t \text{ and } t + 1 \leq k \leq m; \tag{13.1}$$

note that $0 \neq n - m + k + l - 1$ for all $t + 1 \leq k,\, l \leq m$ since $m \leq n$. Also, notice that (13.1) is equivalent to $\lambda_1 \leq n - m + t$. Therefore, we deduce that $Y_{[\![\lambda_1,\dots,\lambda_t]\!]} \in \mathcal{S}(Y_{m,n})$ if and only if $Y_{(\lambda_1,\dots,\lambda_t)} \in \mathcal{S}(Y_{t,n-m+t})$.

Next, we show the assertion on the Grundy values. Assume that $Y_{(\lambda_1,\dots,\lambda_t)} \in \mathcal{S}(Y_{t,n-m+t})$, or equivalently, $Y_{[\![\lambda_1,\dots,\lambda_t]\!]} \in \mathcal{S}(Y_{m,n})$. If $t = 0$ or $\lambda_1 = 0$, then $Y_{[\![\lambda_1,\dots,\lambda_t]\!]} = Y_{(\lambda_1,\dots,\lambda_t)} = \emptyset$ (the empty Young diagram). Thus, both the Grundy value of $Y_{[\![\lambda_1,\dots,\lambda_t]\!]} = \emptyset$ in $\mathcal{S}(Y_{m,n})$ and the Grundy value of $Y_{(\lambda_1,\dots,\lambda_t)} = \emptyset$ in $\mathcal{S}(Y_{t,n-m+t})$ are equal to 0. Assume that $1 \leq t$ and $1 \leq \lambda_1$. Since $m \leq n$ and $1 \leq t$, we get $m - t + 1 \leq n + t - 1$. Hence, we have $c\,(t, 1) = \min\,(1 - t + m, t - 1 + n) = m - t + 1$. Since $m - t + 1 \leq m + \lambda_1 - 1$, and since $\lambda_1 \leq n - m + t$ as seen above, we have $c\,(1, \lambda_1) = \min\,(\lambda_1 - 1 + m, 1 - \lambda_1 + n) \geq m - t + 1$. Thus, we obtain $\min\,\{c(p, q) \mid (p, q) \in Y_{[\![\lambda_1,\dots,\lambda_t]\!]}\} \geq m - t + 1$:
We notice that

(i) in $Y_{[\![\lambda_1,\dots,\lambda_t]\!]} \in \mathcal{S}(Y_{m,n})$ with the unimodal numbering $c\,(p, q)$ for $(p, q) \in Y_{[\![\lambda_1,\dots,\lambda_t]\!]}$, if we replace $c\,(p, q)$ by $c\,(p, q) - m + t$, then we get $Y_{(\lambda_1,\dots,\lambda_t)} \in \mathcal{S}(Y_{t,n-m+t})$ with the unimodal numbering;
(ii) in $Y_{(\lambda_1,\dots,\lambda_t)} \in \mathcal{S}(Y_{t,n-m+t})$ with the unimodal numbering $c'\,(p, q)$ for $(p, q) \in Y_{(\lambda_1,\dots,\lambda_t)}$, if we replace $c'\,(p, q)$ by $c'\,(p, q) + m - t$, then we get $Y_{[\![\lambda_1,\dots,\lambda_t]\!]} \in \mathcal{S}(Y_{m,n})$ with the unimodal numbering.

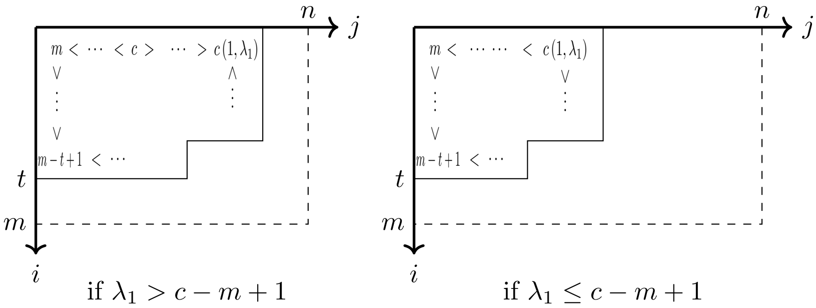
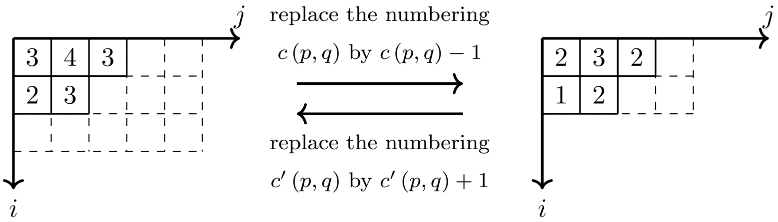


Figure 2. Numbering of $Y_{[\![\lambda_1,\ldots,\lambda_t]\!]}$ in $\mathcal{S}(Y_{m,n})$.

Here we give an example. Let $m = 3, n = 5$, and $t = 2$. Let $\lambda = (3, 2, 0) \in \mathcal{Y}_3(8)$. In $Y_{[\![3,2]\!]} \in \mathcal{S}(Y_{3,5})$ (resp., $Y_{(3,2)} \in \mathcal{S}(Y_{2,4})$) with the unimodal numbering $c\,(p, q)$ for $(p, q) \in Y_{[\![3,2]\!]}$ (resp., $c'\,(p, q)$ for $(p, q) \in Y_{(3,2)}$), if we replace $c\,(p, q)$ by $c\,(p, q) - 1$ (resp., $c'\,(p, q)$ by $c'\,(p, q) + 1$), then we get $Y_{(3,2)} \in \mathcal{S}(Y_{2,4})$ (resp., $Y_{[\![3,2]\!]} \in \mathcal{S}(Y_{3,5})$) with the unimodal numbering:



It is obvious that the operation (i) is the inverse of the operation (ii). Moreover, there exists a natural bijection between $\mathcal{O}(Y_{[\![\lambda_1,\ldots,\lambda_t]\!]}) \subset \mathcal{S}(Y_{m,n})$ and $\mathcal{O}(Y_{(\lambda_1,\ldots,\lambda_t)}) \subset \mathcal{S}(Y_{t,n-m+t})$. Then the inductive argument shows that the Grundy value of $Y_{[\![\lambda_1,\ldots,\lambda_t]\!]}$ in $\mathcal{S}(Y_{m,n})$ is equal to the Grundy value of $Y_{(\lambda_1,\ldots,\lambda_t)}$ in $\mathcal{S}(Y_{t,n-m+t})$. This completes the proof of Theorem 13.1. □

Assume that $m = 2$. Set $c_i(q) \coloneqq c + i + 4q$ for $i \in \mathbb{Z}$ and $q \geq 0$. We know from [1, Theorem 4.13] that a Young diagram $Y_\lambda \in \mathcal{S}(Y_{2,n})$ with $\lambda = (\lambda_1, \lambda_2)$ is a $\mathcal{P}$-position if and only if

$$\lambda \in \begin{cases} \mathcal{C} \cup \{(c_1(q), c_0(q)), (c_2(q), c_1(q)) \mid 0 \leq q \leq (p-1)\,/\,2\} & \text{if } n-2 = 4p, \\ \mathcal{C} \cup \{(c_2(q), c_1(q)), (c_3(q), c_2(q)) \mid 0 \leq q \leq (p-1)\,/\,2\} & \text{if } n-2 = 4p+1, \\ \mathcal{C} \cup \{(c_0(q), c_{-1}(q)), (c_1(q), c_0(q)) \mid 0 \leq q \leq p\,/\,2\} & \text{if } n-2 = 4p+2, \\ \mathcal{C} \cup \{(2p+4, 2p+2), (2p+5, 2p+4)\} & \\ \qquad \cup \{(c_1(q), c_0(q)), (c_2(q), c_1(q)) \mid 1 \leq q \leq p\,/\,2\} & \text{if } n-2 = 4p+3, \end{cases} \tag{13.2}$$

where $p \in \mathbb{N}_0$, and $\mathcal{C} = \mathcal{C}(p) \coloneqq \{(2q, 2q) \mid 0 \leq q \leq p\}$.

The following is an immediate consequence of Theorem 13.1 and (13.2).

Corollary 13.2. We set $d_i(q) \coloneqq c - m + 2 + i + 4q$ for $i \in \mathbb{Z}$ and $q \geq 0$. A Young diagram $Y_\lambda \in \mathcal{S}(Y_{m,n})$ having at most two rows is a $\mathcal{P}$-position if and only if

$$\lambda \in \begin{cases} \mathcal{D} \cup \{[\![d_1(q), d_0(q)]\!], [\![d_2(q), d_1(q)]\!] \mid 0 \leq q \leq (p-1)\,/\,2\} & \text{if } n-m = 4p, \\ \mathcal{D} \cup \{[\![d_2(q), d_1(q)]\!], [\![d_3(q), d_2(q)]\!] \mid 0 \leq q \leq (p-1)\,/\,2\} & \text{if } n-m = 4p+1, \\ \mathcal{D} \cup \{[\![d_0(q), d_{-1}(q)]\!], [\![d_1(q), d_0(q)]\!] \mid 0 \leq q \leq p\,/\,2\} & \text{if } n-m = 4p+2, \\ \mathcal{D} \cup \{[\![2p+4, 2p+2]\!], [\![2p+5, 2p+4]\!]\} & \\ \qquad \cup \{[\![d_1(q), d_0(q)]\!], [\![d_2(q), d_1(q)]\!] \mid 1 \leq q \leq p\,/\,2\} & \text{if } n-m = 4p+3, \end{cases}$$

where $p \in \mathbb{N}_0$, and $\mathcal{D} = \mathcal{D}(p) \coloneqq \{[\![2q, 2q]\!] \mid 0 \leq q \leq p\}$.

Part III

14 Preliminaries.

14.1 Basic notation.

For $a, b \in \mathbb{Z}$, we set $(-\infty, b] := \{x \in \mathbb{Z} \mid x \leq b\}$ and $[a, \infty) := \{x \in \mathbb{Z} \mid a \leq x\}$. Let $c \in \mathbb{N}$. For a subset X of $\mathbb{Z}$, we set $X - c := \{x - c \mid x \in X\}$. For a poset $P = (P, \leq)$, we denote by $P^* = (P, \leq^*)$ the dual poset of P. Namely, $P^* = P$ as sets, and $\leq^*$ is defined by: $u \leq^* v$ if $u \geq v$ for $u, v \in P^* = P$.

14.2 Pre-dominant integral weights.

In Appendix, $\mathfrak{g}$ is the infinite rank affine Lie algebra of type A_∞ over $\mathbb{C}$ associated to the following Dynkin diagram (see [3, Exercise 4.14]):

$$\cdots \;\text{———}\; \underset{-2}{\bullet} \;\text{———}\; \underset{-1}{\bullet} \;\text{———}\; \underset{0}{\bullet} \;\text{———}\; \underset{1}{\bullet} \;\text{———}\; \underset{2}{\bullet} \;\text{———}\; \cdots$$

Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$, $\Pi^\vee = \{\alpha_i^\vee \mid i \in \mathbb{Z}\} \subset \mathfrak{h}$ the set of simple coroots of $\mathfrak{g}$, and $\Pi = \{\alpha_i \mid i \in \mathbb{Z}\} \subset \mathfrak{h}^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ the set of simple roots of $\mathfrak{g}$; recall that $\mathfrak{h} = \oplus_{i \in \mathbb{Z}} \mathbb{C}\alpha_i$, and that $\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij}$ for $i, j \in \mathbb{Z}$, where

$$a_{ij} := \begin{cases} 2 & \text{if } \ i = j, \\ -1 & \text{if } \ |i - j| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

For each $i \in \mathbb{Z}$, define $\Lambda_i \in \mathfrak{h}^*$ by: $\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$ for $j \in \mathbb{Z}$. We denote by $W = \langle s_i \mid i \in \mathbb{Z} \rangle \subset GL(\mathfrak{h}^*)$ the Weyl group of $\mathfrak{g}$. Denote by Φ_+ the set of positive roots for $\mathfrak{g}$; recall that $\Phi_+ = \{\alpha_{p,q} := \alpha_p + \alpha_{p+1} + \cdots + \alpha_q \mid p \leq q\}$. For $\beta \in \Phi_+$, $\beta^\vee \in \mathfrak{h}$ denotes the coroot of β; note that $\alpha_{p,q}^\vee = \alpha_p^\vee + \alpha_{p+1}^\vee + \cdots + \alpha_{q-1}^\vee + \alpha_q^\vee$ for $p \leq q$. For each $\beta \in \Phi_+$, we define $s_\beta \in W$ by: $s_\beta(\mu) = \mu - \langle \mu, \beta^\vee \rangle \beta$ for $\mu \in \mathfrak{h}^*$. For each $w \in W$, we set $\Phi(w) := \{\gamma \in \Phi_+ \mid -w^{-1}(\gamma) \in \Phi_+\}$.

Remark 14.1 ([9, Chapter 5])**.** If $w = s_{i_1} \cdots s_{i_d}$ is a reduced expression of $w \in W$, then $\Phi(w) = \{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \ldots, s_{i_1} \cdots s_{i_{d-1}}(\alpha_{i_d})\}$.

Definition 14.2 ([11, Definitions 1 and 2])**.** An integral weight Λ is said to be *pre-dominant* if $\langle \Lambda, \beta^\vee \rangle \geq -1$ for all $\beta \in \Phi_+$. The set of pre-dominant integral weights is denoted by $P_{\geq -1}$. For $\Lambda \in P_{\geq -1}$, the set $D(\Lambda) := \{\beta \in \Phi_+ \mid \langle \Lambda, \beta^\vee \rangle = -1\}$ is called the *diagram* of Λ. We say that a pre-dominant integral weight Λ is *finite* (resp., *infinite*) if $\#D(\Lambda) < \infty$ (resp., $\#D(\Lambda) = \infty$). The set of finite (resp., infinite) pre-dominant integral weights is denoted by $P_{\geq -1}^{\mathrm{fin}}$ (resp., $P_{\geq -1}^{\mathrm{inf}}$).

Definition 14.3 ([11, Definition 6])**.** For $\Lambda \in P_{\geq -1}$ and $\beta \in D(\Lambda)$, the set $H_\Lambda(\beta) := D(\Lambda) \cap \Phi(s_\beta)$ is called the *hook at* β (*in the diagram* $D(\Lambda)$). The number $\#H_\Lambda(\beta)$ is called the *hook length at* β (*in the diagram* $D(\Lambda)$).

Remark 14.4. Define a partial order $\leq$ on $\mathfrak{h}^*$ by: $\alpha \leq \beta$ if $\beta - \alpha \in \sum_{i \in \mathbb{Z}} \mathbb{N}_0\, \alpha_i$. We regard $D(\Lambda)$ and $H_\Lambda(\beta)$ as subposets of $(\mathfrak{h}^*, \leq)$.

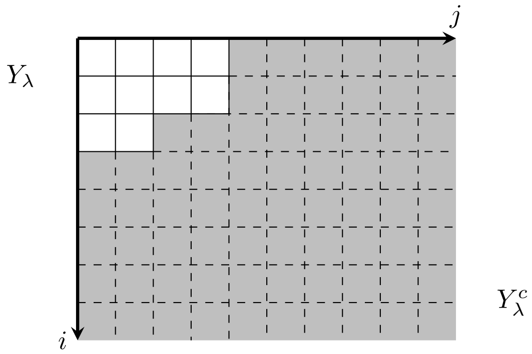
15 Complementary Young diagrams and hooks.

15.1 Complementary Young diagrams.

We define a partial order $\preceq$ on the set $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$ by: $(i, j) \preceq (i', j')$ if $i \geq i'$ and $j \geq j'$. For each partition λ, the Young diagram Y_λ is regarded as a subposet of $\mathbb{N}^2$ in this partial order $\preceq$ (see Subsection 9.1 and Example 15.2 below).

Definition 15.1. The poset $Y_\lambda^c := (\mathbb{N}^2 \setminus Y_\lambda, \preceq)$ is called the *complementary Young diagram of* Y_λ.

Example 15.2. If $\lambda = (4, 4, 2)$, then Y_λ and Y_λ^c are as follows:



15.2 Hooks.

Definition 15.3. Let λ be a partition, and Y_λ the corresponding Young diagram. For $(i,j) \in Y_\lambda$, we set

$$\mathrm{Arm}_{Y_\lambda}(i,j) := \{(i,j') \in Y_\lambda \mid j < j'\}, \quad \mathrm{Leg}_{Y_\lambda}(i,j) := \{(i',j) \in Y_\lambda \mid i < i'\};$$

note that $H_{Y_\lambda}(i,j) = \{(i,j)\} \sqcup \mathrm{Arm}_{Y_\lambda}(i,j) \sqcup \mathrm{Leg}_{Y_\lambda}(i,j)$. The number $h_{Y_\lambda}(i,j) := \# H_{Y_\lambda}(i,j)$ is called the *hook length at* (i,j) in Y_λ. For a subset A of Y_λ, we define $\widetilde{\mathcal{H}}_{Y_\lambda}(A)$ to be the multiset consisting of $h_{Y_\lambda}(i,j)$ for $(i,j) \in A$ (see Example 15.5 below).

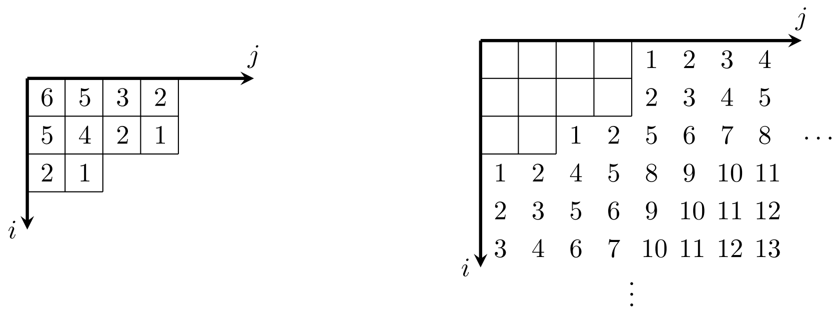
Definition 15.4. Let λ be a partition, Y_λ the corresponding Young diagram, and $Y_\lambda^c = \mathbb{N}^2 \setminus Y_\lambda$ the complementary Young diagram of Y_λ. For $(i,j) \in Y_\lambda^c$, we set

$$\mathrm{Arm}_{Y_\lambda^c}(i,j) := \{(i,j') \in Y_\lambda^c \mid j' < j\}, \quad \mathrm{Leg}_{Y_\lambda^c}(i,j) := \{(i',j) \in Y_\lambda^c \mid i' < i\},$$
$$H_{Y_\lambda^c}(i,j) := \{(i,j)\} \sqcup \mathrm{Arm}_{Y_\lambda^c}(i,j) \sqcup \mathrm{Leg}_{Y_\lambda^c}(i,j).$$

The subset $H_{Y_\lambda^c}(i,j)$ of Y_λ^c is called the *hook at* (i,j) in Y_λ^c, and the number $h_{Y_\lambda^c}(i,j) := \# H_{Y_\lambda^c}(i,j)$ is called the *hook length at* (i,j) in Y_λ^c. For a subset B of Y_λ^c, we define $\widetilde{\mathcal{H}}_{Y_\lambda^c}(B)$ to be the multiset consisting of $h_{Y_\lambda^c}(i,j)$ for $(i,j) \in B$ (see Example 15.5 below).

Example 15.5. If $\lambda = (4, 4, 2)$, then

$$\widetilde{\mathcal{H}}_{Y_\lambda}(Y_\lambda) = \{1,1,2,2,2,3,4,5,5,6\},$$
$$\widetilde{\mathcal{H}}_{Y_\lambda^c}(Y_\lambda^c) = \{1,1,1,2,2,2,2,2,3,3,3,3,4,4,4,4,4,5,5,5,5,5,5,5,6,\dots\}.$$



16 Description of the diagrams for pre-dominant integral weights.

Here we employ the notation in Subsection 9.1. Fix $p \leq q$. Let $\lambda \in \mathcal{Y}_p(q)$. Recall from Subsection 9.1 that there exists a bijection I from $\mathcal{F}(Y_{p,q-p})$ onto $\binom{[1,q]}{p}$. We set $i'_t := \lambda_{p-t+1} + t$ for $1 \leq t \leq p$. Then we get $I_\lambda = I(Y_\lambda) = \{i'_1 < \cdots < i'_p\} \in \binom{[1,q]}{p}$.

Remark 16.1 (see Example 16.2 below)**.** We set $J_\lambda := [1, q] \setminus I_\lambda$. It is obvious that

$$J_\lambda = \bigsqcup_{t=1}^{p+1} [i'_{t-1} + 1, i'_t - 1],$$

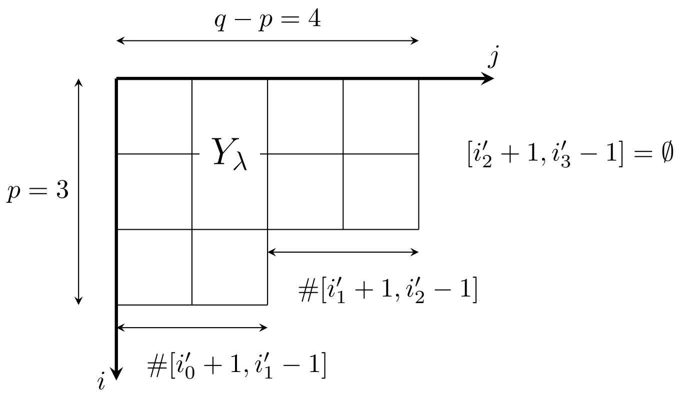
where we set $i'_0 := 0$ and $i'_{p+1} := q+1$ for convention; note that if $i'_t = i'_{t-1}+1$, then $[i'_{t-1}+1, i'_t - 1] = \emptyset$. By definition, we have $[i'_{t-1} + 1, i'_t - 1] = [\lambda_{p-t+2} + t, \lambda_{p-t+1} + t - 1]$ for $1 \leq t \leq p + 1$, where we set $\lambda_0 := q-p$ and $\lambda_{p+1} := 0$ for convention. Notice that $\#[i'_{t-1}+1, i'_t-1] = \#[\lambda_{p-t+2}+t, \lambda_{p-t+1}+t-1]$ is equal to the difference of the number of boxes in the $(p-t+1)$-th row (from the top) in the Young diagram Y_λ and that in the $(p-t+2)$-th row in the Young diagram Y_λ. In particular, for $1 \leq a \leq q-p = \#J_\lambda$, if the a-th smallest element in J_λ is less than or equal to $i'_b - 1$ for some $1 \leq b \leq p+1$, then

$$a \leq \sum_{u=1}^{b} \#[i'_{u-1} + 1, i'_u - 1] = \sum_{u=1}^{b} (\lambda_{p-u+1} - \lambda_{p-u+2}) = \lambda_{p-b+1}. \tag{16.1}$$

Similarly, if the a-th smallest element in J_λ is more than i'_{p-b+1} for some $1 \leq b \leq p+1$, then

$$a > \sum_{u=b}^{p} \#[i'_{p-u} + 1, i'_{p-u+1} - 1] = \sum_{u=b}^{p} (\lambda_u - \lambda_{u+1}) = \lambda_b. \tag{16.2}$$

Example 16.2. Assume that $p = 3$ and $q = 7$. Let $\lambda = (4, 4, 2) \in \mathcal{Y}_3(7)$. Then we have $I_\lambda = \{i'_1 = 3, i'_2 = 6, i'_3 = 7\}$, and $J_\lambda = [1, 7] \setminus I_\lambda = \{1, 2, 4, 5\}$.



Now, fix $k \in \mathbb{N}$, and a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $\lambda_k > 0$. Consider the following composition of bijections:

$$\mathcal{Y}_k(\lambda_1 + k) \longrightarrow \binom{[1, \lambda_1 + k]}{k} \xrightarrow{-k} \binom{[1-k, \lambda_1]}{k}, \ \mu \mapsto I_\mu \mapsto I_\mu - k. \tag{16.3}$$

Note that $\lambda \in \mathcal{Y}_k(\lambda_1 + k)$. Let $\widetilde{I}_\lambda := I_\lambda - k$ be the element in $\binom{[1-k,\lambda_1]}{k}$ corresponding to λ under the bijection above, and write it as: $\widetilde{I}_\lambda = \{i_1 < \cdots < i_k\}$. We set $\widetilde{J}_\lambda := [1-k, \lambda_1] \setminus \widetilde{I}_\lambda = J_\lambda - k$, and write it as: $\widetilde{J}_\lambda = \{j_1 < j_2 < \cdots < j_{\lambda_1}\}$. Note that $i_k = \lambda_1 \in \widetilde{I}_\lambda$ and $j_1 = 1-k \in \widetilde{J}_\lambda$. For each $1-k \leq j \leq \lambda_1 - 1$, we set

$$b_j := \begin{cases} -1 & \text{if } j \in \widetilde{I}_\lambda - 1, \\ 0 & \text{if } j \in \widetilde{J}_\lambda - 1, \end{cases} \qquad c_j := \begin{cases} 1 & \text{if } j \in \widetilde{I}_\lambda, \\ 0 & \text{if } j \in \widetilde{J}_\lambda. \end{cases} \tag{16.4}$$

We put

$$\Lambda_\lambda = \Lambda_{-k} + \sum_{j=1-k}^{\lambda_1 - 1} (b_j + c_j)\Lambda_j + \Lambda_{\lambda_1};$$

we can easily check that $\Lambda_\lambda \in P_{\geq -1}^{\mathrm{fin}}$.

Lemma 16.3. Keep the notation and setting above. It holds that

$$D(\Lambda_\lambda) = \{\alpha_{p,q} \mid p \in \widetilde{J}_\lambda, q \in \widetilde{I}_\lambda - 1, p \leq q\}. \tag{16.5}$$

Proof. Remark that if $1 - k \leq i \leq \lambda_1 - 2$, then $b_i + c_{i+1} = 0$. For $p \in \widetilde{J}_\lambda$, $q \in \widetilde{I}_\lambda - 1$ such that $(1 - k \leq)\ p \leq q\ (\leq \lambda_1 - 1)$, we have $c_p = 0$, $b_q = -1$. Hence it follows that

$$\begin{aligned} \langle \Lambda_\lambda, \alpha_{p,q}^\vee \rangle &= (b_p + c_p) + (b_{p+1} + c_{p+1}) + \cdots + (b_q + c_q) \\ &= c_p + \underbrace{(b_p + c_{p+1})}_{=0} + \cdots + \underbrace{(b_{q-1} + c_q)}_{=0} + b_q \\ &= -1. \end{aligned}$$

Conversely, asssume that $\beta = \alpha_{i,j} \in D(\Lambda_\lambda)$ for $i \leq j$. If $i \leq -k$ or $\lambda_1 \leq j$, then $\langle \Lambda_\lambda, \beta^\vee \rangle \geq 0$, which contradicts the assumption that $\beta \in D(\Lambda_\lambda)$. Thus we get $1 - k \leq i \leq j \leq \lambda_1 - 1$. It follows that

$$-1 = \langle \Lambda_\lambda, \beta^\vee \rangle = (b_i + c_i) + \cdots + (b_j + c_j) = c_i + b_j,$$

and hence $c_i = 0$ and $b_j = -1$. Therefore, by (16.4), we obtain $i \in \widetilde{J}_\lambda$ and $j \in \widetilde{I}_\lambda - 1$, as desired. □

We define a map $\varphi\colon D(\Lambda_\lambda) \to Y_\lambda$ as follows. Let $\alpha_{p,q} \in D(\Lambda_\lambda)$. Recall that $\widetilde{I}_\lambda = \{i_1 < i_2 < \cdots < i_k\}$ denotes the element $I_\lambda - k$ in $\binom{[1-k,\lambda_1]}{k}$ corresponding to λ under the bijection in (16.3), and that $\widetilde{J}_\lambda = [1 - k, \lambda_1] \setminus \widetilde{I}_\lambda = \{j_1 < j_2 < \cdots < j_{\lambda_1}\}$. It follows from Lemma 16.3 that $p = j_s$ for some $1 \leq s \leq \lambda_1$ and $q = i_t - 1$ for some $1 \leq t \leq k$, with $j_s = p \leq q = i_t - 1$. Then we set $\varphi(\alpha_{p,q}) = \varphi(\alpha_{j_s, i_t - 1}) := (k - t + 1, s) \in \mathbb{N}^2$. We claim that $(k - t + 1, s) \in Y_\lambda$. Indeed, since $1 \leq t \leq k$, it follows that $1 \leq k - t + 1 \leq k$. We show that $1 \leq s \leq \lambda_{k-t+1}$. Notice that $I_\lambda = \{i_1 + k < i_2 + k < \cdots < i_k + k\}$ and $J_\lambda = [1, \lambda_1 + k] \setminus I_\lambda$, and that $j_s + k$ is the s-th smallest element in J_λ. Because $j_s + k \leq i_t + k - 1$, we deduce from (16.1) that $1 \leq s \leq \lambda_{k-t+1}$.

Theorem 16.4. The map $\varphi\colon D(\Lambda_\lambda) \to Y_\lambda$, $\alpha_{j_s, i_t - 1} \mapsto (k - t + 1, s)$, is an order isomorphism, and preserves the hooks in the sense that

$$\varphi(H_{\Lambda_\lambda}(\beta)) = H_{Y_\lambda}(\varphi(\beta)) \text{ for all } \beta \in D(\Lambda_\lambda).$$

Proof. First we prove that $\varphi\colon D(\Lambda_\lambda) \to Y_\lambda$, $\alpha_{j_s, i_t - 1} \mapsto (k - t + 1, s)$, is an order isomorphism, that is, φ is bijective, and for $\alpha, \beta \in D(\Lambda_\lambda)$, $\alpha \leq \beta$ if and only if $\varphi(\alpha) \preceq \varphi(\beta)$. For the bijectivity of φ, since it is obvious that φ is injective, it suffices to show that $\#D(\Lambda_\lambda) = |\lambda|\ (= \#Y_\lambda)$. We compute

$$\begin{aligned} \#D(\Lambda_\lambda) &= \#\{\alpha_{p,q} \mid p \in \widetilde{J}_\lambda, q \in \widetilde{I}_\lambda - 1, p \leq q\} \\ &= \sum_{j=1}^{k} \#(\widetilde{J}_\lambda \cap [1 - k, i_j - 1]) = \sum_{j=1}^{k} \#([1 - k, i_j - 1] \setminus \{i_1, i_2, \ldots, i_{j-1}\}) \\ &= \sum_{j=1}^{k} \{(i_j - 1) - (1 - k) + 1 - (j - 1)\} = \sum_{j=1}^{k} (i_j + k - j) \end{aligned}$$

$$= \sum_{j=1}^{k} \lambda_{k-j+1} = |\lambda|.$$

Let us show that φ preserves the orderings. Let $\alpha, \beta \in D(\Lambda_\lambda)$, and write them as $\alpha = \alpha_{j_s,i_t-1}$ and $\beta = \alpha_{j_u,i_v-1}$, respectively. Then,

$$\begin{aligned} &\alpha_{j_s,i_t-1} \le \alpha_{j_u,i_v-1} \iff j_u \le j_s \text{ and } i_t \le i_v \\ \iff\ & u \le s \text{ and } t \le v \iff u \le s \text{ and } k-v+1 \le k-t+1 \\ \iff\ & \varphi(\alpha_{j_s,i_t-1}) = (k-t+1, s) \preceq (k-v+1, u) = \varphi(\alpha_{j_u,i_v-1}). \end{aligned}$$

Thus, we have proved that φ is an order isomorphism.

We prove that φ preserves the hooks. By Remark 14.1, we get

$$\begin{aligned} H_{\Lambda_\lambda}(\alpha_{j_s,i_t-1}) &= D(\Lambda_\lambda) \cap \Phi(s_{\alpha_{j_s,i_t-1}}) \\ &= D(\Lambda_\lambda) \cap \{\underbrace{\alpha_{j_s,j_s}, \alpha_{j_s,j_s+1}, \dots, \alpha_{j_s,i_t-2}}_{=:L}, \alpha_{j_s,i_t-1}, \\ &\qquad \underbrace{\alpha_{i_t-1,i_t-1}, \alpha_{i_t-2,i_t-1}, \dots, \alpha_{j_s+1,i_t-1}}_{=:A}\}. \end{aligned}$$

We show that $\varphi(D(\Lambda_\lambda) \cap L) = \mathrm{Leg}_{Y_\lambda}(k-t+1, s)$. If $\beta \in D(\Lambda_\lambda) \cap L$, then it follows from Lemma 16.3 that $\beta = \alpha_{j_s,i_u-1}$ for some $1 \le u < t$ with $j_s \le i_u - 1$. In this case, $\varphi(\beta) = (k-u+1, s) \in Y_\lambda$ by the definition of $\varphi \colon D(\Lambda_\lambda) \to Y_\lambda$. Since $k-u+1 > k-t+1$, we obtain $\varphi(\beta) \in \mathrm{Leg}_{Y_\lambda}(k-t+1, s)$. Hence, $\varphi(D(\Lambda_\lambda) \cap L) \subset \mathrm{Leg}_{Y_\lambda}(k-t+1, s)$. For the reverse inclusion, let $(x, y) \in \mathrm{Leg}_{Y_\lambda}(k-t+1, s)$. Then we have $y = s$, and $x = k-u+1$ for some $1 \le u < t$. Because $\varphi \colon D(\Lambda_\lambda) \to Y_\lambda$ is bijective, there exists unique $\gamma \in D(\Lambda_\lambda)$ such that $\varphi(\gamma) = (x, y) = (k-u+1, s)$. If we write $\gamma = \alpha_{j_a,i_b-1}$ for some $j_a \in \widetilde{J}_\lambda$ and $i_b - 1 \in \widetilde{I}_\lambda - 1$ such that $j_a \le i_b - 1$ (see Lemma 16.3), then we have $\varphi(\gamma) = (k-b+1, a)$. Thus we get $b = u$ and $a = s$, and hence $\gamma = \alpha_{j_s,i_u-1}$ with $j_s = j_a \le i_b - 1 = i_u - 1$, which implies that $\gamma \in L$. Therefore, we obtain $\gamma \in D(\Lambda_\lambda) \cap L$, and hence $(x, y) = \varphi(\gamma) \in \varphi(D(\Lambda_\lambda) \cap L)$. This proves $\varphi(D(\Lambda_\lambda) \cap L) \supset \mathrm{Leg}_{Y_\lambda}(k-t+1, s)$, and hence $\varphi(D(\Lambda_\lambda) \cap L) = \mathrm{Leg}_{Y_\lambda}(k-t+1, s)$. Similarly, we can show that $\varphi(D(\Lambda_\lambda) \cap A) = \mathrm{Arm}_{Y_\lambda}(k-t+1, s)$. Therefore we conclude that $\varphi(H_{\Lambda_\lambda}(\alpha_{j_s,i_t-1})) = \varphi(D(\Lambda_\lambda) \cap \Phi(s_{\alpha_{j_s,i_t-1}})) = \varphi(D(\Lambda_\lambda) \cap L) \sqcup \varphi(D(\Lambda_\lambda) \cap \{\alpha_{j_s,i_t-1}\}) \sqcup \varphi(D(\Lambda_\lambda) \cap A) = \mathrm{Leg}_{Y_\lambda}(k-t+1, s) \sqcup \{(k-t+1, s)\} \sqcup \mathrm{Arm}_{Y_\lambda}(k-t+1, s) = H_{Y_\lambda}(k-t+1, s) = H_{Y_\lambda}(\varphi(\alpha_{j_s,i_t-1}))$. This completes the proof of Theorem 16.4. □

We put $\Lambda_\lambda^{\mathrm{c}} = -\Lambda_\lambda$; we can easily check that $\Lambda_\lambda^{\mathrm{c}} \in P_{\ge -1}^{\mathrm{inf}}$. Recall that $\widetilde{I}_\lambda = I_\lambda - k = \{i_1 < i_2 < \cdots < i_k\}$, and $\widetilde{J}_\lambda = [1-k, \lambda_1] \setminus \widetilde{I}_\lambda = \{j_1 < j_2 < \cdots < j_{\lambda_1}\}$. We set $I_{\mathrm{row}} := (-\infty, -k] \sqcup \widetilde{I}_\lambda$ and $I_{\mathrm{col}} := [\lambda_1, \infty) \sqcup (\widetilde{J}_\lambda - 1)$, and write them as $I_{\mathrm{row}} = \{x_1 > x_2 > \cdots\}$ and $I_{\mathrm{col}} = \{y_1 < y_2 < \cdots\}$, respectively; note that $x_s = i_{k-s+1}$ for $1 \le s \le k$, $y_t = j_t - 1$ for $1 \le t \le \lambda_1$.

Lemma 16.5. It holds that

$$D(\Lambda_\lambda^{\mathrm{c}}) = \{\alpha_{p,q} \mid p \in I_{\mathrm{row}}, q \in I_{\mathrm{col}}, p \le q\}. \tag{16.6}$$

Proof. Note that $c_{1-k} = 0$ and $b_{\lambda_1 - 1} = -1$ since $1-k \in \widetilde{J}_\lambda$ and $\lambda_1 \in \widetilde{I}_\lambda$. Let us show that $\alpha_{p,q} \in D(\Lambda_\lambda)$ for $p \in I_{\mathrm{row}}$ and $q \in I_{\mathrm{col}}$ such that $p \le q$. We give a proof only for the case that $p \in \widetilde{I}_\lambda$ and $q \in \widetilde{J}_\lambda - 1$; the proofs for the other cases are similar. Because $p \le q$, it follows that $p \ne \lambda_1$ and $q \ne -k$. Since $c_p = 1$ and $b_q = 0$, we have $\langle \Lambda_\lambda^{\mathrm{c}}, \alpha_{p,q}^\vee \rangle = -(c_p + b_q) = -1$. Thus, $\alpha_{p,q} \in D(\Lambda_\lambda)$.

Conversely, let us show that if $\beta = \alpha_{i,j} \in D(\Lambda_\lambda^{\mathrm{c}})$ with $i \le j$, then $i \in I_{\mathrm{row}}$ and $j \in I_{\mathrm{col}}$. We give a proof only for the case that $i \in [1-k, \lambda_1 - 1]$; the proof for the case that $i \notin [1-k, \lambda_1 - 1]$ is simpler. If $j \in [1-k, \lambda_1 - 1]$, then $-1 = \langle \Lambda_\lambda^{\mathrm{c}}, \alpha_{i,j}^\vee \rangle = -(c_i + b_j)$. Hence we get $c_i = 1$ and $b_j = 0$. Therefore, we have $i \in \widetilde{I}_\lambda \subset I_{\mathrm{row}}$ and $j \in \widetilde{J}_\lambda - 1 \subset I_{\mathrm{col}}$. If $j \in [\lambda_1, \infty)$, then $-1 = \langle \Lambda_\lambda^{\mathrm{c}}, \alpha_{i,j}^\vee \rangle = -(c_i + b_{\lambda_1 - 1}) - 1 = -c_i$. Hence we get $c_i = 1$. Therefore, we have $i \in \widetilde{I}_\lambda \subset I_{\mathrm{row}}$ and $j \in [\lambda_1, \infty) \subset I_{\mathrm{col}}$. Thus we have proved the lemma. □

Here we define a map $\varphi^c\colon D(\Lambda_\lambda^c)^* \to Y_\lambda^c$ as follows. Let $\alpha_{p,q} \in D(\Lambda_\lambda^c)^*$. Recall that $I_{\mathrm{row}} = \{x_1 > x_2 > \cdots\}$ and $I_{\mathrm{col}} = \{y_1 < y_2 < \cdots\}$. It follows from Lemma 16.5 that $p = x_s$ for some $1 \le s$ and $q = y_t$ for some $1 \le t$ with $x_s = p \le q = y_t$. Then we set $\varphi^c(\alpha_{p,q}) = \varphi^c(\alpha_{x_s,y_t}) := (s,t) \in \mathbb{N}^2$. We claim that $(s,t) \in Y_\lambda^c$. Indeed, observe that $(s,t) \in Y_\lambda^c$ if and only if $s > k$ or $t > \lambda_s$. Hence it suffices to show that if $x_s \le y_t$ and $1 \le s \le k$, then $t > \lambda_s$. Recall that $I_{\mathrm{col}} = [\lambda_1, \infty) \sqcup (\widetilde{J}_\lambda - 1)$. If $y_t \in [\lambda_1, \infty)$, then it follows from $\#(\widetilde{J}_\lambda - 1) = \lambda_1$ that $t > \lambda_1 \ge \lambda_s$. Assume that $y_t \in \widetilde{J}_\lambda - 1$. Since $1 \le s \le k$ and $y_t \in \widetilde{J}_\lambda - 1$, we can write x_s and y_t as $x_s = i_{k-s+1}$ and $y_t = j_t - 1$. Notice that $I_\lambda = \{i_1 + k < \cdots < i_k + k\}$ and $J_\lambda = [1, \lambda_1 + k] \setminus I_\lambda$, and that $j_t + k$ is the t-th smallest element in J_λ. Because $j_t + k = y_t + 1 + k > x_s + k = i_{k-s+1} + k$, we deduce from (16.2) that $t > \lambda_s$.

Theorem 16.6. The map $\varphi^c\colon D(\Lambda_\lambda^c)^* \to Y_\lambda^c$, $\alpha_{x_s,y_t} \mapsto (s,t)$, is an order isomorphism, and preserves the hooks in the sense that

$$\varphi^c(H_{\Lambda_\lambda^c}(\beta)^*) = H_{Y_\lambda^c}(\varphi^c(\beta)) \text{ for all } \beta \in D(\Lambda_\lambda^c)^*.$$

Proof. We prove that $\varphi^c\colon D(\Lambda_\lambda^c)^* \to Y_\lambda^c$, $\alpha_{x_s,y_t} \mapsto (s,t)$, is an order isomorphism, that is, φ^c is bijective, and for $\alpha, \beta \in D(\Lambda_\lambda^c)^*$, $\alpha \le^* \beta$ if and only if $\varphi^c(\alpha) \preceq \varphi^c(\beta)$. First, let us show that φ^c is bijective. It is obvious that φ^c is injective. Hence, we show that φ^c is surjective. Let $(s,t) \in Y_\lambda^c$. By Lemma 16.5 and the definition of the map φ^c, it suffices to show that $x_s \le y_t$. Recall that $(s,t) \in Y_\lambda^c$ if and only if $s > k$ or $t > \lambda_s$. If $s > k$, then it follows from $\#\widetilde{I}_\lambda = k$ that $x_s \in (-\infty, -k]$. Since $j_1 - 1 = -k \le y_t$, it follows that $x_s \le y_t$. Assume that $1 \le s \le k$ and $t > \lambda_s$. If $y_t \in [\lambda_1, \infty)$, then it is obvious that $x_s \le y_t$. Assume that $y_t \in \widetilde{J}_\lambda - 1$. Since $1 \le s \le k$ and $y_t \in \widetilde{J}_\lambda - 1$, we can write x_s and y_t as $x_s = i_{k-s+1}$ and $y_t = j_t - 1$. Suppose, for a contradiction, that $x_s > y_t$. Since $i_{k-s+1} + k \ne j_t + k$, we obtain $i_{k-s+1} + k - 1 = x_s + k - 1 \ge y_t + 1 + k = j_t + k$. Notice that $I_\lambda = \{i_1 + k < \cdots < i_k + k\}$ and $J_\lambda = [1, \lambda_1 + k] \setminus I_\lambda = \{j_1 + k < \cdots < j_{\lambda_1} + k\}$. By (16.1), we get $\lambda_s \ge t$. This contradicts the assumption that $t > \lambda_s$. Next, let us show that φ^c preserves the orderings. Let $\alpha, \beta \in D(\Lambda_\lambda^c)^*$, and write them as $\alpha = \alpha_{x_s,y_t}$ and $\beta = \alpha_{x_u,y_v}$, respectively. Then,

$$\begin{aligned}
&\alpha_{x_s,y_t} \le^* \alpha_{x_u,y_v} \iff \alpha_{x_s,y_t} \ge \alpha_{x_u,y_v} \\
\iff& x_s \le x_u \text{ and } y_t \ge y_v \iff s \ge u \text{ and } t \ge v \\
\iff& \varphi^c(\alpha_{x_s,y_t}) = (s,t) \preceq (u,v) = \varphi^c(\alpha_{x_u,y_v}).
\end{aligned}$$

Therefore, we have proved that φ^c is an order isomorphism.

We prove that φ^c preserves the hooks. By Remark 14.1, we get

$$\begin{aligned}
H_{\Lambda_\lambda^c}(\alpha_{x_s,y_t})^* &= D(\Lambda_\lambda^c)^* \cap \Phi(s_{\alpha_{x_s,y_t}}) \\
&= D(\Lambda_\lambda^c)^* \cap \{\underbrace{\alpha_{x_s,x_s}, \alpha_{x_s,x_s+1}, \ldots, \alpha_{x_s,y_t-1}}_{=:A^c}, \alpha_{x_s,y_t}, \\
&\qquad \underbrace{\alpha_{y_t,y_t}, \alpha_{y_t-1,y_t}, \ldots, \alpha_{x_s+1,y_t}}_{=:L^c}\}.
\end{aligned}$$

We show that $\varphi^c(D(\Lambda_\lambda^c)^* \cap A^c) = \mathrm{Arm}_{Y_\lambda^c}(s,t)$. If $\beta \in D(\Lambda_\lambda^c)^* \cap A^c$, then it follows from Lemma 16.5 that $\beta = \alpha_{x_s,y_u}$ for some $1 \le u < t$ with $x_s \le y_u$; in this case, $\varphi^c(\beta) = (s,u) \in Y_\lambda^c$ by the definition of $\varphi^c\colon D(\Lambda_\lambda^c)^* \to Y_\lambda^c$. Since $1 \le u < t$, we obtain $\varphi^c(\beta) \in \mathrm{Arm}_{Y_\lambda^c}(s,t)$. Hence, $\varphi^c(D(\Lambda_\lambda^c)^* \cap A^c) \subset \mathrm{Arm}_{Y_\lambda^c}(s,t)$. For the reverse inclusion, let $(x,y) \in \mathrm{Arm}_{Y_\lambda^c}(s,t)$. Then we have $x = s$, and $y = u$ for some $1 \le u < t$. Because $\varphi^c\colon D(\Lambda_\lambda^c)^* \to Y_\lambda^c$ is bijective, there exists unique $\gamma \in D(\Lambda_\lambda^c)^*$ such that $\varphi^c(\gamma) = (x,y) = (s,u)$. If we write $\gamma = \alpha_{x_a,y_b}$ for some $x_a \in I_{\mathrm{row}}$ and $y_b \in I_{\mathrm{col}}$ such that $x_a \le y_b$ (see Lemma 16.5), then we have $\varphi^c(\gamma) = (a,b)$. Thus we get $a = s$ and $b = u$, and hence $\gamma = \alpha_{x_s,y_u}$ with $x_s = x_a \le y_b = y_u$, which implies that $\gamma \in A^c$. Therefore, we obtain $\gamma \in D(\Lambda_\lambda^c)^* \cap A^c$, and hence $(x,y) = \varphi^c(\gamma) \in \varphi^c(D(\Lambda_\lambda^c)^* \cap A^c)$. This proves $\varphi^c(D(\Lambda_\lambda^c)^* \cap A^c) \supset \mathrm{Arm}_{Y_\lambda^c}(s,t)$, and hence $\varphi^c(D(\Lambda_\lambda^c)^* \cap A^c) = \mathrm{Arm}_{Y_\lambda^c}(s,t)$. Similarly, we can show that $\varphi^c(D(\Lambda_\lambda^c)^* \cap L^c) = \mathrm{Leg}_{Y_\lambda^c}(s,t)$. Therefore we conclude that

$$\varphi^c(H_{\Lambda_\lambda}(\alpha_{x_s,y_t})^*) = \varphi^c(D(\Lambda_\lambda^c)^* \cap \Phi(s_{\alpha_{x_s,y_t}}))$$

$$
\begin{aligned}
&= \varphi^c(D(\Lambda_\lambda^c)^* \cap A^c) \sqcup \varphi^c(D(\Lambda_\lambda^c)^* \cap \{\alpha_{x_s,y_t}\}) \sqcup \varphi^c(D(\Lambda_\lambda^c)^* \cap L^c) \\
&= \mathrm{Arm}_{Y_\lambda^c}(s,t) \sqcup \{(s,t)\} \sqcup \mathrm{Leg}_{Y_\lambda^c}(s,t) \\
&= H_{Y_\lambda^c}(s,t) \\
&= H_{Y_\lambda^c}(\varphi^c(\alpha_{x_s,y_t})).
\end{aligned}
$$

This completes the proof of Theorem 16.6. □

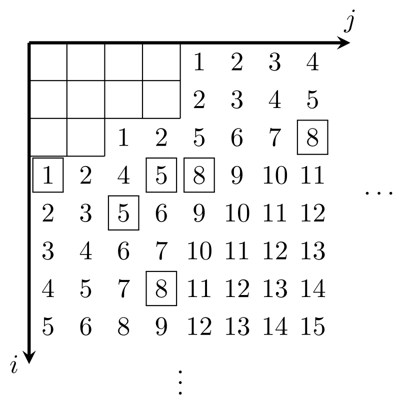
17 Hook length sequences.

Definition 17.1. Let λ be a partition. For a subset A of Y_λ (resp., Y_λ^c), the sequence $(a_n)_{n=1}^\infty$ defined by

$$
a_n := \#\{(i,j) \in A \mid h_{Y_\lambda}(i,j) = n \text{ (resp., } h_{Y_\lambda^c}(i,j) = n)\}
$$

is called the *hook length sequence of* A *in* Y_λ (resp., *in* Y_λ^c).

Example 17.2. Let $\lambda = (4,4,2)$. It can be easily seen that the hook length sequence of Y_λ is $(2,3,1,1,2,1,0,0,0,\ldots)$. Also, the hook length of each box in Y_λ^c is given as follows:



Therefore, the hook length sequence of Y_λ^c is $(3,5,4,5,7,7,7,8,9,\ldots)$. If $A = \{(3,8),(4,1),(4,4),(4,5),(5,3),(7,4)\} \subset Y_\lambda^c$, then the hook length sequence of A (in Y_λ^c) is $(1,0,0,0,2,0,0,3,0,\ldots)$.

Theorem 17.3. Let $(p_n)_{n=1}^\infty$ be the hook length sequence of Y_λ, and let $(q_n)_{n=1}^\infty$ be the hook length sequence of Y_λ^c. Then,

$$
n + p_n = q_n \quad \text{for all } n, \tag{17.1}
$$

or equivalently,

$$
\mathrm{E} + (p_n)_{n=1}^\infty = (q_n)_{n=1}^\infty, \tag{17.2}
$$

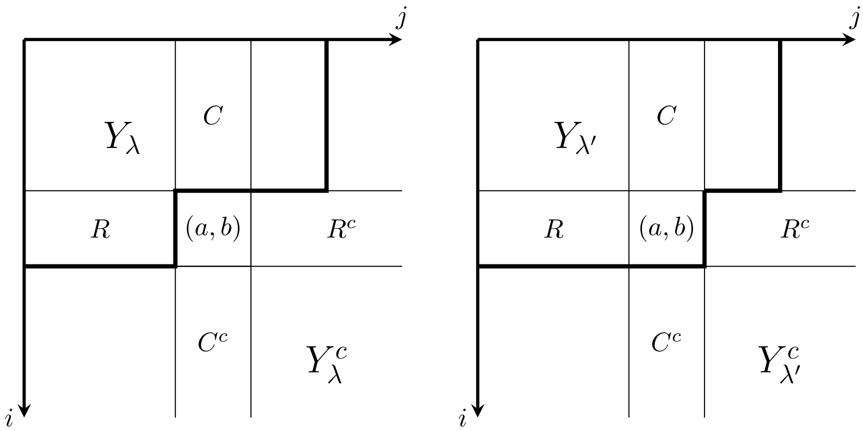
where $\mathrm{E} = (1,2,3,\ldots)$ is the hook length sequence of $Y_\emptyset^c$, where $\emptyset$ is the empty partition.

Example 17.4 (see Example 17.2)**.** For $\lambda = (4,4,2)$, we have

$$
\underbrace{(1,2,3,4,5,6,7,8,9,\ldots)}_{=\,\mathrm{E}} + \underbrace{(2,3,1,1,2,1,0,0,0,\ldots)}_{=\,(p_n)_{n=1}^\infty} = \underbrace{(3,5,4,5,7,7,7,8,9,\ldots)}_{=\,(q_n)_{n=1}^\infty}.
$$

Proof of Theorem 17.3. We can show (17.1) by induction on $m := |\lambda| = \#Y_\lambda$ (showing some lemmas in the proof). If $m = 0$, then (17.1) is obvious. Assume that $m \geq 0$. Let $Y_{\lambda'}$ be the Young diagram obtained from Y_λ by adding a box at (a,b). We set

$$
R := \{(a,j) \mid 1 \leq j < b\}\ ,\ C := \{(i,b) \mid 1 \leq i < a\},
$$
$$
R^c := \{(a,j) \mid b < j\}\ ,\ C^c := \{(i,b) \mid a < i\}.
$$



Recall from Definitions 15.3 and 15.4 that $\widetilde{\mathcal{H}}_{Y_\lambda}(R)$, $\widetilde{\mathcal{H}}_{Y^c_{\lambda'}}(C^c)$, $\widetilde{\mathcal{H}}_{Y_\lambda}(C)$, and $\widetilde{\mathcal{H}}_{Y^c_{\lambda'}}(R^c)$ are the multisets of hook lengths. We set $\widetilde{\mathcal{H}}' \coloneqq \widetilde{\mathcal{H}}_{Y_\lambda}(R) \cup \widetilde{\mathcal{H}}_{Y^c_{\lambda'}}(C^c)$ and $\widetilde{\mathcal{H}}'' \coloneqq \widetilde{\mathcal{H}}_{Y_\lambda}(C) \cup \widetilde{\mathcal{H}}_{Y^c_{\lambda'}}(R^c)$ (the unions of multisets).

Lemma 17.5. For each $n \in \mathbb{N}$, the multiset $\widetilde{\mathcal{H}}'$ contains exactly one n. Similarly, the multiset $\widetilde{\mathcal{H}}''$ contains exactly one n.

Proof. We give a proof only for the former assertion; the proof for the latter assertion is similar. We put $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $\lambda_k > 0$. Set $T \coloneqq \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid 1 \le i \le k+2, 1 \le j \le b\}$. By [25, Lemma 2], we have $\widetilde{\mathcal{H}}_{Y_\lambda}(R) \cup \widetilde{\mathcal{H}}_{Y^c_{\lambda'}}(C^c \cap T) = \{1, 2, \ldots, k-a+b+1\} = [1, k-a+b+1]$. Note that if $(i,j) \in C^c \setminus T$, then $h_{Y^c_{\lambda'}}(i,j) = h_{Y^c_{\lambda'}}(i-1,j) + 1$, and $k-a+b+1 \in \widetilde{\mathcal{H}}_{Y^c_{\lambda'}}(C^c \cap T)$. Hence we have $\widetilde{\mathcal{H}}_{Y^c_{\lambda'}}(C^c \setminus T) = \{k-a+b+2, k-a+b+3, \ldots\} = [k-a+b+2, \infty)$. It is obvious that $\widetilde{\mathcal{H}}_{Y^c_{\lambda'}}(C^c) = \widetilde{\mathcal{H}}_{Y^c_{\lambda'}}(C^c \cap T) \cup \widetilde{\mathcal{H}}_{Y^c_{\lambda'}}(C^c \setminus T)$. Combining these equalities, we obtain

$$\begin{aligned}\widetilde{\mathcal{H}}_{Y_\lambda}(R) \cup \widetilde{\mathcal{H}}_{Y^c_{\lambda'}}(C^c) &= \widetilde{\mathcal{H}}_{Y_\lambda}(R) \cup (\widetilde{\mathcal{H}}_{Y^c_{\lambda'}}(C^c \cap T) \cup \widetilde{\mathcal{H}}_{Y^c_{\lambda'}}(C^c \setminus T)) \\ &= [1, k-a+b+1] \cup [k-a+b+2, \infty) = [1, \infty) = \mathbb{N},\end{aligned}$$

as desired. ☐

Let $(x_n)_{n=1}^\infty$ (resp., $(x^c_n)_{n=1}^\infty$) be the hook length sequence of R in Y_λ (resp., R^c in Y^c_λ), and let $(y_n)_{n=1}^\infty$ (resp., $(y^c_n)_{n=1}^\infty$) be the hook length sequence of C in Y_λ (resp., C^c in Y^c_λ). We set $Z \coloneqq Y_\lambda \setminus (R \cup C)$ and $Z^c \coloneqq Y^c_\lambda \setminus (R^c \cup C^c \cup \{(a,b)\})$. Let $(z_n)_{n=1}^\infty$ (resp., $(z^c_n)_{n=1}^\infty$) be the hook length sequence of Z in Y_λ (resp., Z^c in Y^c_λ). Observe that the hook length sequence of R in $Y_{\lambda'}$ (resp., C in $Y_{\lambda'}$, R^c in $Y^c_{\lambda'}$, and C^c in $Y^c_{\lambda'}$) is equal to $(x_{n-1})_{n=1}^\infty$ (resp., $(y_{n-1})_{n=1}^\infty$, $(x^c_{n+1})_{n=1}^\infty$, and $(y^c_{n+1})_{n=1}^\infty$), where we set $x_0 \coloneqq 0$ and $y_0 \coloneqq 0$ for convention.

Lemma 17.6. For each $n \ge 1$, it holds that $x_n + y_n + x^c_{n+1} + y^c_{n+1} = 2$.

Proof. It is obvious from Lemma 17.5 that $x_n + y^c_{n+1} = 1$ and $y_n + x^c_{n+1} = 1$ for each $n \ge 1$. ☐

By definition, we have $p_n = x_n + y_n + z_n$ and $q_n = x^c_n + y^c_n + z^c_n$ for $n \ge 2$. Let $(p'_n)_{n=1}^\infty$ be the hook length sequence of $Y_{\lambda'}$, and let $(q'_n)_{n=1}^\infty$ be the hook length sequence of $Y^c_{\lambda'}$. Notice that $p'_n = x_{n-1} + y_{n-1} + z_n$ and $q'_n = x^c_{n+1} + y^c_{n+1} + z^c_n$ for $n \ge 2$. We show that $q'_n - p'_n = n$ for all $n \ge 1$. If $n = 1$, then we have $q'_1 - p'_1 = 1$ because $q'_1 = 2 + \#\{t \in [1, k-1] \mid \lambda_t \ne \lambda_{t+1}\}$ and $p'_1 = 1 + \#\{t \in [1, k-1] \mid \lambda_t \ne \lambda_{t+1}\}$. Assume that $n \ge 2$. By Lemma 17.6 and the induction hypothesis, we have

$$q'_n - p'_n = \underbrace{(x_n + y_n + x^c_{n+1} + y^c_{n+1})}_{=2} - \underbrace{(x_{n-1} + y_{n-1} + x^c_n + y^c_n)}_{=2} + \underbrace{(x^c_n + y^c_n + z^c_n)}_{=q_n} - \underbrace{(x_n + y_n + z_n)}_{=p_n} = n.$$

This completes the proof of Theorem 17.3. ☐

18 Application.

By applying [13, Corollary 6.8] to the case of A_∞, we have

$$U(D(\Lambda), \leq) = T(D(\Lambda), \leq)|_{q_i \to q} = \prod_{\beta \in D(\Lambda)} \frac{1}{1 - q^{\# H_\Lambda(\beta)}}, \tag{18.1}$$

where Λ is a finite pre-dominant integral weight, and $T(D(\Lambda), \leq)$ is the trace generating function of the poset $(D(\Lambda), \leq)$ (see [13, Section 2]) corresponding to the coloring $c_\Lambda \colon D(\Lambda) \to I$ in [13, Definition 6.4]. Also, in the case of A_∞, the same formula as (18.1) holds for an infinite pre-dominant integral weight Λ satisfying the following conditions: If we write Λ as $\Lambda = \sum_{i \in \mathbb{Z}} d_i \Lambda_i$ with $d_i \in \mathbb{Z}$ for $i \in \mathbb{Z}$, then
(IP1) it holds that $1 \leq \#\{i \in \mathbb{Z} \mid d_i \neq 0\} < \infty$;
(IP2) for all $i \in \mathbb{Z}$, we have $d_i \in \{0, \pm 1\}$;
(IP3) for each $i, j \in \mathbb{Z}$ with $i < j$ such that $d_i = d_j \neq 0$, there exists $i < m < j$ such that $d_m = -d_i = -d_j$;
(IP4) if we set $u \coloneqq \min\{i \in \mathbb{Z} \mid d_i \neq 0\}$ and $v \coloneqq \max\{i \in \mathbb{Z} \mid d_i \neq 0\}$, then $d_u = -1$ and $d_v = -1$;
(IP5) it holds that $\sum_{i \in \mathbb{Z}} d_i \cdot i = 0$.

Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition such that $\lambda_k > 0$; recall from Section 16 that $\Lambda_\lambda^{\mathrm{c}} = -\Lambda_{-k} - \sum_{i=1-k}^{\lambda_1 - 1} (b_i + c_i)\Lambda_1 - \Lambda_{\lambda_1}$. We can easily check that $\Lambda_\lambda^{\mathrm{c}}$ is an infinite pre-dominant integral weight satisfying the conditions (IP1)–(IP4) above. Let us show that $\Lambda_\lambda^{\mathrm{c}}$ satisfies the condition (IP5) above. Recall that $c_{1-k} = 0$ and that $b_{\lambda_1 - 1} = -1$, and if $1 - k \leq i \leq \lambda_1 - 2$, then $b_i + c_{i+1} = 0$. Thus we obtain

$$\begin{aligned}
&(-1) \cdot (-k) - \sum_{i=1-k}^{\lambda_1 - 1} (b_i + c_i) \cdot i + (-1) \cdot \lambda_1 \\
&= k - \Bigg\{ \sum_{i=1-k}^{\lambda_1 - 1} b_i \cdot i + \sum_{i=1-k}^{\lambda_1 - 1} c_i \cdot (i-1) + \underbrace{\sum_{i=1-k}^{\lambda_1 - 1} c_i}_{= \#(I \setminus \{\lambda_1\})} \Bigg\} - \lambda_1 \\
&= k - \Bigg\{ \sum_{i=1-k}^{\lambda_1 - 2} \underbrace{(b_i + c_{i+1})}_{=0} \cdot i + \underbrace{b_{\lambda_1 - 1}}_{=-1} \cdot (\lambda_1 - 1) + (k-1) \Bigg\} - \lambda_1 \\
&= 0,
\end{aligned}$$

as desired. In particular, $-\Lambda_0$ satisfies the conditions above, and its diagram is $D(-\Lambda_0) = \{\alpha_{i,j} \mid i \leq 0 \leq j\}$. We can easily check that the map $\psi' \colon D(-\Lambda_0)^* \to Y_\emptyset^c = \mathbb{N}^2$, $\alpha_{i,j} \mapsto (-i+1, j+1)$, is an order isomorphism, and preserves the hooks in the sense that $\psi'(H_{-\Lambda_0}(\beta)^*) = H_{Y_\emptyset^c}(\psi'(\beta))$ for all $\beta \in D(-\Lambda_0)^*$. Hence we deduce by (18.1) that

$$U(\mathbb{N}^2, \preceq^*) = U(D(-\Lambda_0), \leq) = \prod_{i \leq 0 \leq j} \frac{1}{1 - q^{j-i+1}} = \prod_{n=1}^{\infty} \left(\frac{1}{1 - q^n} \right)^n, \tag{18.2}$$

which is known as MacMahon's identity (see [8] and also [7, Chapter 1, Section 5, Example 13(c)]). The following formula includes MacMahon's identity as a special case where $\lambda = \emptyset$.

Corollary 18.1 (see also [17, Theorem 2.1])**.** For any partition λ, it holds that

$$\frac{U(Y_\lambda^c, \preceq^*)}{U(Y_\lambda, \preceq)} = \prod_{n=1}^{\infty} \left(\frac{1}{1 - q^n} \right)^n. \tag{18.3}$$

In particular, the left-hand side is independent of λ.

Proof. If $\lambda = \emptyset$, then the formula (18.3) is obvious from (18.2) and $U(Y_\lambda, \preceq) = 1$. Assume that $\lambda \neq \emptyset$. Let $(p_n)_{n=1}^{\infty}$ (resp., $(q_n)_{n=1}^{\infty}$) be the hook length sequence of Y_λ (resp., Y_λ^c). By Theorems 16.4 and 16.6, together with (18.1), we have

$$U(Y_\lambda, \preceq) = U(D(\Lambda_\lambda), \leq) = \prod_{n=1}^{\infty} \left(\frac{1}{1-q^n} \right)^{p_n},$$

$$U(Y_\lambda^c, \preceq^*) = U(D(\Lambda_\lambda^c), \leq) = \prod_{n=1}^{\infty} \left(\frac{1}{1-q^n} \right)^{q_n}.$$

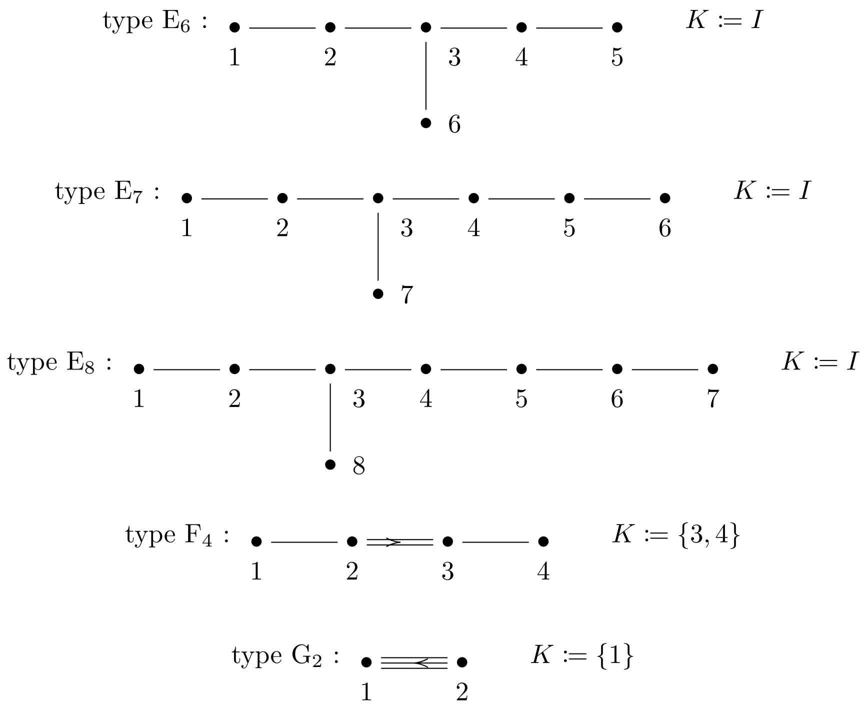
By Theorem 17.3, we obtain

$$\frac{U(Y_\lambda^c, \preceq^*)}{U(Y_\lambda, \preceq)} = \prod_{n=1}^{\infty} \left(\frac{1}{1-q^n} \right)^{q_n - p_n} = \prod_{n=1}^{\infty} \left(\frac{1}{1-q^n} \right)^{n},$$

as desired. □

A Appendix to Part I.

In Appendix A, we assume that $\mathfrak{g}$ is the exceptional finite-dimensional simple Lie algebra of type E_6, E_7, E_8, F_4, or G_2. The Dynkin diagram for $\mathfrak{g}$ and $K \subset I$ are given as follows.



Also for $\mathfrak{g}$ of exceptional type, we can show statements similar to Theorems 5.1, 5.2, 5.3, and Corollary 8.3, as seen below; we prove them by using computer programs due to Kawai and Tada [4].

Define $v_i \in W$ for $i \in K$ as Table 1 below. Then we deduce that the same statement as Lemma 6.4 holds also in these exceptional cases; in particular, we have

$$\mathcal{SM} = \bigsqcup_{i \in K} \mathcal{SM}_i,$$

where $\mathcal{SM}_i = \{w \in \mathcal{SM} \mid \Lambda_w = \Lambda_i\}$ for $i \in K$. Then we see that $\#\mathcal{SM}_i$ is given as Table 2 below. Let $i \in K$ be such that Λ_i is a minuscule weight; in this case, $\mathfrak{g}$ is of type E_6 and $i = 1$ or 5, or $\mathfrak{g}$ is of type E_7 and $i = 6$. If $\mathfrak{g}$ is of type E_6, then $\mathcal{SM}_1 = [v_1, w_0^{J_1}]^{J_1}$, $\mathcal{SM}_5 = [v_5, w_0^{J_5}]^{J_5}$, and $\dim E_{\overline{v_1}}(\Lambda_1) = \dim E_{\overline{v_5}}(\Lambda_5) = 16$. If $\mathfrak{g}$ is of type E_7, then $\mathcal{SM}_6 = [v_6, w_0^{J_6}]^{J_6}$, and $\dim E_{\overline{v_6}}(\Lambda_6) = 43$.

v_i	E$_6$	E$_7$	E$_8$	F$_4$	G$_2$
$i=1$	$s_6s_5s_4s_3s_2s_1$	$s_7s_6s_5s_4s_3s_2s_1$	$s_8s_7s_6s_5s_4s_3s_2s_1$	$-$	s_2s_1
$i=2$	$s_6s_5s_4s_3s_1s_2$	$s_7s_6s_5s_4s_3s_1s_2$	$s_8s_7s_6s_5s_4s_3s_1s_2$	$-$	$-$
$i=3$	$s_6s_5s_4s_1s_2s_3$	$s_7s_6s_5s_4s_1s_2s_3$	$s_8s_7s_6s_5s_4s_1s_2s_3$	$s_1s_2s_4s_3$	
$i=4$	$s_6s_5s_1s_2s_3s_4$	$s_7s_6s_5s_1s_2s_3s_4$	$s_8s_7s_6s_5s_1s_2s_3s_4$	$s_1s_2s_3s_4$	
$i=5$	$s_6s_1s_2s_3s_4s_5$	$s_7s_6s_1s_2s_3s_4s_5$	$s_8s_7s_6s_1s_2s_3s_4s_5$		
$i=6$	$s_1s_2s_5s_4s_3s_6$	$s_7s_1s_2s_3s_4s_5s_6$	$s_8s_7s_1s_2s_3s_4s_5s_6$		
$i=7$		$s_1s_2s_6s_5s_4s_3s_7$	$s_8s_1s_2s_3s_4s_5s_6s_7$		
$i=8$			$s_1s_2s_7s_6s_5s_4s_3s_8$		

Table 1. Definition of $v_i \in W$.

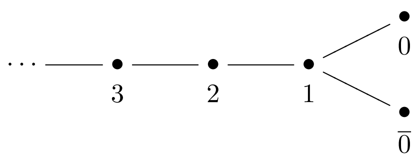
$\#\mathcal{SM}_i$	E$_6$	E$_7$	E$_8$	F$_4$	G$_2$
$i=1$	16	35	71	$-$	1
$i=2$	4	5	6	$-$	$-$
$i=3$	1	1	1	1	
$i=4$	4	5	6	6	
$i=5$	16	11	16		
$i=6$	12	43	27		
$i=7$		20	105		
$i=8$			30		

Table 2. The number of strong minuscule elements in $\mathcal{SM}_i$.

B Appendix to Part III.

B.1 Description of the diagrams.

In Appendix B, we assume that $\mathfrak{g}$ is the infinite rank affine Lie algebra of type D$_\infty$ over $\mathbb{C}$ associated to the following Dynkin diagram (see [3, Exercise 4.14]):



We use the same notation as for type A$_\infty$ (see Subsection 14.2), with $I=\mathbb{Z}$ replaced by $I=\mathbb{N}_0 \cup \{\bar{0}\}$. We set

$$\alpha_{i,j} \coloneqq \alpha_i + \alpha_{i-1} + \cdots + \alpha_{j+1} + \alpha_j \text{ for } i \ge j \ge 0\ ,\quad \theta_i \coloneqq \alpha_{i,1} \text{ for } i \ge 1,$$
$$\beta_{i,j} \coloneqq \alpha_{i,j+1} + 2\theta_j + \alpha_0 + \alpha_{\bar{0}} = \alpha_i + \cdots + \alpha_{j+1} + 2\alpha_j + \cdots + 2\alpha_1 + \alpha_0 + \alpha_{\bar{0}} \text{ for } i > j \ge 1,$$
$$\gamma_0 \coloneqq \alpha_{\bar{0}}\ ,\quad \gamma_i \coloneqq \theta_i + \alpha_{\bar{0}} \text{ for } i \ge 1\ ,\quad \delta_i \coloneqq \theta_i + \alpha_0 + \alpha_{\bar{0}} \text{ for } i \ge 1;$$

we note that $\Phi_+ = \{\alpha_{i,j} \mid i \ge j \ge 0\} \sqcup \{\beta_{i,j} \mid i > j \ge 1\} \sqcup \{\gamma_i \mid i \ge 0\} \sqcup \{\delta_i \mid i \ge 1\}$.

Definition B.1 (cf. Subsection 9.1)**.** Let $k \in \mathbb{N}$. A partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ is said to be *strict* if $\lambda_1 > \lambda_2 > \cdots > \lambda_k$. For a strict partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, we set

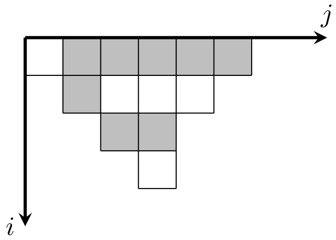
$$S_\lambda \coloneqq \{(i,j) \in \mathbb{N}^2 \mid 1 \le i \le k,\ i \le j \le \lambda_i + i - 1\}.$$

We identify $(i,j) \in S_\lambda$ with the square in $\mathbb{R}^2$ whose vertices are $(i-1,j-1), (i-1,j), (i,j-1)$, and (i,j); elements in S_λ are called *boxes* in S_λ. The set S_λ is called the *shifted Young diagram* associated to a strict partition λ. For $(i,j) \in S_\lambda$, we set

$$\mathrm{Arm}_{S_\lambda}(i,j) := \{(i,j') \in S_\lambda \mid j < j'\}\ , \quad \mathrm{Leg}_{S_\lambda}(i,j) := \{(i',j) \in S_\lambda \mid i < i'\},$$
$$\mathrm{Tail}_{S_\lambda}(i,j) := \{(j+1,j') \in S_\lambda \mid j < j'\}\ , \quad H_{S_\lambda}(i,j) := \{(i,j)\} \sqcup \mathrm{Arm}_{S_\lambda}(i,j) \sqcup \mathrm{Leg}_{S_\lambda}(i,j) \sqcup \mathrm{Tail}_{S_\lambda}(i,j).$$

The subset $H_{S_\lambda}(i,j)$ of S_λ is called the *hook at* (i,j) in S_λ, and the number $h_{S_\lambda}(i,j) := \#H_{S_\lambda}(i,j)$ is called the *hook length at* (i,j) in S_λ.

Example B.2. If $\lambda = (6,4,2,1)$, then S_λ and the hook at $(1,2) \in S_\lambda$ (grayed boxes) are as follows:



Definition B.3. A sequence $\nu = (\nu_n)_{n=1}^\infty$ is called an *infinite strict partition* if $\nu_i < \nu_{i+1}$ for all $i \geq 1$, and there exists $j \geq 1$ such that $\nu_{i+1} = \nu_i + 1$ for all $i \geq j$. For an infinite strict partition $\nu = (\nu_n)_{n=1}^\infty$, we set $s(\nu) = s := \min\{j \geq 1 \mid \nu_{i+1} = \nu_i + 1 \text{ for all } i \geq j\}$, and

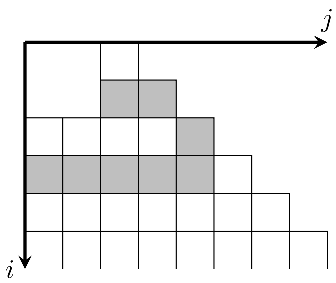
$$S_\nu^\infty := \{(i,j) \in \mathbb{N}^2 \mid 1 \leq i,\ \nu_s - \nu_i + i - s + 1 \leq j \leq \nu_s + i - s\}.$$

We identify $(i,j) \in S_\nu^\infty$ with the square in $\mathbb{R}^2$ whose vertices are $(i-1,j-1), (i-1,j), (i,j-1)$, and (i,j); elements in S_ν^∞ are called *boxes* in S_ν^∞. The set S_ν^∞ is called the *infinite shifted Young diagram* associated to an infinite strict partition $\nu = (\nu_n)_{n=1}^\infty$ (see Example B.4 below). For $(i,j) \in S_\nu^\infty$, we set

$$\mathrm{Arm}_{S_\nu^\infty}(i,j) := \{(i,j') \in S_\nu^\infty \mid j' < j\}\ , \qquad \mathrm{Leg}_{S_\nu^\infty}(i,j) := \{(i',j) \in S_\nu^\infty \mid i' < i\}\ ,$$
$$\mathrm{Tail}_{S_\nu^\infty}(i,j) := \{(s - \nu_s + j - 1, j') \in S_\nu^\infty \mid j' < j\}\ ,$$
$$H_{S_\nu^\infty}(i,j) := \{(i,j)\} \sqcup \mathrm{Arm}_{S_\nu^\infty}(i,j) \sqcup \mathrm{Leg}_{S_\nu^\infty}(i,j) \sqcup \mathrm{Tail}_{S_\nu^\infty}(i,j).$$

The subset $H_{S_\nu^\infty}(i,j)$ of S_ν^∞ is called the *hook at* (i,j) in S_ν^∞, and the number $h_{S_\nu^\infty}(i,j) := \#H_{S_\nu^\infty}(i,j)$ is called the *hook length at* (i,j) in S_ν^∞.

Example B.4. A sequence $\nu = (1,2,5,6,7,8,9,\ldots)$ is an infinite strict partition with $s(\nu) = s = 3$. The infinite shifted Young diagram S_ν^∞ and the hook at $(4,5) \in S_\nu^\infty$ (grayed boxes) are as follows:



Recall from Subsection 15.1 that the set $(\mathbb{N}^2, \preceq)$ is a poset. For each strict partition λ (resp., infinite strict partition ν), the shifted Young diagram S_λ (resp., the infinite shifted Young diagram S_ν^∞) is regarded as a subposet of $\mathbb{N}^2$ in the partial order $\preceq$.

Now, fix $k \in \mathbb{N}$, and a strict partition $\lambda = (\lambda_k, \lambda_{k-1}, \ldots, \lambda_1)$ such that $\lambda_1 > 0$. We can easily check that $\lambda^p := (\lambda_k - k + 1, \lambda_{k-1} - k + 2, \ldots, \lambda_2 - 1, \lambda_1) \in \mathcal{Y}_k(\lambda_k + 1)$. Recall from Subsecction 9.1 that there exists a bijection from $\mathcal{Y}_k(\lambda_k+1)$ onto $\binom{[1,\lambda_k+1]}{k}$. Let U be the element in $\binom{[1,\lambda_k+1]}{k}$ corresponding to λ^p

under this bijection, and write it as: $U = \{u'_1 < \cdots < u'_k\}$. We set $V := [1, \lambda_k + 1] \setminus U$, and write it as: $V = \{v'_1 < \cdots < v'_{\lambda_k - k+1}\}$. For each $0 \leq j \leq \lambda_k - 1$, we set

$$\widetilde{b}_j := \begin{cases} -1 & \text{if } j \in U - 2, \\ 0 & \text{if } j \in V - 2. \end{cases}$$

We set $\widetilde{c}_0 := \widetilde{b}_0 + 1$, and for each $1 \leq j \leq \lambda_k - 1$, we set

$$\widetilde{c}_j := \begin{cases} 1 & \text{if } j \in U - 1, \\ 0 & \text{if } j \in V - 1. \end{cases}$$

We put

$$\widetilde{\Lambda}_\lambda = \Lambda_{\lambda_k} + \sum_{j=0}^{\lambda_k - 1} (\widetilde{b}_j + \widetilde{c}_j)\Lambda_j;$$

we can easily check that $\widetilde{\Lambda}_\lambda \in P^{\mathrm{fin}}_{\geq -1}$. We can prove the following lemma in exactly the same way as Lemma 16.3 and Theorem 16.4 for type A_∞.

Lemma B.5. Keep the notation and setting above.

(1) If $\lambda_1 = 1$, then $D(\widetilde{\Lambda}_\lambda) = \{\alpha_{p,q} \mid p \in U - 2, q \in V - 1, p \geq q\} \sqcup \{\beta_{p,q} \mid p, q \in U - 2, p > q \geq 1\} \sqcup \{\delta_p \mid p \in U - 2, p \geq 1\}$.
(2) If $\lambda_1 > 1$, then $D(\widetilde{\Lambda}_\lambda) = \{\alpha_{p,q} \mid p \in U - 2, q \in (V \setminus \{1\}) - 1, p \geq q\} \sqcup \{\beta_{p,q} \mid p, q \in U - 2, p > q\} \sqcup \{\gamma_p \mid p \in U - 2\}$.

Moreover, the map $\psi \colon D(\widetilde{\Lambda}_\lambda) \to S_\lambda$,

$$\begin{cases} \alpha_{u'_i - 2, v'_j - 1} & \mapsto (k - i + 1, k + j - 1), \\ \beta_{u'_i - 2, u'_j - 2} & \mapsto (k - i + 1, k - j), \\ \gamma_{u'_i - 2} & \mapsto (k - i + 1, k), \\ \delta_{u'_i - 2} & \mapsto (k - i + 1, k - 1), \end{cases}$$

is an order isomorphism, and preserves the hooks in the sense that

$$\psi(H_{\widetilde{\Lambda}_\lambda}(\beta)) = H_{S_\lambda}(\psi(\beta)) \text{ for all } \beta \in D(\widetilde{\Lambda}_\lambda).$$

Definition B.6. Let $\lambda = (\lambda_k, \ldots, \lambda_1)$ be a strict partition. We set $M_\lambda := \mathbb{N} \setminus \{\lambda_1, \ldots, \lambda_k\}$ $(\subset \mathbb{N})$, and write it as: $M_\lambda = \{m_1 < m_2 < \cdots\}$. We set $\lambda^c := (m_n)_{n=1}^\infty$; note that λ^c is an infinite strict partition. The infinite shifted Young diagram $S^c_\lambda := S^\infty_{\lambda^c}$ is called the *complemetary shifted Young diagram* of S_λ.

Example B.7. If $\lambda = (6, 4, 2, 1)$, then $M_\lambda = \mathbb{N} \setminus \{1, 2, 4, 6\} = \{3 < 5 < 7 < 8 < 9 < 10 < \cdots\}$. Therefore, we get $\lambda^c = (3, 5, 7, 8, 9, 10, \ldots)$.

We put $\widetilde{\Lambda}^c_\lambda = -\widetilde{\Lambda}_\lambda$; we can easily check that $\widetilde{\Lambda}^c_\lambda \in P^{\mathrm{inf}}_{\geq -1}$. Let $U = \{u'_1 < \cdots < u'_k\}$, and $V = [1, \lambda_k + 1] \setminus U = \{v'_1 < \cdots < v'_{\lambda_k - k+1}\}$ as above. We set $\widetilde{V} := (V - 2) \sqcup [\lambda_k, \infty)$, and write them as $\widetilde{V} = \{\widetilde{v}_1 < \widetilde{v}_2 < \cdots\}$. We can prove the following lemma in exactly the same way as Lemma 16.5 and Theorem 16.6 for type A_∞.

Lemma B.8. Keep the notation and setting above.

(1') If $\lambda_1 = 1$, then $D(\widetilde{\Lambda}^c_\lambda) = \{\alpha_{p,q} \mid p \in \widetilde{V} \setminus \{\widetilde{v}_1\}, q \in U - 1, p \geq q\} \sqcup \{\beta_{p,q} \mid p, q \in \widetilde{V} \setminus \{\widetilde{v}_1\}, p > q\} \sqcup \{\gamma_p \mid p \in \widetilde{V} \setminus \{\widetilde{v}_1\}\}$.
(2') If $\lambda_1 > 1$, then $D(\widetilde{\Lambda}^c_\lambda) = \{\alpha_{p,q} \mid p \in \widetilde{V}, q \in (U - 1) \cup \{0\}, p \geq q\} \sqcup \{\beta_{p,q} \mid p, q \in \widetilde{V}, p > q \geq 1\} \sqcup \{\delta_p \mid p \in \widetilde{V}, p \geq 1\}$;

if $\lambda_1 > 1$, then we set $u'_0 \coloneqq 1$ (and hence $(U-1)\cup\{0\} = \{u'_0 - 1 < u'_1 - 1 < \cdots < u'_k - 1\}$) for convention. Moreover, the map $\psi^c\colon D(\widetilde{\Lambda}^c_\lambda)^* \to S^c_\lambda$,

$$\begin{cases} \alpha_{\widetilde{v}_i, u'_j - 1} & \mapsto (i-1, k-j+1), \\ \beta_{\widetilde{v}_i, \widetilde{v}_j} & \mapsto (i-1, k+j), \\ \gamma_{\widetilde{v}_i} & \mapsto (i-1, k+1), \\ \delta_{\widetilde{v}_i} & \mapsto (i-1, k+2), \end{cases}$$

is an order isomorphism, and preserves the hooks in the sense that

$$\psi^c(H_{\widetilde{\Lambda}^c_\lambda}(\beta)^*) = H_{S^c_\lambda}(\psi^c(\beta)) \text{ for all } \beta \in D(\widetilde{\Lambda}^c_\lambda)^*.$$

B.2 Hook length sequences.

Definition B.9. Let λ be a strict partition. The sequence $(a_n)_{n=1}^\infty$ defined by

$$a_n \coloneqq \#\{(i,j) \in S_\lambda \text{ (resp., } S^c_\lambda) \mid h_{S_\lambda}(i,j) = n \text{ (resp., } h_{S^c_\lambda}(i,j) = n)\}$$

is called the *hook length sequence of* S_λ (resp., S^c_λ).

Let $\lambda = (\lambda_k, \ldots, \lambda_1)$ be a strict partition. We set

$$\widetilde{x}_i \coloneqq \begin{cases} 1 & \text{if } i \in \{\lambda_1, \ldots, \lambda_k\}, \\ 0 & \text{if } i \notin \{\lambda_1, \ldots, \lambda_k\}, \end{cases} \quad \widetilde{y}_i \coloneqq \begin{cases} -1 & \text{if } i \in \{2\lambda_1, \ldots, 2\lambda_k\}, \\ 0 & \text{if } i \notin \{2\lambda_1, \ldots, 2\lambda_k\}, \end{cases} \quad (\widetilde{s}_n)_{n=1}^\infty \coloneqq (\widetilde{x}_n)_{n=1}^\infty + (\widetilde{y}_n)_{n=1}^\infty.$$

We can show the following theorem by use of some facts mentioned in [14, Chapter 1, Section 4].

Theorem B.10. Let $(\widetilde{p}_n)_{n=1}^\infty$ be the hook length sequence of S_λ, and let $(\widetilde{q}_n)_{n=1}^\infty$ be the hook length sequence of S^c_λ. Then,

$$\widetilde{q}_n - \widetilde{p}_n = \left\lceil \frac{n}{2} \right\rceil - \widetilde{s}_n \text{ for all } n, \tag{B.1}$$

where $\lceil\cdot\rceil$ is the ceiling function.

By Lemma B.8 and [17, Theorem 3.1], we have

$$U(D(\widetilde{\Lambda}^c_\lambda), \leq) = \prod_{\beta \in D(\widetilde{\Lambda}^c_\lambda)} \frac{1}{1 - q^{\# H_{\widetilde{\Lambda}^c_\lambda}(\beta)}}. \tag{B.2}$$

By the same argument as for Corollary 18.1, we obtain the following formula.

Corollary B.11. For any strict partition λ, it holds that

$$\frac{U(S^c_\lambda, \preceq^*)}{U(S_\lambda, \preceq)} = \prod_{n=1}^\infty \left(\frac{1}{1-q^n}\right)^{\lceil \frac{n}{2} \rceil - \widetilde{s}_n}.$$

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