Combinatorial realizations of the crystal bases for extremal weight modules over quantized hyperbolic Kac-Moody algebras of rank 2

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Contents

1 Introduction.

Let $A = (a_{ij})_{i,j \in I}$ be a symmetrizable generalized Cartan matrix, where I is the index set. Let $\mathfrak{g} = \mathfrak{g}(A)$ be the Kac-Moody algebra associated to A over $\mathbb{C}$, and $U_q(\mathfrak{g})$ the quantized universal enveloping algebra over $\mathbb{C}(q)$ associated to $\mathfrak{g}$. We denote by W the Weyl group of $\mathfrak{g}$. Let P be an integral weight lattice of $\mathfrak{g}$, and P^+ (resp., $-P^+$) the set of dominant (resp., antidominant) integral weights in P. Let $\mu \in P$ be an arbitrary integral weight. The extremal weight module $V(\mu)$ of extremal weight μ is the integrable $U_q(\mathfrak{g})$-module generated by a single element v_μ with the defining relation that v_μ is an extremal weight vector of weight μ in the sense of [6, Definition 8.1.1]. This module was introduced by Kashiwara [6] as a natural generalization of integrable highest (or lowest) weight modules; in fact, if $\mu \in P^+$ (resp., $\mu \in -P^+$), then the extremal weight module of extremal weight μ is isomorphic, as a $U_q(\mathfrak{g})$-module, to the integrable highest (resp., lowest) weight module of highest (resp., lowest) weight μ. Also, Kashiwara proved that $V(\mu)$ has a crystal basis $\mathcal{B}(\mu)$ for all $\mu \in P$; let u_μ denote the element of $\mathcal{B}(\mu)$ corresponding to $v_\mu \in V(\mu)$. We know from [6, Proposition 8.2.2 (iv) and (v)] that $V(\mu) \cong V(w\mu)$ as $U_q(\mathfrak{g})$-modules, and $\mathcal{B}(\mu) \cong \mathcal{B}(w\mu)$ as crystals for all $\mu \in P$ and $w \in W$. Hence we are interested in the case that μ is an integral weight such that

$$W\mu \cap (P^+ \cup -P^+) = \emptyset. \tag{1.1}$$

If $\mathfrak{g}$ is of finite type, then $W\mu \cap P^+ \neq \emptyset$ for all $\mu \in P$. Assume that $\mathfrak{g}$ is of affine type. Then, $W\mu \cap (P^+ \cup -P^+) = \emptyset$ if and only if ($\mu \neq 0$, and) μ is of level-zero. In this case, Naito and Sagaki proved in [11] and [12] that if μ is a positive integer multiple of a level-zero fundamental weight, then the crystal basis $\mathcal{B}(\mu)$ of the extremal weight module $V(\mu)$ is isomorphic, as a crystal, to the crystal $\mathbb{B}(\mu)$ of Lakshmibai-Seshadri (LS for short) paths, which was introduced by Littelmann in [9] and [10]. Then, Ishii, Naito, and Sagaki [4] introduced the notion of semi-infinite LS paths of shape μ for a level-zero dominant integral weight μ, and proved that the crystal basis $\mathcal{B}(\mu)$ of the extremal weight module $V(\mu)$ is isomorphic, as a crystal, to the crystal $\mathbb{B}^{\frac{\infty}{2}}(\mu)$ of semi-infinite LS paths of shape μ. On the other hand, in the case that $\mathfrak{g}$ is of indefinite type, there are few studies on the combinatorial realization of the crystal basis $\mathcal{B}(\mu)$ (for $\mu \in P$ satisfying (1.1)). As a special case (which is one of most fundamental and interesting cases), let us assume that $\mathfrak{g}$ is the hyperbolic Kac-Moody algebra associated to the generalized Cartan matrix

$$A = \begin{pmatrix} 2 & -a_1 \\ -a_2 & 2 \end{pmatrix}, \quad \text{where } a_1, a_2 \in \mathbb{Z}_{\geq 1} \text{ with } a_1 a_2 > 4. \tag{1.2}$$

Sagaki and Yu [15] proved that if $\mu = \Lambda_1 - \Lambda_2$ (Λ_1, Λ_2 are the fundamental weights of $\mathfrak{g}$) then the crystal basis $\mathcal{B}(\mu)$ is isomorphic, as a crystal, to the crystal $\mathbb{B}(\mu)$ of LS paths in the case that $a_1, a_2 \geq 2$; note that $\mu = \Lambda_1 - \Lambda_2$ does not satisfy condition (1.1) if $a_1 = 1$ or $a_2 = 1$ (see [16, Remark 3.1.2]).

In this thesis, we classify the integral weights satisfying condition (1.1) as follows.

Theorem 1.1 (=Theorem 3.1)**.** *Let $\mathbb{O} := \{W\mu \mid \mu \in P\}$ be the set of W-orbits in P.*

(1) *Assume that $a_1, a_2 \geq 2$. Then, $O \in \mathbb{O}$ satisfies condition* (1.1)*, that is, $O \cap (P^+ \cup -P^+) = \emptyset$ if and only if O contains an integral weight λ of the form either* (i) *or* (ii)*:*

   (i) *$\lambda = k_1\Lambda_1 - k_2\Lambda_2$ for some $k_1, k_2 \in \mathbb{Z}_{>0}$ such that $k_2 \leq k_1 < (a_1 - 1)k_2$;*

   (ii) *$\lambda = k_1\Lambda_1 - k_2\Lambda_2$ for some $k_1, k_2 \in \mathbb{Z}_{>0}$ such that $k_1 < k_2 \leq (a_2 - 1)k_1$.*

(2) *Assume that $a_1 = 1$. Then, $O \in \mathbb{O}$ satisfies condition* (1.1) *if and only if O contains an integral weight λ of the form $\lambda = k_1\Lambda_1 - k_2\Lambda_2$ for some $k_1, k_2 \in \mathbb{Z}_{>0}$ such that $2k_1 \leq k_2 \leq (a_2 - 2)k_1$.*

(3) *Assume that $a_2 = 1$. Then, $O \in \mathbb{O}$ satisfies condition* (1.1) *if and only if O contains an integral weight λ of the form $\lambda = k_1\Lambda_1 - k_2\Lambda_2$ for some $k_1, k_2 \in \mathbb{Z}_{>0}$ such that $2k_2 \leq k_1 \leq (a_1 - 2)k_2$.*

Let $\lambda = k_1\Lambda_1 - k_2\Lambda_2 \in P$ be an integral weight of the form mentioned in Theorem 3.1 above. As a further study after [15, 16], we study the crystal structure of the crystal $\mathbb{B}(\lambda)$ of LS paths, and its relationship to the crystal basis $\mathcal{B}(\lambda)$ of extremal weight module $V(\lambda)$.

Theorem 1.2 (=Theorem 3.2). *The crystal graph of $\mathbb{B}(\lambda)$ is connected if and only if $k_1 = 1$ or $k_2 = 1$. Otherwise, the crystal graph of $\mathbb{B}(\lambda)$ has infinitely many connected components.*

Theorem 1.3 (=Theorem 3.3). *Let $\mathbb{B}_0(\lambda)$ (resp., $\mathcal{B}_0(\lambda)$) be the connected component of $\mathbb{B}(\lambda)$ (resp., $\mathcal{B}(\lambda)$) containing the "straight line" $\pi_\lambda := (\lambda; 0, 1)$ (resp., u_λ). There exists an isomorphism $\mathbb{B}_0(\lambda) \to \mathcal{B}_0(\lambda)$ of crystals that sends π_λ to u_λ.*

Theorem 1.4 (=Corollary 3.5). *If $k_1 = 1$ or $k_2 = 1$, then there exists an isomorphism $\mathbb{B}(\lambda) \to \mathcal{B}(\lambda)$ of crystals that sends π_λ to u_λ.*

Again, let us assume that $\mathfrak{g}$ is a general Kac-Moody algebra, and μ is an arbitrary integral weight of $\mathfrak{g}$. In the latter half of this thesis, we introduce and use the following embedding of the crystal basis $\mathcal{B}(\mu)$ for the extremal weight module $V(\mu)$ into an infinite $\mathbb{Z}$-lattice. Let $\mathcal{B}(\infty)$ (resp., $\mathcal{B}(-\infty)$) be the crystal basis of the negative (resp., positive) part of $U_q(\mathfrak{g})$. Nakashima and Zelevinsky [13] introduced an embedding $\Psi^+_{\iota^+} : \mathcal{B}(\infty) \hookrightarrow \mathbb{Z}^{+\infty}_{\geq 0,\iota^+}$ of crystals, where ι^+ is an infinite sequence of elements in the index set I satisfying a certain condition, and $\mathbb{Z}^{+\infty}_{\geq 0,\iota^+} := \{(\ldots, x_k, \ldots, x_2, x_1) \mid x_k \in \mathbb{Z}_{\geq 0}$ and $x_k = 0$ for $k \gg 0\}$ is the semi-infinite $\mathbb{Z}$-lattice together with a crystal structure associated to ι^+. Assuming a certain positivity condition on ι^+, they gave a combinatorial description of $\mathcal{B}(\infty)$ (which is called a polyhedral realization of $\mathcal{B}(\infty)$) as a polyhedral convex cone in $\mathbb{Z}^{+\infty}_{\geq 0,\iota^+}$. Namely, they found the set Ξ_{ι^+} of linear functions on $\mathbb{R}^{+\infty}$ such that the image $\mathrm{Im}(\Psi^+_{\iota^+}) \cong \mathcal{B}(\infty)$ is identical to the set

$$\{\hat{x} \in \mathbb{Z}^{+\infty}_{\geq 0,\iota^+} \mid \phi(\hat{x}) \geq 0 \text{ for all } \phi \in \Xi_{\iota^+}\}. \tag{1.3}$$

Similarly, there exists an embedding $\Psi^-_{\iota^-} : \mathcal{B}(-\infty) \hookrightarrow \mathbb{Z}^{-\infty}_{\leq 0,\iota^-}$ of crystals, where ι^- is an infinite sequence of elements in the index set I satisfying a certain condition, and $\mathbb{Z}^{-\infty}_{\leq 0,\iota^-} := \{(x_0, x_{-1}, \ldots, x_k, \ldots) \mid x_k \in \mathbb{Z}_{\leq 0}$ and $x_k = 0$ for $k \ll 0\}$ is the semi-infinite $\mathbb{Z}$-lattice together with a crystal structure associated to ι^-. Hence there exists an embedding

$$\Psi^\mu_\iota : \mathcal{B}(\infty) \otimes \mathcal{T}_\mu \otimes \mathcal{B}(-\infty) \hookrightarrow \mathbb{Z}^{+\infty}_{\geq 0,\iota^+} \otimes \mathcal{T}_\mu \otimes \mathbb{Z}^{-\infty}_{\leq 0,\iota^-} =: \mathbb{Z}_\iota(\mu)$$

of crystals, where $\mathcal{T}_\mu$ is the crystal consisting of a single element of weight μ, and $\iota := (\iota^+, \iota^-)$. Now, in [6, Proposition 8.2.2 (and Theorem 3.1.1)], Kashiwara showed that $\mathcal{B}(\mu)$ is isomorphic, as a crystal, to the subcrystal $\{b \in \mathcal{B}(\infty) \otimes \mathcal{T}_\mu \otimes \mathcal{B}(-\infty) \mid b^* \text{ is extremal}\}$ of $\mathcal{B}(\infty) \otimes \mathcal{T}_\mu \otimes \mathcal{B}(-\infty)$, where $*\colon \mathcal{B}(\infty) \otimes \mathcal{T}_\mu \otimes \mathcal{B}(-\infty) \to \mathcal{B}(\infty) \otimes \mathcal{T}_\mu \otimes \mathcal{B}(-\infty)$ is the $*$-operation (see [6, Theorem 4.3.2]). Therefore the crystal basis $\mathcal{B}(\mu)$ is isomorphic, as a crystal, to the subcrystal $\{\vec{x} \in \mathrm{Im}(\Psi^\mu_\iota) \mid \vec{x}^* \text{ is extremal}\}$ of $\mathrm{Im}(\Psi^\mu_\iota) = \mathrm{Im}(\Psi^+_{\iota^+}) \otimes \mathcal{T}_\mu \otimes \mathrm{Im}(\Psi^-_{\iota^-}) \cong \mathcal{B}(\infty) \otimes \mathcal{T}_\mu \otimes \mathcal{B}(-\infty)$.

We return to be the case that A is as (1.2). The next purpose of this thesis is to give a polyhedral realization (such as (1.3)) of $\mathcal{B}(\lambda) \hookrightarrow \mathrm{Im}(\Psi^\lambda_\iota)$ for $\lambda = k_1\Lambda_1 - k_2\Lambda_2 \in P$ of the form mentioned in Theorem 3.1 above. Let $\iota = (\iota^+, \iota^-)$ with $\iota^+ = (\ldots, i_2, i_1) := (\ldots, 2, 1, 2, 1)$ and $\iota^- = (i_0, i_{-1}, \ldots) := (2, 1, 2, 1, \ldots)$. For $k \in \mathbb{Z}$, we define the linear function $\zeta_k \in (\mathbb{R}^\infty)^*$ by $\zeta_k(\vec{x}) := x_k$ for $\vec{x} = (\ldots, x_2, x_1) \otimes t_\lambda \otimes (x_0, x_{-1}, \ldots) \in \mathbb{R}^\infty$, and set

$$\begin{aligned}\Xi_\iota[\lambda] = &\{\gamma_0 p_0 + \gamma_0\zeta_0 - \zeta_1,\ \gamma_1 p_1 + \zeta_0 - \gamma_1\zeta_1\}\\ &\cup \{p_k - \zeta_k,\ \gamma_k\zeta_k - \zeta_{k+1},\ \gamma_{k+1}p_{k+1} - p_k + \zeta_k - \gamma_{k+1}\zeta_{k+1} \mid k \geq 1\}\\ &\cup \{p_k + \zeta_k,\ \zeta_{k-1} - \gamma_k\zeta_k,\ \gamma_{k-1}p_{k-1} - p_k + \gamma_{k-1}\zeta_{k-1} - \zeta_k \mid k \leq 0\},\end{aligned}$$

where the numbers $\gamma_k \in \mathbb{R} \setminus \mathbb{Q}$, $k \in \mathbb{Z}$, are defined by (3.3), and the sequence $\{p_m\}_{m\in\mathbb{Z}}$ are defined by (3.1) and (3.2). We set

$$\Sigma_\iota(\lambda) := \{\vec{x} \in \mathbb{Z}_\iota(\lambda) \mid \varphi(\vec{x}) \geq 0 \text{ for all } \varphi \in \Xi_\iota[\lambda]\}.$$

Theorem 1.5 (=Theorem 3.6)**.** *The set* $\Sigma_\iota(\lambda)$ *is a subcrystal of* $\mathrm{Im}(\Psi_\iota^\lambda)$*.*

Theorem 1.6 (=Theorem 3.7)**.** *The equality* $\Sigma_\iota(\lambda) = \{\vec{x} \in \mathrm{Im}(\Psi_\iota^\lambda) \mid \vec{x}^* \text{ is extremal}\}$ *holds. Therefore,* $\Sigma_\iota(\lambda)$ *is isomorphic, as a crystal, to the crystal basis* $\mathcal{B}(\lambda)$ *of the extremal weight module* $V(\lambda)$ *of extremal weight* λ*.*

This paper is organized as follows. In §2, we fix our notation, and recall some basic facts about extremal weight modules and their crystal bases. Also, we recall the definition of LS paths. In §4, we prove Theorem 1.1. In §5, we prove Theorem 1.2. In §6, we prove Theorems 1.3 and 1.4. In §7, we prove Theorems 1.5 and 1.6. In Appendix A, we give some formulas of the operators F_k (which is defined in §7.2) on $\Xi_\iota[\lambda]$.

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2 Preliminaries.

2.1 Kac-Moody algebras.

Let $A = (a_{ij})_{i,j\in I}$ be a symmetrizable generalized Cartan matrix, and $\mathfrak{g} = \mathfrak{g}(A)$ the Kac-Moody algebra associated to A over $\mathbb{C}$. We denote by $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{g}$, $\{\alpha_i\}_{i\in I} \subset \mathfrak{h}^*$ the set of simple roots, and $\{\alpha_i^\vee\}_{i\in I} \subset \mathfrak{h}$ the set of simple coroots. Let s_i be the simple reflection with respect to α_i for $i \in I$, and let $W = \langle s_i \mid i \in I\rangle$ be the Weyl group of $\mathfrak{g}$. Let Δ_{re}^+ denote the set of positive real roots. For a positive real root $\beta \in \Delta_{\mathrm{re}}^+$, we denote by $\beta^\vee$ the dual root of β, and by $s_\beta \in W$ the reflection with respect to β. Let $\{\Lambda_i\}_{i\in I} \subset \mathfrak{h}^*$ be the fundamental weights for $\mathfrak{g}$, i.e., $\langle \Lambda_i, \alpha_j^\vee\rangle = \delta_{i,j}$ for $i, j \in I$, where $\langle\cdot,\cdot\rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$ is the canonical pairing of $\mathfrak{h}^*$ and $\mathfrak{h}$. We take an integral weight lattice P containing α_i and Λ_i for all $i \in I$. We denote by P^+ (resp., $-P^+$) the set of dominant (resp., antidominant) integral weights in P.

Let $U_q(\mathfrak{g})$ be the quantized universal enveloping algebra over $\mathbb{C}(q)$ associated to $\mathfrak{g}$, and let $U_q^+(\mathfrak{g})$ (resp., $U_q^-(\mathfrak{g})$) be the positive (resp., negative) part of $U_q(\mathfrak{g})$, that is, $\mathbb{C}(q)$-subalgebra generated by the Chevalley generators E_i (resp., F_i) of $U_q(\mathfrak{g})$ corresponding to the positive (resp., negative) simple roots α_i (resp., $-\alpha_i$) for $i \in I$.

2.2 Crystal bases and crystals.

For details on crystal bases and crystals, we refer the reader to [7] and [1]. Let $\mathcal{B}(\infty)$ (resp., $\mathcal{B}(-\infty)$) be the crystal basis of $U_q^-(\mathfrak{g})$ (resp., $U_q^+(\mathfrak{g})$), and let $u_\infty \in \mathcal{B}(\infty)$ (resp., $u_{-\infty} \in \mathcal{B}(-\infty)$) be the element corresponding to $1 \in U_q^-(\mathfrak{g})$ (resp., $1 \in U_q^+(\mathfrak{g})$). Denote by $* : \mathcal{B}(\pm\infty) \to \mathcal{B}(\pm\infty)$ the $*$-operation on $\mathcal{B}(\pm\infty)$; see [5, Theorem 2.1.1] and [7, §8.3]. For $\mu \in P$, let $\mathcal{T}_\mu = \{t_\mu\}$ be the crystal consisting of a single element t_μ such that

$$\mathrm{wt}(t_\mu) = \mu, \quad \tilde{e}_i t_\mu = \tilde{f}_i t_\mu = \mathbf{0}, \quad \varepsilon_i(t_\mu) = \varphi_i(t_\mu) = -\infty \text{ for } i \in I,$$

where $\mathbf{0}$ is an extra element not contained in any crystal.

Let B be a normal crystal in the sense of [6, §1.5]. We know from [6, §7] (see also [7, Theorem 11.1]) that B has the following action of the Weyl group W. For $i \in I$ and $b \in B$, we set

$$S_i b := \begin{cases} \tilde{f}_i^{\langle \mathrm{wt}(b), \alpha_i^\vee\rangle} b & \text{if } \langle \mathrm{wt}(b), \alpha_i^\vee\rangle \geq 0, \\ \tilde{e}_i^{-\langle \mathrm{wt}(b), \alpha_i^\vee\rangle} b & \text{if } \langle \mathrm{wt}(b), \alpha_i^\vee\rangle \leq 0. \end{cases}$$

Then, for $w \in W$, we set $S_w := S_{i_1} \cdots S_{i_k}$ if $w = s_{i_1} \cdots s_{i_k}$. Notice that $\mathrm{wt}(S_w b)= w\mathrm{wt}(b)$ for $w \in W$ and $b \in B$.

Definition 2.1. An element b of a normal crystal B is said to be extremal if for each $w \in W$ and $i \in I$,

$$\tilde{e}_i(S_w b) = \mathbf{0} \text{ if } \langle \mathrm{wt}(S_w b), \alpha_i^\vee\rangle \geq 0,$$
$$\tilde{f}_i(S_w b) = \mathbf{0} \text{ if } \langle \mathrm{wt}(S_w b), \alpha_i^\vee\rangle \leq 0.$$

Let B be a normal crystal. For $b \in B$ and $i \in I$, we set

$$\tilde{e}_i^{\max} b := \tilde{e}_i^{\varepsilon_i(b)} b \quad \text{and} \quad \tilde{f}_i^{\max} b := \tilde{f}_i^{\varphi_i(b)} b.$$

2.3 Crystal bases of extremal weight modules.

Let $\mu \in P$ be an arbitrary integral weight. The extremal weight module $V(\mu)$ of extremal weight μ is, by definition, the integrable $U_q(\mathfrak{g})$-module generated by a single element v_μ with the defining relation that v_μ is an extremal weight vector of weight μ in the sense of [6, Definition 8.1.1]. We know from [6, Proposition 8.2.2] that $V(\mu)$ has a crystal basis $\mathcal{B}(\mu)$. Let u_μ denote the element of $\mathcal{B}(\mu)$ corresponding to v_μ.

Remark 2.2. We see from [6, Proposition 8.2.2 (iv) and (v)] that $V(\mu) \cong V(w\mu)$ as $U_q(\mathfrak{g})$-modules, and $\mathcal{B}(\mu) \cong \mathcal{B}(w\mu)$ as crystals for all $\mu \in P$ and $w \in W$. Also, we know from the comment at the end of [6, §8.2] that if $\mu \in P^+$ (resp., $\mu \in -P^+$), then $V(\mu)$ is isomorphic, as a $U_q(\mathfrak{g})$-module, to the integrable highest (resp., lowest) weight module of highest (resp., lowest) weight μ, and $\mathcal{B}(\mu)$ is isomorphic, as a crystal, to its crystal basis. So, we focus on those $\mu \in P$ satisfying the condition that

$$W\mu \cap (P^+ \cup -P^+) = \emptyset. \tag{2.1}$$

The crystal basis $\mathcal{B}(\mu)$ of $V(\mu)$ can be realized (as a crystal) as follows. We set

$$\mathcal{B} := \bigsqcup_{\mu \in P} \mathcal{B}(\infty) \otimes \mathcal{T}_\mu \otimes \mathcal{B}(-\infty);$$

in fact, $\mathcal{B}$ is isomorphic, as a crystal, to the crystal basis $\mathcal{B}(\tilde{U}_q(\mathfrak{g}))$ of the modified quantized universal enveloping algebra $\tilde{U}_q(\mathfrak{g})$ associated to $\mathfrak{g}$ (see [6, Theorem 3.1.1]). Denote by $* : \mathcal{B} \to \mathcal{B}$ the $*$-operation on $\mathcal{B}$ (see [6, Theorem 4.3.2]); we know from [6, Corollary 4.3.3] that for $b_1 \in \mathcal{B}(\infty)$, $b_2 \in \mathcal{B}(-\infty)$, and $\mu \in P$,

$$(b_1 \otimes t_\mu \otimes b_2)^* = b_1^* \otimes t_{-\mu-\mathrm{wt}(b_1)-\mathrm{wt}(b_2)} \otimes b_2^*. \tag{2.2}$$

Remark 2.3. The weight of $(b_1 \otimes t_\mu \otimes b_2)^*$ is equal to $-\mu$ for all $b_1 \in \mathcal{B}(\infty)$ and $b_2 \in \mathcal{B}(-\infty)$ since $\mathrm{wt}(b_1^*) = \mathrm{wt}(b_1)$ and $\mathrm{wt}(b_2^*) = \mathrm{wt}(b_2)$.

Because $\mathcal{B}$ is a normal crystal by [6, §2.1 and Theorem 3.1.1], $\mathcal{B}$ has the action of the Weyl group W (see §2.2). We know the following theorem from [6, Proposition 8.2.2 (and Theorem 3.1.1)].

Theorem 2.4. *For $\mu \in P$, the set $\{b \in \mathcal{B}(\infty) \otimes \mathcal{T}_\mu \otimes \mathcal{B}(-\infty) \mid b^* \text{ is extremal}\}$ is a subcrystal of $\mathcal{B}(\infty) \otimes \mathcal{T}_\mu \otimes \mathcal{B}(-\infty)$, and is isomorphic, as a crystal, to the crystal basis $\mathcal{B}(\mu)$ of the extremal weight module $V(\mu)$ of extremal weight μ. In particular, $u_\infty \otimes t_\mu \otimes u_{-\infty} \in \mathcal{B}(\infty) \otimes \mathcal{T}_\mu \otimes \mathcal{B}(-\infty)$ is contained in the set above, and corresponds to $u_\mu \in \mathcal{B}(\mu)$ under this isomorphism.*

2.4 Realizations of $\mathcal{B}(\pm\infty)$ and $\mathcal{B}(\mu)$.

Let us recall realizations of $\mathcal{B}(\pm\infty)$ from [13]. We fix an infinite sequence $\iota^+ = (\dots, i_k, \dots, i_2, i_1)$ of elements of I such that $i_k \neq i_{k+1}$ for $k \in \mathbb{Z}_{\geq 1}$, and $\#\{k \in \mathbb{Z}_{\geq 1} \mid i_k = i\} = \infty$ for each $i \in I$. Similarly, we fix an infinite sequence $\iota^- = (i_0, i_{-1}, \dots, i_k, \dots)$ of elements of I such that $i_k \neq i_{k-1}$ for $k \in \mathbb{Z}_{\leq 0}$, and $\#\{k \in \mathbb{Z}_{\leq 0} \mid i_k = i\} = \infty$ for each $i \in I$. We set

$$\mathbb{Z}_{\geq 0}^{+\infty} := \{(\dots, x_k, \dots, x_2, x_1) \mid x_k \in \mathbb{Z}_{\geq 0} \text{ and } x_k = 0 \text{ for } k \gg 0\},$$
$$\mathbb{Z}_{\leq 0}^{-\infty} := \{(x_0, x_{-1}, \dots, x_k, \dots) \mid x_k \in \mathbb{Z}_{\leq 0} \text{ and } x_k = 0 \text{ for } k \ll 0\}.$$

We endow $\mathbb{Z}_{\geq 0}^{+\infty}$ and $\mathbb{Z}_{\leq 0}^{-\infty}$ with crystal structures as follows. Let $\hat{x}^+ = (\dots, x_k, \dots, x_2, x_1) \in \mathbb{Z}_{\geq 0}^{+\infty}$ and $\hat{x}^- = (x_0, x_{-1} \dots, x_k \dots) \in \mathbb{Z}_{\leq 0}^{-\infty}$. For $k \geq 1$, we set

$$\sigma_k^+(\hat{x}^+) = x_k + \sum_{j>k} \langle \alpha_{i_j}, \alpha_{i_k}^\vee \rangle x_j,$$

and for $k \leq 0$, we set

$$\sigma_k^-(\hat{x}^-) = -x_k - \sum_{j<k} \langle \alpha_{i_j}, \alpha_{i_k}^\vee \rangle x_j;$$

since $x_j = 0$ for $|j| \gg 0$, we see that $\sigma_k^{\pm}(\hat{x}^{\pm})$ is well-defined, and $\sigma_k^{\pm}(\hat{x}^{\pm}) = 0$ for $|k| \gg 0$. For $i \in I$, we set $\sigma_{(i)}^{+}(\hat{x}^{+}) \coloneqq \max\{\sigma_k^{+}(\hat{x}^{+}) \mid k \geq 1, i_k = i\}$ and $\sigma_{(i)}^{-}(\hat{x}^{-}) \coloneqq \max\{\sigma_k^{-}(\hat{x}^{-}) \mid k \leq 0, i_k = i\}$, and define

$$\begin{aligned} M_{(i)}^{+} &= M_{(i)}^{+}(\hat{x}^{+}) \coloneqq \{k \mid k \geq 1, i_k = i, \sigma_k^{+}(\hat{x}^{+}) = \sigma_{(i)}^{+}(\hat{x}^{+})\}, \\ M_{(i)}^{-} &= M_{(i)}^{-}(\hat{x}^{-}) \coloneqq \{k \mid k \leq 0, i_k = i, \sigma_k^{-}(\hat{x}^{-}) = \sigma_{(i)}^{-}(\hat{x}^{-})\}. \end{aligned}$$

Note that $\sigma_{(i)}^{\pm}(\hat{x}^{\pm}) \geq 0$, and that $M_{(i)}^{\pm} = M_{(i)}^{\pm}(\hat{x}^{\pm})$ is a finite set if and only if $\sigma_{(i)}^{\pm}(\hat{x}^{\pm}) > 0$. We define the maps $\tilde{e}_i, \tilde{f}_i : \mathbb{Z}_{\geq 0}^{+\infty} \to \mathbb{Z}_{\geq 0}^{+\infty} \sqcup \{\mathbf{0}\}$ and $\tilde{e}_i, \tilde{f}_i : \mathbb{Z}_{\leq 0}^{-\infty} \to \mathbb{Z}_{\leq 0}^{-\infty} \sqcup \{\mathbf{0}\}$ by

$$\begin{aligned} \tilde{e}_i \hat{x}^{+} &\coloneqq \begin{cases} (\ldots, x'_k, \ldots, x'_2, x'_1) \text{ with } x'_k \coloneqq x_k - \delta_{k, \max M_{(i)}^{+}} & \text{if } \sigma_{(i)}^{+}(\hat{x}^{+}) > 0, \\ \mathbf{0} & \text{if } \sigma_{(i)}^{+}(\hat{x}^{+}) = 0, \end{cases} \\ \tilde{f}_i \hat{x}^{+} &\coloneqq (\ldots, x'_k, \ldots, x'_2, x'_1) \text{ with } x'_k \coloneqq x_k + \delta_{k, \min M_{(i)}^{+}}, \\ \tilde{e}_i \hat{x}^{-} &\coloneqq (x'_0, x'_{-1}, \ldots, x'_k, \ldots) \text{ with } x'_k \coloneqq x_k - \delta_{k, \max M_{(i)}^{-}}, \\ \tilde{f}_i \hat{x}^{-} &\coloneqq \begin{cases} (x'_0, x'_{-1}, \ldots, x'_k, \ldots) \text{ with } x'_k \coloneqq x_k + \delta_{k, \min M_{(i)}^{-}} & \text{if } \sigma_{(i)}^{-}(\hat{x}^{-}) > 0, \\ \mathbf{0} & \text{if } \sigma_{(i)}^{-}(\hat{x}^{-}) = 0, \end{cases} \end{aligned}$$

respectively. Moreover, we define

$$\begin{aligned} \mathrm{wt}(\hat{x}^{+}) &\coloneqq -\sum_{j \geq 1} x_j \alpha_{i_j}, \quad \varepsilon_i(\hat{x}^{+}) \coloneqq \sigma_{(i)}^{+}(\hat{x}^{+}), \quad \varphi_i(\hat{x}^{+}) \coloneqq \varepsilon_i(\hat{x}^{+}) + \langle \mathrm{wt}(\hat{x}^{+}), \alpha_i^{\vee} \rangle, \\ \mathrm{wt}(\hat{x}^{-}) &\coloneqq -\sum_{j \leq 0} x_j \alpha_{i_j}, \quad \varphi_i(\hat{x}^{-}) \coloneqq \sigma_{(i)}^{-}(\hat{x}^{-}), \quad \varepsilon_i(\hat{x}^{-}) \coloneqq \varphi_i(\hat{x}^{-}) - \langle \mathrm{wt}(\hat{x}^{-}), \alpha_i^{\vee} \rangle. \end{aligned}$$

These maps make $\mathbb{Z}_{\geq 0}^{+\infty}$ (resp., $\mathbb{Z}_{\leq 0}^{-\infty}$) into a crystal for $\mathfrak{g}$; we denote this crystal by $\mathbb{Z}_{\geq 0, \iota^{+}}^{+\infty}$ (resp., $\mathbb{Z}_{\leq 0, \iota^{-}}^{-\infty}$).

Theorem 2.5 ([13, Theorem 2.5])**.** *There exists an embedding $\Psi_{\iota^{+}}^{+} : \mathcal{B}(\infty) \hookrightarrow \mathbb{Z}_{\geq 0, \iota^{+}}^{+\infty}$ of crystals which sends $u_{\infty} \in \mathcal{B}(\infty)$ to $z_{\infty} \coloneqq (\ldots, 0 \ldots, 0, 0) \in \mathbb{Z}_{\geq 0, \iota^{+}}^{+\infty}$. Similarly, there exists an embedding $\Psi_{\iota^{-}}^{-} : \mathcal{B}(-\infty) \hookrightarrow \mathbb{Z}_{\leq 0, \iota^{-}}^{-\infty}$ of crystals which sends $u_{-\infty} \in \mathcal{B}(-\infty)$ to $z_{-\infty} \coloneqq (0, 0, \ldots, 0, \ldots) \in \mathbb{Z}_{\leq 0, \iota^{-}}^{-\infty}$.*

We define the $*$-operations on $\mathrm{Im}(\Psi_{\iota^{\pm}}^{\pm})$ by the following commutative diagram:

$$\begin{tikzcd} \mathcal{B}(\pm\infty) \arrow[r, "*"] \arrow[d, "\Psi_{\iota^{\pm}}^{\pm}"'] & \mathcal{B}(\pm\infty) \arrow[d, "\Psi_{\iota^{\pm}}^{\pm}"] \\ \mathrm{Im}(\Psi_{\iota^{\pm}}^{\pm}) \arrow[r, "*"] & \mathrm{Im}(\Psi_{\iota^{\pm}}^{\pm}). \end{tikzcd}$$

We know the following proposition from [13, Remark in §2.4].

Proposition 2.6. *Keep the notation and setting above; recall that $\iota^{+} = (\ldots, i_2, i_1)$ and $\iota^{-} = (i_0, i_{-1}, \ldots)$.*

(1) *Let $\hat{x} = (\ldots, x_2, x_1) \in \mathbb{Z}_{\geq 0, \iota^{+}}^{+\infty}$. Then, $\hat{x} \in \mathrm{Im}(\Psi_{\iota^{+}}^{+})$ if and only if*

$$0 = \varepsilon_{i_k}(\tilde{f}_{i_{k+1}}^{x_{k+1}} \tilde{f}_{i_{k+2}}^{x_{k+2}} \cdots z_{\infty})$$

for all $k \geq 1$. Furthermore, if $\hat{x} \in \mathrm{Im}(\Psi_{\iota^{+}}^{+})$, then $\hat{x}^{*} = \tilde{f}_{i_1}^{x_1} \tilde{f}_{i_2}^{x_2} \cdots z_{\infty}$, and

$$x_k = \varepsilon_{i_k}(\tilde{e}_{i_{k-1}}^{x_{k-1}} \cdots \tilde{e}_{i_2}^{x_2} \tilde{e}_{i_1}^{x_1} \hat{x}^{*})$$

for $k \geq 1$.

(2) *Let* $\hat{x} = (x_0, x_{-1}, \ldots) \in \mathbb{Z}^{-\infty}_{\leq 0,\iota^-}$. *Then,* $\hat{x} \in \mathrm{Im}(\Psi^-_{\iota^-})$ *if and only if*

$$0 = \varphi_{i_k}(\tilde{e}_{i_{k-1}}^{-x_{k-1}} \tilde{e}_{i_{k-2}}^{-x_{k-2}} \cdots z_{-\infty})$$

for all $k \leq 0$. *Furthermore, if* $\hat{x} \in \mathrm{Im}(\Psi^-_{\iota^-})$, *then* $\hat{x}^* = \tilde{e}_{i_0}^{-x_0} \tilde{e}_{i_{-1}}^{-x_{-1}} \cdots z_{-\infty}$, *and*

$$-x_k = \varphi_{i_k}(\tilde{f}_{i_{k+1}}^{-x_{k+1}} \cdots \tilde{f}_{i_{-1}}^{-x_{-1}} \tilde{f}_{i_0}^{-x_0} \hat{x}^*)$$

for $k \leq 0$.

We set $\iota \coloneqq (\iota^+, \iota^-)$, and $\mathbb{Z}_\iota(\mu) \coloneqq \mathbb{Z}^{+\infty}_{\geq 0,\iota^+} \otimes \mathcal{T}_\mu \otimes \mathbb{Z}^{-\infty}_{\leq 0,\iota^-}$ for $\mu \in P$. By the tensor product rule of crystals, we can describe the crystal structure of $\mathbb{Z}_\iota(\mu)$ as follows. Let $\vec{x} = \hat{x}^+ \otimes t_\mu \otimes \hat{x}^- \in \mathbb{Z}_\iota(\mu)$ with $\hat{x}^+ = (\ldots, x_2, x_1) \in \mathbb{Z}^{+\infty}_{\geq 0,\iota^+}$ and $\hat{x}^- = (x_0, x_{-1}, \ldots) \in \mathbb{Z}^{-\infty}_{\leq 0,\iota^-}$. For $k \in \mathbb{Z}$, we set

$$\sigma_k(\vec{x}) \coloneqq \begin{cases} \sigma_k^+(\hat{x}^+) & \text{if } k \geq 1, \\ \sigma_k^-(\hat{x}^-) - \langle \mathrm{wt}(\vec{x}), \alpha_{i_k}^\vee \rangle & \text{if } k \leq 0. \end{cases}$$

For $i \in I$, we set $\sigma_{(i)}(\vec{x}) \coloneqq \max\{\sigma_k(\vec{x}) \mid k \in \mathbb{Z}, i_k = i\}$, and

$$M_{(i)} = M_{(i)}(\vec{x}) \coloneqq \{k \mid i_k = i, \sigma_k(\vec{x}) = \sigma_{(i)}(\vec{x})\}. \tag{2.3}$$

Then we see that

$$\mathrm{wt}(\vec{x}) = \mu - \sum_{j \in \mathbb{Z}} x_j \alpha_{i_j}; \quad \varepsilon_i(\vec{x}) = \sigma_{(i)}(\vec{x}); \quad \varphi_i(\vec{x}) = \varepsilon_i(\vec{x}) + \langle \mathrm{wt}(\vec{x}), \alpha_i^\vee \rangle;$$

if $\varepsilon_i(\vec{x}) = 0$, then $\tilde{e}_i \vec{x} = \mathbf{0}$; if $\varepsilon_i(\vec{x}) > 0$, then

$$\tilde{e}_i \vec{x} = (\ldots, x'_2, x'_1) \otimes t_\mu \otimes (x'_0, x'_{-1}, \ldots) \text{ with } x'_k \coloneqq x_k - \delta_{k, \max M_{(i)}};$$

if $\varphi_i(\vec{x}) = 0$, then $\tilde{f}_i \vec{x} = \mathbf{0}$; if $\varphi_i(\vec{x}) > 0$, then

$$\tilde{f}_i \vec{x} = (\ldots, x'_2, x'_1) \otimes t_\mu \otimes (x'_0, x'_{-1}, \ldots) \text{ with } x'_k \coloneqq x_k + \delta_{k, \min M_{(i)}}. \tag{2.4}$$

The next corollary follows immediately from Theorem 2.5.

Corollary 2.7. *For each* $\mu \in P$, *there exists an embedding* $\Psi^\mu_\iota \coloneqq \Psi^+_{\iota^+} \otimes \mathrm{id} \otimes \Psi^-_{\iota^-} : \mathcal{B}(\infty) \otimes \mathcal{T}_\mu \otimes \mathcal{B}(-\infty) \hookrightarrow \mathbb{Z}_\iota(\mu)$ *of crystals which sends* $u_\infty \otimes t_\mu \otimes u_{-\infty} \in \mathcal{B}(\infty) \otimes \mathcal{T}_\mu \otimes \mathcal{B}(-\infty)$ *to* $z_\mu \coloneqq z_\infty \otimes t_\mu \otimes z_{-\infty} \in \mathbb{Z}_\iota(\mu)$.

We also define the $*$-operation on $\mathrm{Im}(\Psi^\mu_\iota) = \mathrm{Im}(\Psi^+_{\iota^+}) \otimes \mathcal{T}_\mu \otimes \mathrm{Im}(\Psi^-_{\iota^-})$ by the following commutative diagram:

$$\begin{tikzcd} \mathcal{B}(\infty) \otimes \mathcal{T}_\mu \otimes \mathcal{B}(-\infty) \arrow[r, "*"] \arrow[d, "\Psi^\mu_\iota"'] & \mathcal{B}(\infty) \otimes \mathcal{T}_\mu \otimes \mathcal{B}(-\infty) \arrow[d, "\Psi^\mu_\iota"] \\ \mathrm{Im}(\Psi^+_{\iota^+}) \otimes \mathcal{T}_\mu \otimes \mathrm{Im}(\Psi^-_{\iota^-}) \arrow[r, "*"] & \mathrm{Im}(\Psi^+_{\iota^+}) \otimes \mathcal{T}_\mu \otimes \mathrm{Im}(\Psi^-_{\iota^-}). \end{tikzcd}$$

We see by (2.2) that if $z_1 \in \mathrm{Im}(\Psi^+_{\iota^+})$ and $z_2 \in \mathrm{Im}(\Psi^-_{\iota^-})$, then

$$(z_1 \otimes t_\mu \otimes z_2)^* = z_1^* \otimes t_{-\mu - \mathrm{wt}(z_1) - \mathrm{wt}(z_2)} \otimes z_2^*. \tag{2.5}$$

The next corollary is a consequence of Theorem 2.4 and Corollary 2.7.

Corollary 2.8. *For* $\mu \in P$, *the set* $\{\vec{x} \in \mathrm{Im}(\Psi^\mu_\iota) \mid \vec{x}^* \text{ is extremal}\}$ *is a subcrystal of* $\mathrm{Im}(\Psi^\mu_\iota)$, *and is isomorphic, as a crystal, to the crystal basis* $\mathcal{B}(\mu)$ *of the extremal weight module* $V(\mu)$ *of extremal weight* μ.

2.5 Lakshmibai-Seshadri paths.

We recall Lakshmibai-Seshadri paths from [10, §2 and §4]. In this subsection, we fix an integral weight $\mu \in P$.

Definition 2.9. For $\nu, \nu' \in W\mu$, we write $\nu \geq \nu'$ if there exist a sequence $\nu = \nu_0, \nu_1, \dots, \nu_u = \nu'$ of elements in $W\mu$ and a sequence $\beta_1, \beta_2, \dots, \beta_u$ of positive real roots such that $\nu_k = s_{\beta_k}(\nu_{k-1})$ and $\langle \nu_{k-1}, \beta_k^\vee \rangle < 0$ for each $k = 1, 2, \dots, u$. If $\nu \geq \nu'$, then we define $\mathrm{dist}(\nu, \nu')$ to be the maximal length u of all possible such sequences $\nu = \nu_0, \nu_1, \dots, \nu_u = \nu'$.

Remark 2.10. For $\nu, \nu' \in W\mu$ such that $\nu > \nu'$ and $\mathrm{dist}(\nu, \nu') = 1$, there exists a unique positive real root $\beta \in \Delta_{\mathrm{re}}^+$ such that $\nu' = s_\beta(\nu)$.

The Hasse diagram of $W\mu$ is, by definition, the Δ_{re}^+-labeled, directed graph with vertex set $W\mu$, and edges of the following form: $\nu \xleftarrow{\beta} \nu'$ for $\nu, \nu' \in W\mu$ and $\beta \in \Delta_{\mathrm{re}}^+$ such that $\nu > \nu'$ with $\mathrm{dist}(\nu, \nu') = 1$ and $\nu' = s_\beta(\nu)$.

Definition 2.11. Let $\nu, \nu' \in W\mu$ with $\nu > \nu'$, and let $0 < \sigma < 1$ be a rational number. A σ-chain for (ν, ν') is a sequence $\nu = \nu_0, \dots, \nu_u = \nu'$ of elements of $W\mu$ such that $\mathrm{dist}(\nu_{k-1}, \nu_k) = 1$ and $\sigma \langle \nu_{k-1}, \beta_k^\vee \rangle \in \mathbb{Z}_{<0}$ for all $k = 1, 2, \dots, u$, where β_k is the unique positive real root satisfying $\nu_k = s_{\beta_k}(\nu_{k-1})$.

Definition 2.12. Let $\nu_1 > \cdots > \nu_u$ be a finite sequence of elements in $W\mu$, and let $0 = \sigma_0 < \cdots < \sigma_u = 1$ be a finite sequence of rational numbers. The pair $\pi = (\nu_1, \dots, \nu_u; \sigma_0, \dots, \sigma_u)$ is called a Lakshmibai-Seshadri (LS for short) path of shape ν if there exists a σ_k-chain for (ν_k, ν_{k+1}) for each $k = 1, \dots, u-1$. We denote by $\mathbb{B}(\mu)$ the set of LS paths of shape μ.

Let $[0,1] := \{t \in \mathbb{R} \mid 0 \leq t \leq 1\}$. We identify $\pi = (\nu_1, \dots, \nu_u; \sigma_0, \dots, \sigma_u) \in \mathbb{B}(\mu)$ with the following piecewise-linear continuous map $\pi : [0,1] \to \mathbb{R} \otimes_{\mathbb{Z}} P$:

$$\pi(t) = \sum_{k=1}^{j-1} (\sigma_k - \sigma_{k-1})\nu_k + (t - \sigma_{j-1})\nu_j \quad \text{for } \sigma_{j-1} \leq t \leq \sigma_j,\ 1 \leq j \leq u.$$

We endow $\mathbb{B}(\mu)$ with a crystal structure as follows. First, we define $\mathrm{wt}(\pi) := \pi(1)$ for $\pi \in \mathbb{B}(\mu)$; we know from [10, Lemma 4.5 (a)] that $\pi(1) \in P$. Next, for $\pi \in \mathbb{B}(\mu)$ and $i \in I$,

$$\begin{aligned} H_i^\pi(t) &:= \langle \pi(t), \alpha_i^\vee \rangle \quad \text{for } 0 \leq t \leq 1, \\ m_i^\pi &:= \min\{H_i^\pi(t) \mid 0 \leq t \leq 1\}. \end{aligned} \tag{2.6}$$

From [10, Lemma 4.5 (d)], we know that

$$\text{all local minimum values of } H_i^\pi(t) \text{ are integers;} \tag{2.7}$$

in particular, $m_i^\pi \in \mathbb{Z}_{\leq 0}$ and $H_i^\pi(1) - m_i^\pi \in \mathbb{Z}_{\geq 0}$. We define $\tilde{e}_i\pi$ as follows. If $m_i^\pi = 0$, then we set $\tilde{e}_i\pi := \mathbf{0}$. If $m_i^\pi \leq -1$, then we set

$$t_1 := \min\{t \in [0,1] \mid H_i^\pi(t) = m_i^\pi\}, \tag{2.8}$$

$$t_0 := \max\{t \in [0,t_1] \mid H_i^\pi(t) = m_i^\pi + 1\}; \tag{2.9}$$

we see by (2.7) that

$$H_i^\pi(t) \text{ is strictly decreasing on } [t_0, t_1]. \tag{2.10}$$

We define

$$(\tilde{e}_i\pi)(t) := \begin{cases} \pi(t) & \text{if } 0 \leq t \leq t_0, \\ s_i(\pi(t) - \pi(t_0)) + \pi(t_0) & \text{if } t_0 \leq t \leq t_1, \\ \pi(t) + \alpha_i & \text{if } t_1 \leq t \leq 1; \end{cases}$$

we know from [10, §4] that $\tilde{e}_i\pi \in \mathbb{B}(\mu)$. Similarly, we define $\tilde{f}_i\pi$ as follows. If $H_i^\pi(1) - m_i^\pi = 0$, then we set $\tilde{f}_i\pi := \mathbf{0}$. If $H_i^\pi(1) - m_i^\pi \geq 1$, then we set

$$t_0 := \max\{t \in [0,1] \mid H_i^\pi(t) = m_i^\pi\}, \tag{2.11}$$
$$t_1 := \min\{t \in [t_0,1] \mid H_i^\pi(t) = m_i^\pi + 1\}; \tag{2.12}$$

we see by (2.7) that $H_i^\pi(t)$ is strictly increasing on $[t_0, t_1]$. We define

$$(\tilde{f}_i\pi)(t) := \begin{cases} \pi(t) & \text{if } 0 \leq t \leq t_0, \\ s_i(\pi(t) - \pi(t_0)) + \pi(t_0) & \text{if } t_0 \leq t \leq t_1, \\ \pi(t) - \alpha_i & \text{if } t_1 \leq t \leq 1; \end{cases}$$

we know from [10, §4] that $\tilde{f}_i\pi \in \mathbb{B}(\mu)$. We set $\tilde{e}_i\mathbf{0} = \tilde{f}_i\mathbf{0} := \mathbf{0}$ for $i \in I$. Finally, for $\pi \in \mathbb{B}(\mu)$ and $i \in I$, we set

$$\varepsilon_i(\pi) := \max\{k \in \mathbb{Z}_{\geq 0} \mid \tilde{e}_i^k\pi \neq \mathbf{0}\}, \quad \varphi_i(\pi) := \max\{k \in \mathbb{Z}_{\geq 0} \mid \tilde{f}_i^k\pi \neq \mathbf{0}\}.$$

We know from [10, Lemma 2.1 (c)] that

$$\varepsilon_i(\pi) = -m_i^\pi, \quad \varphi_i(\pi) = H_i^\pi(1) - m_i^\pi. \tag{2.13}$$

Theorem 2.13 ([10, §2, §4])**.** *The set $\mathbb{B}(\mu)$, together with the maps $\mathrm{wt} : \mathbb{B}(\mu) \to P$, $\tilde{e}_i, \tilde{f}_i : \mathbb{B}(\mu) \to \mathbb{B}(\mu) \cup \{\mathbf{0}\}$, $i \in I$, and $\varepsilon_i, \varphi_i : \mathbb{B}(\mu) \to \mathbb{Z}_{\geq 0}$, $i \in I$, is a crystal.*

For $\pi = (\nu_1, \dots, \nu_u; \sigma_0, \dots, \sigma_u) \in \mathbb{B}(\mu)$, we set $\iota(\pi) := \nu_1$ and $\kappa(\pi) := \nu_u$. For $\pi \in \mathbb{B}(\mu)$ and $i \in I$, we set $\tilde{e}_i^{\max}\pi := \tilde{e}_i^{\varepsilon_i(\pi)}\pi$ and $\tilde{f}_i^{\max}\pi := \tilde{f}_i^{\varphi_i(\pi)}\pi$.

Lemma 2.14 ([10, Proposition 4.7])**.** *Let $\pi \in \mathbb{B}(\mu)$, and $i \in I$. If $\langle \iota(\pi), \alpha_i^\vee \rangle < 0$, then $\iota(\tilde{e}_i^{\max}\pi) = s_i\iota(\pi)$. If $\langle \kappa(\pi), \alpha_i^\vee \rangle > 0$, then $\kappa(\tilde{f}_i^{\max}\pi) = s_i\kappa(\pi)$.*

3 Main results.

In what follows, we assume that the generalized Cartan matrix A is

$$A = \begin{pmatrix} 2 & -a_1 \\ -a_2 & 2 \end{pmatrix}, \quad \text{where } a_1, a_2 \in \mathbb{Z}_{\geq 1} \text{ with } a_1 a_2 > 4.$$

Let Λ_1, Λ_2 denote the fundamental weights for $\mathfrak{g} = \mathfrak{g}(A)$; note that $P = \mathbb{Z}\Lambda_1 \oplus \mathbb{Z}\Lambda_2$ and $\alpha_1 = 2\Lambda_1 - a_2\Lambda_2$, $\alpha_2 = -a_1\Lambda_1 + 2\Lambda_2$.

Theorem 3.1 (will be proved in §4)**.** *Let $\mathbb{O} := \{W\mu \mid \mu \in P\}$ be the set of W-orbits in P.*

(1) *Assume that $a_1, a_2 \geq 2$. Then, $O \in \mathbb{O}$ satisfies condition* (2.1)*, that is, $O \cap (P^+ \cup -P^+) = \emptyset$ if and only if O contains an integral weight λ of the form either* (i) *or* (ii)*:*

 (i) *$\lambda = k_1\Lambda_1 - k_2\Lambda_2$ for some $k_1, k_2 \in \mathbb{Z}_{>0}$ such that $k_2 \leq k_1 < (a_1 - 1)k_2$;*

 (ii) *$\lambda = k_1\Lambda_1 - k_2\Lambda_2$ for some $k_1, k_2 \in \mathbb{Z}_{>0}$ such that $k_1 < k_2 \leq (a_2 - 1)k_1$.*

(2) *Assume that $a_1 = 1$. Then, $O \in \mathbb{O}$ satisfies condition* (2.1) *if and only if O contains an integral weight λ of the form $\lambda = k_1\Lambda_1 - k_2\Lambda_2$ for some $k_1, k_2 \in \mathbb{Z}_{>0}$ such that $2k_1 \leq k_2 \leq (a_2 - 2)k_1$.*

(3) *Assume that $a_2 = 1$. Then, $O \in \mathbb{O}$ satisfies condition* (2.1) *if and only if O contains an integral weight λ of the form $\lambda = k_1\Lambda_1 - k_2\Lambda_2$ for some $k_1, k_2 \in \mathbb{Z}_{>0}$ such that $2k_2 \leq k_1 \leq (a_1 - 2)k_2$.*

Let $\lambda = k_1\Lambda_1 - k_2\Lambda_2 \in P$ be an integral weight of the form mentioned in Theorem 3.1 above.

Theorem 3.2 (will be proved in §5)**.** *The crystal graph of $\mathbb{B}(\lambda)$ is connected if and only if $k_1 = 1$ or $k_2 = 1$. Otherwise, the crystal graph of $\mathbb{B}(\lambda)$ has infinitely many connected components.*

Let $\mathbb{B}_0(\lambda)$ (resp., $\mathcal{B}_0(\lambda)$) be the connected component of $\mathbb{B}(\lambda)$ (resp., $\mathcal{B}(\lambda)$) containing $\pi_\lambda := (\lambda; 0, 1)$ (resp., u_λ).

Theorem 3.3 (will be proved in §6.2)**.** *There exists an isomorphism $\mathbb{B}_0(\lambda) \to \mathcal{B}_0(\lambda)$ of crystals that sends π_λ to u_λ.*

Theorem 3.4 (will be proved in §6.3)**.** *Assume that $k_1 = 1$ or $k_2 = 1$. For $b \in \mathcal{B}(\lambda)$, there exist $i_1, \ldots, i_r \in I$ such that $b = \tilde{f}_{i_r} \cdots \tilde{f}_{i_1} u_\lambda$ or $b = \tilde{e}_{i_r} \cdots \tilde{e}_{i_1} u_\lambda$. In particular, the crystal graph of $\mathcal{B}(\lambda)$ is connected.*

If $k_1 = 1$ or $k_2 = 1$, then we see by Theorem 3.4 (resp., Theorem 3.2) that $\mathcal{B}(\lambda) = \mathcal{B}_0(\lambda)$ (resp., $\mathbb{B}(\lambda) = \mathbb{B}_0(\lambda)$). Therefore, by Theorem 3.3, we obtain the following corollary.

Corollary 3.5. *If $k_1 = 1$ or $k_2 = 1$, then there exists an isomorphism $\mathbb{B}(\lambda) \to \mathcal{B}(\lambda)$ of crystals that sends π_λ to u_λ.*

We define the sequence $\{p_m\}_{m \in \mathbb{Z}}$ of integers by the following recursive formulas: for $m \geq 0$,

$$p_0 := k_2, \quad p_1 := k_1, \quad p_{m+2} := \begin{cases} a_2 p_{m+1} - p_m & \text{if } m \text{ is even,} \\ a_1 p_{m+1} - p_m & \text{if } m \text{ is odd;} \end{cases} \tag{3.1}$$

for $m < 0$,

$$p_m = \begin{cases} a_2 p_{m+1} - p_{m+2} & \text{if } m \text{ is even,} \\ a_1 p_{m+1} - p_{m+2} & \text{if } m \text{ is odd.} \end{cases} \tag{3.2}$$

We assume that $\iota = (\iota^+, \iota^-)$ with $\iota^+ = (\ldots, i_2, i_1) := (\ldots, 2, 1, 2, 1)$ and $\iota^- = (i_0, i_{-1}, \ldots) := (2, 1, 2, 1, \ldots)$. We set

$$\alpha := \frac{a_1 a_2 + \sqrt{a_1^2 a_2^2 - 4a_1 a_2}}{2a_2}, \quad \beta := \frac{a_1 a_2 + \sqrt{a_1^2 a_2^2 - 4a_1 a_2}}{2a_1},$$

and

$$\gamma_k \coloneqq \begin{cases} \alpha & \text{if } k \text{ is even,} \\ \beta & \text{if } k \text{ is odd} \end{cases} \tag{3.3}$$

for $k \in \mathbb{Z}$; note that $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$ and $\alpha, \beta > 0$. By the definition, we have

$$\frac{1}{\gamma_k} + \gamma_{k+1} = a_{i_k}. \tag{3.4}$$

We regard $\mathbb{R}^\infty \coloneqq \{\vec{x} = (\ldots, x_2, x_1) \otimes t_\lambda \otimes (x_0, x_{-1}, \ldots) \mid x_k \in \mathbb{R} \text{ and } x_k = 0 \text{ for } |k| \gg 0\}$ as an infinite dimensional vector space over $\mathbb{R}$; note that $\mathbb{Z}_\iota(\lambda) \subset \mathbb{R}^\infty$. Let $(\mathbb{R}^\infty)^* \coloneqq \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^\infty, \mathbb{R})$ be its dual space. For $k \in \mathbb{Z}$, we define the linear function $\zeta_k \in (\mathbb{R}^\infty)^*$ by $\zeta_k(\vec{x}) \coloneqq x_k$ for $\vec{x} = (\ldots, x_2, x_1) \otimes t_\lambda \otimes (x_0, x_{-1}, \ldots) \in \mathbb{R}^\infty$. Set

$$\Sigma_\iota(\lambda) \coloneqq \{\vec{x} \in \mathbb{Z}_\iota(\lambda) \mid \varphi(\vec{x}) \geq 0 \text{ for all } \varphi \in \Xi_\iota[\lambda]\},$$

where

$$\begin{aligned} \Xi_\iota[\lambda] = & \{\gamma_0 p_0 + \gamma_0 \zeta_0 - \zeta_1,\ \gamma_1 p_1 + \zeta_0 - \gamma_1 \zeta_1\} \\ & \cup \{p_k - \zeta_k,\ \gamma_k \zeta_k - \zeta_{k+1},\ \gamma_{k+1} p_{k+1} - p_k + \zeta_k - \gamma_{k+1}\zeta_{k+1} \mid k \geq 1\} \\ & \quad \cup \{p_k + \zeta_k,\ \zeta_{k-1} - \gamma_k \zeta_k,\ \gamma_{k-1} p_{k-1} - p_k + \gamma_{k-1}\zeta_{k-1} - \zeta_k \mid k \leq 0\}. \end{aligned}$$

Theorem 3.6 (will be proved in §7.2)**.** *The set $\Sigma_\iota(\lambda)$ is a subcrystal of $\mathrm{Im}(\Psi_\iota^\lambda)$.*

Theorem 3.7 (will be proved in §7.3)**.** *The equality $\Sigma_\iota(\lambda) = \{\vec{x} \in \mathrm{Im}(\Psi_\iota^\lambda) \mid \vec{x}^* \text{ is extremal}\}$ holds. Therefore, $\Sigma_\iota(\lambda)$ is isomorphic, as a crystal, to the crystal basis $\mathcal{B}(\lambda)$ of the extremal weight module $V(\lambda)$ of extremal weight λ.*

4 Weyl group orbit O satisfying $O \cap (P^+ \cup -P^+) = \emptyset$.

4.1 Proof of Theorem 3.1 (1).

Lemma 4.1. *An orbit $O \in \mathbb{O}$ contains $\lambda \in P$ of the form either* (i) *or* (ii) *in* Theorem 3.1 (1) *if and only if O contains $\lambda' \in P$ of the form either* (a)–(d)*:*

(a) $\lambda' = k\Lambda_1 - k\Lambda_2$ *for some* $k \in \mathbb{Z}_{>0}$*;*

(b) $\lambda' = k\Lambda_1 - (a_2 - 1)k\Lambda_2$ *for some* $k \in \mathbb{Z}_{>0}$*;*

(c) $\lambda' = k\Lambda_1 - l\Lambda_2$ *for some* $k, l \in \mathbb{Z}_{>0}$ *such that* $l < k < (a_1 - 1)l$*;*

(d) $\lambda' = k\Lambda_1 - l\Lambda_2$ *for some* $k, l \in \mathbb{Z}_{>0}$ *such that* $k < l < (a_2 - 1)k$*.*

Proof. The "only if" part is obvious. We show the "if" part. If λ' is of the form (c) (resp., (d)), then it is obvious that λ' is of the form (i) (resp., (ii)). Assume that λ' is of the form (a). If $a_1 \geq 3$, then λ' is of the form (i). If $a_1 = 2$, then we see that $a_2 \geq 3$ and O contains $s_1 s_2 \lambda' = s_1 s_2 (k\Lambda_1 - k\Lambda_2) = k\Lambda_1 - (a_2 - 1)k\Lambda_2$, which is of the form (ii). Assume that λ' is of the form (b). If $a_2 \geq 3$, then λ' is of the form (ii). If $a_2 = 2$, then we see that $a_1 \geq 3$ and $\lambda' = k\Lambda_1 - (a_2 - 1)k\Lambda_2 = k\Lambda_1 - k\Lambda_2$ is of the form (i). Thus we have proved the lemma. $\square$

For $\lambda \in P$ of the form $\lambda = k\Lambda_1 - l\Lambda_2$ with $k, l \in \mathbb{Z}$, we define the sequence $\{p_m\}_{m \in \mathbb{Z}}$ of integers by the following recursive formulas: For $m \geq 0$,

$$p_0 = l, \quad p_1 = k, \quad p_{m+2} = \begin{cases} a_2 p_{m+1} - p_m & \text{if } m \text{ is even,} \\ a_1 p_{m+1} - p_m & \text{if } m \text{ is odd;} \end{cases} \tag{4.1}$$

for $m < 0$,

$$p_m = \begin{cases} a_2 p_{m+1} - p_{m+2} & \text{if } m \text{ is even,} \\ a_1 p_{m+1} - p_{m+2} & \text{if } m \text{ is odd.} \end{cases} \tag{4.2}$$

For $m \in \mathbb{Z}$, we set

$$w_m := \begin{cases} (s_2 s_1)^n & \text{if } m = 2n \text{ with } n \in \mathbb{Z}_{\geq 0}, \\ s_1 (s_2 s_1)^n & \text{if } m = 2n + 1 \text{ with } n \in \mathbb{Z}_{\geq 0}, \\ (s_1 s_2)^{-n} & \text{if } m = 2n \text{ with } n \in \mathbb{Z}_{\leq 0}, \\ s_2 (s_1 s_2)^{-n} & \text{if } m = 2n - 1 \text{ with } n \in \mathbb{Z}_{\leq 0}; \end{cases}$$

note that $W = \{w_m \mid m \in \mathbb{Z}\}$. By induction on $|m|$, we can show the following lemma.

Lemma 4.2. *For $m \in \mathbb{Z}$,*

$$w_m \lambda = \begin{cases} p_{m+1}\Lambda_1 - p_m \Lambda_2 & \text{if } m \text{ is even,} \\ -p_m \Lambda_1 + p_{m+1}\Lambda_2 & \text{if } m \text{ is odd.} \end{cases} \tag{4.3}$$

Corollary 4.3. *Let $\lambda = k\Lambda_1 - l\Lambda_2 \in P$ be an integral weight. The Weyl group orbit $W\lambda \in \mathbb{O}$ satisfies condition* (2.1) *if and only if $p_m > 0$ for all $m \in \mathbb{Z}$ or $p_m < 0$ for all $m \in \mathbb{Z}$.*

Lemma 4.4. *Let $\lambda = k\Lambda_1 - l\Lambda_2 \in P$.*

(1) *If there exists $n' \in \mathbb{Z}_{\geq 0}$ such that $0 < p_{n'} < p_{n'+1}$, then $0 < p_n < p_{n+1}$ for all $n \geq n'$.*

(2) *If there exists $n' \in \mathbb{Z}_{\leq 1}$ such that $0 < p_{n'} < p_{n'-1}$, then $0 < p_n < p_{n-1}$ for all $n \leq n'$.*

Proof. We give a proof only for part (1); the proof for part (2) is similar. We proceed by induction on n. The assertion is trivial when $n = n'$. Assume that $n > n'$. We set

$$a' := \begin{cases} a_1 & \text{if } n \text{ is even,} \\ a_2 & \text{if } n \text{ is odd;} \end{cases}$$

note that $p_{n+1} = a'p_n - p_{n-1}$. Then we compute

$$p_{n+1} - p_n = (a'p_n - p_{n-1}) - p_n = (a' - 1)(p_n - p_{n-1}) + (a' - 2)p_{n-1}.$$

Because $a' \geq 2$, and $p_n > p_{n-1} > 0$ by the induction hypothesis, we obtain $p_{n+1} > p_n$ as desired. $\square$

Proposition 4.5. *Let $\lambda = k\Lambda_1 - l\Lambda_2 \in P$ with $k, l > 0$. If $p_m \neq p_{m+1}$ for any $m \in \mathbb{Z}$, then the following are equivalent.*

(1) *The Weyl group orbit $W\lambda$ satisfies condition* (2.1)*, or equivalently, $p_m > 0$ for all $m \in \mathbb{Z}$ by* Corollary 4.3 *and the assumption that $k, l > 0$.*

(2) *There exists an element $\lambda' = k'\Lambda_1 - l'\Lambda_2$ in $W\lambda$ satisfying the conditions that $k', l' \in \mathbb{Z}_{>0}$, and $l' < k' < (a_1 - 1)l'$ or $k' < l' < (a_2 - 1)k'$.*

Proof. (1) $\Rightarrow$ (2): Since p_m is a positive integer for every $m \in \mathbb{Z}$ by the assumption in (1), and since $p_m \neq p_{m+1}$ for any $m \in \mathbb{Z}$ by the assumption, there exists $n \in \mathbb{Z}$ such that $p_{n-1} > p_n < p_{n+1}$. If n is even, then we have $(a_1 - 1)p_n - p_{n+1} = p_{n-1} - p_n > 0$ by (4.1) and (4.2). Hence, $\lambda' := w_n\lambda = p_{n+1}\Lambda_1 - p_n\Lambda_2$ satisfies the condition $p_n < p_{n+1} < (a_1 - 1)p_n$. Similarly, if n is odd, then we have $(a_2 - 1)p_n - p_{n-1} = p_{n+1} - p_n > 0$ by (4.1) and (4.2). Hence, $\lambda' := w_{n-1}\lambda = p_n\Lambda_1 - p_{n-1}\Lambda_2$ satisfies the condition $p_n < p_{n-1} < (a_2 - 1)p_n$.

(2) $\Rightarrow$ (1): Assume that $W\lambda$ contains an element λ' of the form $\lambda' = k'\Lambda_1 - l'\Lambda_2 \in W\lambda$ with $k', l' \in \mathbb{Z}_{>0}$ such that $l' < k' < (a_1 - 1)l'$ (resp., $k' < l' < (a_2 - 1)k'$). We define the sequence $\{p'_m\}_{m\in\mathbb{Z}}$ for λ' in the same manner as (4.1) and (4.2):

$$p'_0 = l', \quad p'_1 = k', \quad p'_{m+2} = \begin{cases} a_2 p'_{m+1} - p'_m & \text{if } m \text{ is even,} \\ a_1 p'_{m+1} - p'_m & \text{if } m \text{ is odd.} \end{cases}$$

Since $l' < k' < (a_1 - 1)l'$ (resp., $k' < l' < (a_2 - 1)k'$), it is easy to check that $p'_{-1} > p'_0 < p'_1$ (resp., $p'_0 > p'_1 < p'_2$). By Lemma 4.4, we obtain $p'_m > 0$ for all $m \in \mathbb{Z}$. Hence, we see from Corollary 4.3 that $W\lambda' = W\lambda$ satisfies condition (2.1). Thus, we have proved the proposition. $\square$

Remark 4.6. By Lemma 4.4 and the proof of Proposition 4.5, we see that if λ is of the form (c) (resp., (d)) in Lemma 4.1, then

$$\cdots > p_{-1} > p_0 = l < p_1 = k < p_2 < \cdots$$
$$(\text{resp., } \cdots > p_{-1} > p_0 = l > p_1 = k < p_2 < \cdots),$$

where the sequence $\{p_m\}_{m\in\mathbb{Z}}$ is defined by the recursive formulas (4.1) and (4.2) for λ.

Proof of Theorem 3.1 (1). By Lemma 4.1, it suffices to show that O satisfies condition (2.1) if and only if O contains $\lambda \in P$ of the form either (a)–(d) in Lemma 4.1.

First, we prove the "if" part. We know from [16, Proposition 3.1.1] that if $\mu = \Lambda_1 - \Lambda_2$, then $W\mu$ satisfies condition (2.1). Hence, $W(k\mu)$ also satisfies condition (2.1) for every $k \in \mathbb{Z}\backslash\{0\}$. Since $s_1(k\Lambda_1 - (a_2 - 1)k\Lambda_2) = -k\Lambda_1 + k\Lambda_2 = -k\mu$, we see that for λ of the form (b), $W\lambda$ satisfies condition (2.1). Also, we see from (2) $\Rightarrow$ (1) in Proposition 4.5 that for λ of the form (c) or (d), $W\lambda$ satisfies condition (2.1). Thus we have proved the "if" part.

Next, we prove the "only if" part. Assume that $O \in \mathbb{O}$ satisfies condition (2.1). By Lemma 4.2, we see that O contains $\lambda = k\Lambda_1 - l\Lambda_2$ such that $k, l > 0$. Then we define the sequence $\{p_m\}_{m\in\mathbb{Z}}$ by the recursive formulas (4.1) and (4.2) for this λ. If $p_m = p_{m+1}$ for some $m \in \mathbb{Z}$, we see by Lemma 4.2 that $O = W\lambda$ contains $p_m\Lambda_1 - p_m\Lambda_2$ or $-p_m\Lambda_1 + p_m\Lambda_2 = s_1(p_m\Lambda_1 - (a_2 - 1)p_m\Lambda_2)$. Hence, $W\lambda$ contains an integral weight of the form either (a) or (b). If $p_m \neq p_{m+1}$ for any $m \in \mathbb{Z}$, then we see from (1) $\Rightarrow$ (2) in Proposition 4.5 that $O = W\lambda$ contains an integral weight of the form (c) or (d). Thus we have proved Theorem 3.1 (1). $\square$

4.2 Proofs of Theorem 3.1 (2) and (3).

We give a proof only for part (3); the proof for part (2) is similar. For $\mu = k\Lambda_1 - l\Lambda_2 \in P$, we define the sequence $\{p^\mu_m\}_{m\in\mathbb{Z}}$ of integers by the following recursive formulas: for $m \geq 0$,

$$p^\mu_0 := l, \quad p^\mu_1 := k, \quad p^\mu_{m+2} := \begin{cases} a_2 p^\mu_{m+1} - p^\mu_m & \text{if } m \text{ is even,} \\ a_1 p^\mu_{m+1} - p^\mu_m & \text{if } m \text{ is odd;} \end{cases} \tag{4.4}$$

for $m < 0$,

$$p^\mu_m = \begin{cases} a_2 p^\mu_{m+1} - p^\mu_{m+2} & \text{if } m \text{ is even,} \\ a_1 p^\mu_{m+1} - p^\mu_{m+2} & \text{if } m \text{ is odd;} \end{cases} \tag{4.5}$$

note that for $m \in \mathbb{Z}$,

$$w_m\mu = \begin{cases} p^\mu_{m+1}\Lambda_1 - p^\mu_m\Lambda_2 & \text{if } m \text{ is even,} \\ -p^\mu_m\Lambda_1 + p^\mu_{m+1}\Lambda_2 & \text{if } m \text{ is odd.} \end{cases} \tag{4.6}$$

Lemma 4.7. *Assume that $a_1 \geq 5$ and $a_2 = 1$. Let $\mu \in P$.*

(1) *If there exists $n \in \mathbb{Z}$ such that $0 < p^\mu_{2n} \leq p^\mu_{2n+2}$, then $0 < p^\mu_{2m} \leq p^\mu_{2m+2}$ for all $m \geq n$.*

(2) *If there exists $n \in \mathbb{Z}$ such that $0 < p^\mu_{2n} \leq p^\mu_{2n-2}$, then $0 < p^\mu_{2m} \leq p^\mu_{2m-2}$ for all $m \leq n$.*

Proof. We give a proof only for part (1); the proof for part (2) is similar. We proceed by induction on m. If $m = n$, then the assertion is trivial. Assume that $m > n$. By (4.4) and (4.5), we have $p^\mu_{2m+2} - p^\mu_{2m} = (a_1 - 3)(p^\mu_{2m} - p^\mu_{2m-2}) + (a_1 - 4)p^\mu_{2m-2}$. Since $p^\mu_{2m} - p^\mu_{2m-2} \geq 0$ and $p^\mu_{2m-2} > 0$ by the induction hypothesis, we obtain $p^\mu_{2m+2} - p^\mu_{2m} > 0$. $\square$

Proof of Theorem 3.1 (3). Assume that $O \in \mathbb{O}$ satisfies condition (2.1). We can take $\mu = k\Lambda_1 - l\Lambda_2 \in O$ such that $k, l > 0$. Then we see by the assumption and (4.6) that $p^\mu_m > 0$ for all $m \in \mathbb{Z}$. Hence it follows from Lemma 4.7 that there exists $n \in \mathbb{Z}$ such that

$$\cdots \geq p^\mu_{2n-4} \geq p^\mu_{2n-2} \geq p^\mu_{2n} \leq p^\mu_{2n+2} \leq p^\mu_{2n+4} \leq \cdots. \tag{4.7}$$

By (4.4) and (4.5), we have $p^\mu_{2n-2} - p^\mu_{2n} = (a_1 - 2)p^\mu_{2n} - p^\mu_{2n+1}$ and $p^\mu_{2n+2} - p^\mu_{2n} = p^\mu_{2n+1} - 2p^\mu_{2n}$. Hence we see by (4.7) that $2p^\mu_{2n} \leq p^\mu_{2n+1} \leq (a_1 - 2)p^\mu_{2n}$. Then, $\lambda := w_{2n}\mu = p^\mu_{2n+1}\Lambda_1 - p^\mu_{2n}\Lambda_2 \in W\mu = O$ satisfies the desired condition.

Let $\lambda = k_1\Lambda_1 - k_2\Lambda_2$ for some $k_1, k_2 \in \mathbb{Z}_{>0}$ such that $2k_2 \leq k_1 \leq (a_1 - 2)k_2$; we show that $O := W\lambda$ satisfies condition (2.1). By (4.6), it suffices to show that $p^\lambda_m > 0$ for all $m \in \mathbb{Z}$. By (4.4), (4.5), and the assumption that $2k_2 \leq k_1 \leq (a_1 - 2)k_2$, we obtain $p^\lambda_2 - p^\lambda_0 = p^\lambda_1 - 2p^\lambda_0 = k_1 - 2k_2 \geq 0$ and $p^\lambda_{-2} - p^\lambda_0 = (a_1 - 2)p^\lambda_0 - p^\lambda_1 = (a_1 - 2)k_2 - k_1 \geq 0$. Hence we see by Lemma 4.7 that $p^\lambda_{2m} > 0$ for all $m \in \mathbb{Z}$. Note that $p^\lambda_{2m-1} = p^\lambda_{2m+2} + p^\lambda_{2m}$ by (4.4) and (4.5). Since $p^\lambda_{2m}, p^\lambda_{2m+2} > 0$ as seen above, we get $p^\lambda_{2m-1} = p^\lambda_{2m+2} + p^\lambda_{2m} > 0$ for all $m \in \mathbb{Z}$. Thus we have proved Theorem 3.1 (3). $\square$

Remark 4.8. By the argument above, we see that if $\lambda = k_1\Lambda_1 - k_2\Lambda_2$ satisfies $2k_2 \leq k_1 \leq (a_1 - 2)k_2$, then $p^\lambda_{2m} < p^\lambda_{2m+1}$ for all $m \in \mathbb{Z}$, and

$$\cdots \geq p^\lambda_{-2} \geq p^\lambda_0 = k_2 \leq p^\lambda_2 \leq p^\lambda_4 \leq \cdots.$$

In particular, we have $p^\lambda_0 \leq p^\lambda_n$ for all $n \in \mathbb{Z}$.

5 Connectedness of the crystals of LS paths.

Throughout this section, we assume that $\lambda = k_1\Lambda_1 - k_2\Lambda_2$ is of the form mentioned in Theorem 3.1. Theorem 3.2 is a consequence of the following three propositions.

Proposition 5.1 (will be proved in §5.2). *If $k_1 = 1$ or $k_2 = 1$, then the crystal graph of $\mathbb{B}(\lambda)$ is connected.*

Proposition 5.2 (will be proved in §5.3). *Assume that k_1 and k_2 are relatively prime. If $k_1 \neq 1$ and $k_2 \neq 1$, then the crystal graph of $\mathbb{B}(\lambda)$ has infinitely many connected components.*

Proposition 5.3 (will be proved in §5.4). *If k_1 and k_2 are not relatively prime, then the crystal graph of $\mathbb{B}(\lambda)$ has infinitely many connected components.*

5.1 Hasse diagram of $W\lambda$.

We draw the Hasse diagram of $W\lambda$ (in the ordering of Definition 2.9). Recall that $p_m > 0$ for all $m \in \mathbb{Z}$ (by Corollary 4.3).

Proposition 5.4 (cf. [16, Proposition 3.2.5]). *The Hasse diagram of $W\lambda$ is*

$$\cdots \xleftarrow{\alpha_1} w_2\lambda \xleftarrow{\alpha_2} w_1\lambda \xleftarrow{\alpha_1} w_0\lambda \xleftarrow{\alpha_2} w_{-1}\lambda \xleftarrow{\alpha_1} w_{-2}\lambda \xleftarrow{\alpha_2} \cdots.$$

Proof. For $m \in \mathbb{Z}$, we set

$$i := \begin{cases} 2 & \text{if } m \text{ is even,} \\ 1 & \text{if } m \text{ is odd.} \end{cases}$$

Since $s_i w_m\lambda = w_{m-1}\lambda$ and $\langle w_m\lambda, \alpha_i^\vee \rangle = -p_m < 0$ for every $m \in \mathbb{Z}$ by Lemma 4.2, we have $w_m\lambda > w_{m-1}\lambda$. Hence, we have

$$\cdots > w_2\lambda > w_1\lambda > w_0\lambda > w_{-1}\lambda > w_{-2}\lambda > \cdots; \tag{5.1}$$

it is obvious from (5.1) that $\mathrm{dist}(w_m\lambda, w_{m-1}\lambda) = 1$. Thus, we have proved the proposition. $\square$

For each $\nu \in W\lambda$, there exists unique $m \in \mathbb{Z}$ such that $\nu = w_m\lambda$. Then we define $z(\nu) := m$. By the definition of LS paths and Proposition 5.4, we have $z(\nu_1) > z(\nu_2) > \cdots > z(\nu_u)$ for

$$\pi = (\nu_1, \nu_2, \ldots, \nu_u; \sigma_0, \sigma_1, \ldots, \sigma_u) \in \mathbb{B}(\lambda).$$

We define the subset $\mathbb{B}_1(\lambda)$ of $\mathbb{B}(\lambda)$ by

$$\mathbb{B}_1(\lambda) := \{(\nu_1, \ldots, \nu_u; \sigma_0, \ldots, \sigma_u) \in \mathbb{B}(\lambda) \mid z(\nu_v) - z(\nu_{v+1}) = 1 \text{ for } v = 1, \ldots, u-1\}.$$

Remark 5.5. Let $\pi = (\nu_1, \nu_2, \ldots, \nu_s; \sigma_0, \sigma_1, \ldots, \sigma_s) \in \mathbb{B}_1(\lambda)$. By Lemma 4.2, we see that the function $H_i^\pi(t)$ for $i \in I$ attains its minimal and maximal values at $t = \sigma_u$, $u = 0, 1, \ldots, s$, alternately. Namely, if $H_i^\pi(t)$ for $i \in I$ attains a minimal (resp., maximal) value at $t = \sigma_v$, then $H_i^\pi(t)$ attains a minimal (resp., maximal) value at $t = \sigma_u$ for all $u = 0, 1, \ldots, s$ such that $u \equiv v \bmod 2$.

Proposition 5.6. *Let $\pi \in \mathbb{B}_1(\lambda)$, and $i \in I$. If $\tilde{e}_i\pi \neq \mathbf{0}$, then $\tilde{e}_i\pi \in \mathbb{B}_1(\lambda)$. If $\tilde{f}_i\pi \neq \mathbf{0}$, then $\tilde{f}_i\pi \in \mathbb{B}_1(\lambda)$. Therefore, $\mathbb{B}_1(\lambda)$ is a subcrystal of $\mathbb{B}(\lambda)$.*

Proof. Let $\pi = (\nu_1, \ldots, \nu_u; \sigma_0, \ldots, \sigma_u) \in \mathbb{B}_1(\lambda)$ with $u \geq 1$. We show that $\tilde{e}_i\pi \in \mathbb{B}_1(\lambda)$ if $\tilde{e}_i\pi \neq \mathbf{0}$. Take t_1 and t_0 as (2.8) and (2.9), respectively; note that $t_1 = \sigma_n$ for some $1 \leq n \leq u$. Since $\langle \nu_n, \alpha_i^\vee \rangle < 0$ by the definition of t_1, we see from Lemma 4.2 that $z(\nu_n)$ is odd if $i = 1$ and even if $i = 2$. Hence we have

$$s_i\nu_n = s_i w_{z(\nu_n)}\lambda = w_{z(\nu_n)-1}\lambda. \tag{5.2}$$

If $u = 1$, then $n = 1$. By (5.2),

$$\tilde{e}_i\pi = \begin{cases} (\nu_1, w_{z(\nu_1)-1}\lambda; 0, t_0, 1) & \text{if } 0 < t_0, \\ (w_{z(\nu_1)-1}\lambda; 0, 1) & \text{if } 0 = t_0, \end{cases}$$

and hence $\tilde{e}_i\pi \in \mathbb{B}_1(\lambda)$. Assume that $u \geq 2$. If $n \leq u-1$, then it follows from the definition if $\mathbb{B}_1(\lambda)$ that $\nu_{n+1} = w_{z(\nu_{n+1})}\lambda = w_{z(\nu_n)-1}\lambda$. By (5.2), we obtain $s_i\nu_n = \nu_{n+1}$. If $n \geq 2$, then we see by Remark 5.5 that $H_i^\pi(\sigma_{n-2})$ attains a minimal value. By the definition of t_1, we have $H_i^\pi(\sigma_{n-2}) > H_i^\pi(\sigma_n)$. Since $H_i^\pi(\sigma_n), H_i^\pi(\sigma_{n-2}) \in \mathbb{Z}$ by (2.7), we obtain $H_i^\pi(\sigma_{n-2}) \geq H_i^\pi(\sigma_n) + 1$. Hence we see by Remark 5.5 that $\sigma_{n-1} < t_0$. Therefore,

$$\tilde{e}_i\pi = \begin{cases} (\nu_1, \dots, \nu_u; \sigma_0, t_0, \sigma_2, \dots, \sigma_u) & \text{if } n = 1 \text{ and } 0 < t_0, \\ (\nu_2, \dots, \nu_u; \sigma_0, \sigma_2, \dots, \sigma_u) & \text{if } n = 1 \text{ and } 0 = t_0, \\ (\nu_1, \dots, \nu_u; \sigma_0, \dots, \sigma_{n-1}, t_0, \sigma_{n+1}, \dots, \sigma_u) & \text{if } 2 \leq n \leq u-1, \\ (\nu_1, \dots, \nu_u, w_{z(\nu_u)-1}\lambda; \sigma_0, \dots, \sigma_{u-1}, t_0, \sigma_u) & \text{if } n = u, \end{cases}$$

and hence $\tilde{e}_i\pi \in \mathbb{B}_1(\lambda)$. Similarly, we can show that $\tilde{f}_i\pi \in \mathbb{B}_1(\lambda)$ if $\tilde{f}_i\pi \neq \mathbf{0}$. Thus we have proved the proposition. □

Remark 5.7. The element $\pi_\lambda = (\lambda; 0, 1)$ is contained in $\mathbb{B}_1(\lambda)$. Hence, by Proposition 5.6, it follows that $\mathbb{B}_0(\lambda) \subset \mathbb{B}_1(\lambda)$.

5.2 Proof of Proposition 5.1.

Lemma 5.8 (cf. [16, Lemma 4.1.1, Theorem 4.1.2])**.** *Assume that $\lambda = k_1\Lambda_1 - k_2\Lambda_2 \in P$ is as in* Theorem 3.1. *In addition, assume that k_1 and k_2 are relatively prime.*

(1) *For every $m \in \mathbb{Z}$, the numbers p_m and p_{m+1} (defined by* (3.1) *and* (3.2) *for λ) are relatively prime.*

(2) *Let $0 < \sigma < 1$ be a rational number, and let $\mu, \nu \in W\lambda$ be such that $\mu > \nu$. If $\mu = \mu_0 > \mu_1 > \cdots > \mu_s = \nu$ is a σ-chain for (μ, ν), then $s = 1$.*

(3) *An LS path π of shape λ is of the form*

$$(w_{m+s-1}\lambda, \dots, w_{m+1}\lambda, w_m\lambda; \sigma_0, \sigma_1, \dots, \sigma_s),$$

where $m \in \mathbb{Z}$, $s \geq 0$, and $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_s = 1$ satisfy the condition that $p_{m+s-v}\sigma_v \in \mathbb{Z}$ for $1 \leq v \leq s-1$.

Proof. Part (1) can be easily shown by induction on $|m|$.

Let us show part (2). Suppose, for a contradiction, that $s \geq 2$. Since $\mathrm{dist}(\mu_{v-1}, \mu_v) = 1$ for every $v = 1, 2, \dots, s$ by the definition of a σ-chain, it follows from Proposition 5.4 that there exists $m \in \mathbb{Z}$ such that $\mu_v = w_{m-v}\lambda$ for $v = 0, 1, \dots, s$. We set

$$i := \begin{cases} 2 & \text{if } m \text{ is even}, \\ 1 & \text{if } m \text{ is odd}, \end{cases} \qquad j := \begin{cases} 1 & \text{if } m \text{ is even}, \\ 2 & \text{if } m \text{ is odd}. \end{cases}$$

By Lemma 4.2, we see that $\langle w_m\lambda, \alpha_i^\vee \rangle = -p_m$, $\langle w_{m-1}\lambda, \alpha_j^\vee \rangle = -p_{m-1}$. Since $-p_m$ and $-p_{m-1}$ are relatively prime by part (1), there does not exist $0 < \sigma < 1$ such that $\sigma\langle w_m\lambda, \alpha_i^\vee \rangle = -\sigma p_m \in \mathbb{Z}_{<0}$ and $\sigma\langle w_{m-1}\lambda, \alpha_j^\vee \rangle = -\sigma p_{m-1} \in \mathbb{Z}_{<0}$. This contradicts the assumption that the sequence is a σ-chain.

Part (3) follows immediately from the definition of an LS path and part (2). □

Remark 5.9. Assume that k_1 and k_2 are relatively prime. We see from Lemma 5.8 (3) that $\mathbb{B}_1(\lambda) = \mathbb{B}(\lambda)$. In particular, we see from Remark 5.5 that for $\pi = (\nu_1, \nu_2, \dots, \nu_s; \sigma_0, \sigma_1, \dots, \sigma_s) \in \mathbb{B}(\lambda)$, the function $H_i^\pi(t)$ for $i \in I$ attains its minimal and maximal values at $t = \sigma_u$, $u = 0, 1, \dots, s$, alternately.

In the remainder of this subsection, we give a proof of Proposition 5.1 for the case that $k_2 = 1$; the proof for the case that $k_1 = 1$ is similar. Our proof is essentially the same as [16, Proof of Theorem 3.2.1].

Proposition 5.10. *Let $\pi \in \mathbb{B}(\lambda)$, and write it as (see* Lemma 5.8 (3)*):*

$$\pi = (w_m\lambda, w_{m-1}\lambda, \dots, w_{n+1}\lambda, w_n\lambda; \sigma_0, \dots, \sigma_{m-n+1}) \tag{5.3}$$

for some $n \le m$ and $0 = \sigma_0 < \cdots < \sigma_{m-n+1} = 1$. Then, either $0 \le n \le m$ or $n \le m \le -1$ holds.

Proof. Suppose, for a contradiction, that $m \ge 0$ and $n \le -1$. By the definition of an LS path, there exists a σ_{m+1}-chain for $(\lambda, w_{-1}\lambda)$. It follows from Proposition 5.4 that $\operatorname{dist}(\lambda, w_{-1}\lambda) = 1$ and $s_2\lambda = w_{-1}\lambda$. Thus, we obtain $\langle \lambda, \alpha_2^\vee \rangle = -k_2 = -1$ and $0 < \sigma_{m+1} < 1$, which contradicts $\sigma_{m+1}\langle \lambda, \alpha_2^\vee \rangle \in \mathbb{Z}$. $\square$

Theorem 5.11. *For each $\pi \in \mathbb{B}(\lambda)$, $\pi = \tilde{f}_{i_r} \cdots \tilde{f}_{i_1}\pi_\lambda$ or $\pi = \tilde{e}_{i_r} \cdots \tilde{e}_{i_1}\pi_\lambda$ for some $i_1, \dots, i_r \in I$, where $\pi_\lambda := (\lambda; 0, 1)$. In particular, the crystal graph of $\mathbb{B}(\lambda)$ is connected (which proves* Proposition 5.1*).*

Proof. Write $\pi \in \mathbb{B}(\lambda)$ as (5.3). By Proposition 5.10, either $0 \le n \le m$ or $n \le m \le -1$ holds. We show by induction on m that if $0 \le n \le m$, then $\pi = \tilde{f}_{i_r} \cdots \tilde{f}_{i_1}\pi_\lambda$ for some $i_1, \dots, i_r \in I$. If $m = 0$, then $n = 0$, and hence $\pi = \pi_\lambda$. Thus the claim is obvious. Assume that $m > 0$. We set

$$i := \begin{cases} 2 & \text{if } m \text{ is even,} \\ 1 & \text{if } m \text{ is odd;} \end{cases}$$

note that $\langle w_m\lambda, \alpha_i^\vee \rangle < 0$ and $s_i w_m\lambda = w_{m-1}\lambda$ by Proposition 5.4. By Lemma 2.14, we see that $\tilde{e}_i^{\max}\pi \in \mathbb{B}(\lambda)$ satisfies $\iota(\tilde{e}_i^{\max}\pi) = s_i\iota(\pi) = s_i w_m\lambda = w_{m-1}\lambda$. Hence, by the induction hypothesis, $\tilde{e}_i^{\max}\pi = \tilde{f}_{i_r} \cdots \tilde{f}_{i_1}\pi_\lambda$ for some $i_1, \dots, i_r \in I$. Hence, we obtain $\pi = \tilde{f}_i^{\varepsilon_i(\pi)}\tilde{f}_{i_r} \cdots \tilde{f}_{i_1}\pi_\lambda$, as desired. Similarly, we can show that if $n \le m \le -1$, then $\pi = \tilde{e}_{i_r} \cdots \tilde{e}_{i_1}\pi_\lambda$ for some $i_1, \dots, i_r \in I$. Thus we have proved Theorem 5.11. $\square$

5.3 Proof of Proposition 5.2.

We give a proof of Proposition 5.2 only for the case that $k_2 \le k_1$; the proof for the case that $k_1 \le k_2$ is similar. It follows from Theorem 3.1 and the assumption of Proposition 5.2 that either (5.4) or (5.5) holds:

$$a_1, a_2 \ge 2 \quad \text{and} \quad 1 < k_2 < k_1 < (a_1 - 1)k_2; \tag{5.4}$$

$$a_1 \ge 5, a_2 = 1 \quad \text{and} \quad 2 < 2k_2 < k_1 < (a_1 - 2)k_2. \tag{5.5}$$

There exists a (unique) integer $c \in \{1, 2, \dots, k_1 - 1\}$ such that

$$\frac{c}{k_1} < \frac{1}{k_2} < \frac{c+1}{k_1}. \tag{5.6}$$

Then we define the sequence $\{q_m\}_{m\in\mathbb{Z}}$ of integers by the following recursive formula:

$$q_0 = 1, \quad q_1 = c, \quad q_{m+2} = \begin{cases} a_2 q_{m+1} - q_m & \text{if } m \text{ is even,} \\ a_1 q_{m+1} - q_m & \text{if } m \text{ is odd.} \end{cases} \tag{5.7}$$

Recall that $\{p_m\}_{m\in\mathbb{Z}}$ is defined by the recursive formulas (3.1) and (3.2).

Lemma 5.12. *It hold that $0 < q_m < p_m$ and $q_m p_{m+1} - q_{m+1}p_m = k_1 - k_2 c$ for all $m \in \mathbb{Z}$. In particular, we have $0 < q_{m+1}/p_{m+1} < q_m/p_m < 1$ for all $m \in \mathbb{Z}$.*

Proof. Recall that either (5.4) or (5.5) holds.

Case 1. Assume that (5.4) holds. First, let us show that $q_m > 0$ for all $m \in \mathbb{Z}$. Since $q_1 = c \geq 1 = q_0$, we see by the same argument as Lemma 4.4 (1) that $q_{m+1} \geq q_m$ for all $m \geq 0$; in particular, $q_m > 0$ for all $m \geq 0$. Since $k_1/k_2 < a_1 - 1$ by (5.4), and $c < k_1/k_2$ by (5.6), we see that

$$q_{-1} - q_0 = (a_1 - c) - 1 = (a_1 - 1) - c \geq \frac{k_1}{k_2} - \frac{k_1}{k_2} = 0,$$

and hence $q_{-1} \geq q_0$. By the same argument as Lemma 4.4 (2), we see that $q_{m-1} \geq q_m$ for all $m \leq -1$; in particular, $q_m > 0$ for all $m \leq -1$.

Next, let us show that $q_m < p_m$ for all $m \in \mathbb{Z}$. If we set $d_m := p_m - q_m$ for $m \in \mathbb{Z}$, then we have

$$d_0 = p_0 - q_0, \quad d_1 = p_1 - q_1, \quad d_{m+2} = \begin{cases} a_2 d_{m+1} - d_m & \text{if } m \text{ is even,} \\ a_1 d_{m+1} - d_m & \text{if } m \text{ is odd.} \end{cases}$$

By the same argument as above, it suffices to show that $d_1 \geq d_0$ and $d_0 \leq d_{-1}$. First, we show that $d_1 - d_0 = (k_1 - c) - (k_2 - 1) \geq 0$. Since $c < k_1/k_2$ by (5.6), and $k_2 < k_1$ by (5.4), we have

$$(k_1 - c) - (k_2 - 1) > k_1 - \frac{k_1}{k_2} - k_2 + 1 = k_1\left(1 - \frac{1}{k_2}\right) - k_2 + 1 > k_2\left(1 - \frac{1}{k_2}\right) - k_2 + 1 = 0.$$

Next, we show that $d_{-1} - d_0 = (a_1k_2 - k_1 - a_1 + c) - (k_2 - 1) \geq 0$. Note that (5.4) implies $(k_1 + k_2)/k_2 < a_1$, and (5.6) implies $(k_1 - k_2)/k_2 < c$. Then we compute

$$\begin{aligned}(a_1k_2 - k_1 - a_1 + c) - (k_2 - 1) &= a_1(k_2 - 1) - k_1 + c - k_2 + 1 \\ &> \frac{k_1 + k_2}{k_2}(k_2 - 1) - k_1 + \frac{k_1 - k_2}{k_2} - k_2 + 1 = -1.\end{aligned}$$

Because $d_{-1} - d_0$ is an integer, it follows that $d_{-1} \geq d_0$.

Finally, the equality $q_m p_{m+1} - q_{m+1} p_m = k_1 - k_2 c$ for $m \in \mathbb{Z}$ can be easily shown by induction on $|m|$.

Case 2. Assume that (5.5) holds. First, let us show that $q_m > 0$ for all $m \in \mathbb{Z}$. By the same argument as for p_m^λ in the proof of Theorem 3.1 (3), we see that the assertion follows from $q_{-2} \geq q_0 \leq q_2$. Since $k_1/k_2 > 2$ by (5.5), and $c > k_1/k_2 - 1$ by (5.6), we see that

$$q_2 - q_0 = (c - 1) - 1 = c - 2 > \left(\frac{k_1}{k_2} - 1\right) - 2 > 2 - 1 - 2 = -1.$$

Because $q_2 - q_0$ is an integer, we have $q_2 \geq q_0$. Similarly, since $k_1/k_2 < a_1 - 2$ by (5.5), and $c < k_1/k_2$ by (5.6), we see that

$$q_{-2} - q_0 = (a_1 - 2) - c > \frac{k_1}{k_2} - \frac{k_1}{k_2} = 0,$$

and hence $q_{-2} \geq q_0$.

Next, let us show that $q_m < p_m$ for all $m \in \mathbb{Z}$. If we set $d_m := p_m - q_m$ for $m \in \mathbb{Z}$, then we have

$$d_0 = p_0 - q_0, \quad d_1 = p_1 - q_1, \quad d_{m+2} = \begin{cases} a_2 d_{m+1} - d_m & \text{if } m \text{ is even,} \\ a_1 d_{m+1} - d_m & \text{if } m \text{ is odd.} \end{cases}$$

By the same argument as above, it suffices to show that $d_2 \geq d_0 \leq d_{-2}$. We first show that $d_2 - d_0 = k_1 - 2k_2 - c + 2 \geq 0$. Since $c < k_1/k_2$ by (5.6), and $2k_2 < k_1$ by (5.5), we have

$$k_1 - 2k_2 - c + 2 > k_1 - 2k_2 - \frac{k_1}{k_2} + 2 = k_1\left(1 - \frac{1}{k_2}\right) - 2k_2 + 2 > 2k_2\left(1 - \frac{1}{k_2}\right) - 2k_2 + 2 = 0.$$

Hence we obtain $d_2 \geq d_0$. We next show that $d_{-2} - d_0 \geq 0$. Since $(k_1 + 2k_2)/k_2 < a_1$ by (5.5), and $(k_1 - k_2)/k_2 < c$ by (5.6), we have

$$\begin{aligned} d_{-2} - d_0 &= (a_1 - 2)k_2 - k_1 - a_1 + 2 + c \\ &= a_1(k_2 - 1) - 2k_2 - k_1 + 2 + c \\ &> \frac{k_1 + 2k_2}{k_2}(k_2 - 1) - 2k_2 - k_1 + 2 + \frac{k_1 - k_2}{k_2} = -1. \end{aligned}$$

Because $d_{-2} - d_0$ is an integer, it follows that $d_{-2} \geq d_0$.

Finally, the equality $q_m p_{m+1} - q_{m+1} p_m = k_1 - k_2 c$ for $m \in \mathbb{Z}$ can be easily shown by induction on $|m|$. □

For $m \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 1}$, we say that $\pi = (\nu_1, \nu_2, \dots \nu_u; \sigma_0, \sigma_1, \dots, \sigma_u) \in \mathbb{B}(\lambda)$ satisfies the condition $C(m, n)$ if $u \geq 2n + 1$, and there exists $v \in \mathbb{Z}$ such that $n < v < u - n + 1$, $\nu_v = w_m \lambda$, and $\sigma_{v+s} = q_{m-s}/p_{m-s}$ for $s = -n, -n+1, \dots, n-1$; in this case, we see from Lemma 5.8 (3), along with $\nu_v = w_m \lambda$, that

$$\nu_{v+s} = w_{m-s}\lambda \quad \text{for} \quad v + s = 1, 2, \dots, u. \tag{5.8}$$

Thus, π is of the form

$$\pi = \Big(w_{m+v-1}\lambda, w_{m+v-2}\lambda, \dots, w_{m+v-u}\lambda; \\ \sigma_0, \sigma_1, \dots, \sigma_{v-n-1}, \frac{q_{m+n}}{p_{m+n}}, \frac{q_{m+n-1}}{p_{m+n-1}}, \dots, \frac{q_{m-n+1}}{p_{m-n+1}}, \sigma_{v+n}, \dots, \sigma_u \Big).$$

We set

$$j := \begin{cases} 2 & \text{if } m \text{ is even}, \\ 1 & \text{if } m \text{ is odd}, \end{cases} \qquad j' := \begin{cases} 1 & \text{if } m \text{ is even}, \\ 2 & \text{if } m \text{ is odd}; \end{cases} \tag{5.9}$$

note that

$$\langle w_{m+v-h}\lambda, \alpha_j^\vee \rangle = \begin{cases} -p_{m+v-h} & \text{if } h \equiv v \bmod 2, \\ p_{m+v-h+1} & \text{if } h \not\equiv v \bmod 2 \end{cases} \tag{5.10}$$

by (5.8) and Lemma 4.2.

Lemma 5.13. *If* $\pi = (w_{m+v-1}\lambda, w_{m+v-2}\lambda, \dots, w_{m+v-u}\lambda; \sigma_0, \sigma_1, \dots, \sigma_u) \in \mathbb{B}(\lambda)$ *satisfies the condition* $C(m, n)$*, then for each* $r = v - n, v - n + 1, \dots, v + n - 1$*,*

(1) $H_j^\pi(\sigma_r) = H_j^\pi(\sigma_v) \in \mathbb{Z}$ *if* $r \equiv v \bmod 2$*,*

(2) $H_j^\pi(\sigma_v) < H_j^\pi(\sigma_r) < H_j^\pi(\sigma_v) + 1$ *if* $r \not\equiv v \bmod 2$*;*

in particular, $\{H_j^\pi(t) \mid \sigma_{v-n} \leq t \leq \sigma_{v+n-1}\} \subset [H_j^\pi(\sigma_v), H_j^\pi(\sigma_v)+1)$*, where* $[a, b) := \{t \in \mathbb{R} \mid a \leq t < b\}$ *for* $a, b \in \mathbb{R}$*.*

Proof. Assume that $\pi = (w_{m+v-1}\lambda, w_{m+v-2}\lambda, \dots, w_{m+v-u}\lambda; \sigma_0, \sigma_1, \dots, \sigma_u) \in \mathbb{B}(\lambda)$ satisfies the condition $C(m, n)$. For $h = v - n + 1, v - n + 2, \dots, v + n - 1$, we see that

$$\begin{aligned} a(h) &:= H_j^\pi(\sigma_h) - H_j^\pi(\sigma_{h-1}) \\ &= \big(H_j^\pi(\sigma_{h-1}) + \langle w_{m+v-h}\lambda, \alpha_j^\vee \rangle (\sigma_h - \sigma_{h-1})\big) - H_j^\pi(\sigma_{h-1}) \\ &= \langle w_{m+v-h}\lambda, \alpha_j^\vee \rangle (\sigma_h - \sigma_{h-1}). \end{aligned}$$

By (5.10) and the assumption that π satisfies the condition $C(m,n)$, we obtain

$$a(h) = \begin{cases} -p_{m+v-h}\left(\dfrac{q_{m+v-h}}{p_{m+v-h}} - \dfrac{q_{m+v-h+1}}{p_{m+v-h+1}}\right) & \text{if } h \equiv v \bmod 2, \\ p_{m+v-h+1}\left(\dfrac{q_{m+v-h}}{p_{m+v-h}} - \dfrac{q_{m+v-h+1}}{p_{m+v-h+1}}\right) & \text{if } h \not\equiv v \bmod 2. \end{cases} \tag{5.11}$$

We see from Lemma 5.12 that

$$p_z\left(\frac{q_z}{p_z} - \frac{q_{z+1}}{p_{z+1}}\right) = \frac{k_1 - k_2c}{p_{z+1}} \quad \text{and} \quad p_{z+1}\left(\frac{q_z}{p_z} - \frac{q_{z+1}}{p_{z+1}}\right) = \frac{k_1 - k_2c}{p_z} \tag{5.12}$$

for each $z \in \mathbb{Z}$. Here we recall form Remark 4.6 that $p_z \geq p_0 = k_2$ (see also Remark 4.8). By (5.6), we have

$$\frac{k_1 - k_2c}{p_z} \leq \frac{k_1 - k_2c}{p_0} < 1 \tag{5.13}$$

for all $z \in \mathbb{Z}$. Combining (5.11)–(5.13), we deduce that

$$0 < a(h) < 1 \text{ if } h \not\equiv v \bmod 2, \quad \text{and } -1 < a(h) < 0 \text{ if } h \equiv v \bmod 2. \tag{5.14}$$

Let $M := \{x \in \mathbb{Z} \mid v - n \leq x \leq v + n - 1\}$; note that $\{v-1, v\} \subset M$ for all $n \in \mathbb{Z}_{\geq 1}$. Let $r \in M$ be such that $r \equiv v \bmod 2$. We see from Remark 5.9 and (2.7) that $H_j^{\pi}(\sigma_r) \in \mathbb{Z}$. If $r+2 \in M$ (resp., $r-2 \in M$), then we see by (5.14) that $|H_j^{\pi}(\sigma_{r+2}) - H_j^{\pi}(\sigma_r)| = |a(r+1) + a(r+2)| < 1$ (resp., $|H_j^{\pi}(\sigma_r) - H_j^{\pi}(\sigma_{r-2})| = |a(r-1) + a(r)| < 1$). We see by (2.7) that $H_j^{\pi}(\sigma_r) = H_j^{\pi}(\sigma_{r+2})$ (resp., $H_j^{\pi}(\sigma_r) = H_j^{\pi}(\sigma_{r-2})$). Hence, we obtain $H_j^{\pi}(\sigma_r) = H_j^{\pi}(\sigma_v)$ for all $r \in M$ such that $r \equiv v \bmod 2$. Thus we have shown part (1).

Let $r \in M$ be such that $r \not\equiv v \bmod 2$. If $r+1 \in M$ (resp., $r-1 \in M$), then we see that $H_j^{\pi}(\sigma_r) = H_j^{\pi}(\sigma_{r+1}) - a(r+1)$ (resp., $H_j^{\pi}(\sigma_r) = H_j^{\pi}(\sigma_{r-1}) + a(r)$). By part (1), we obtain $H_j^{\pi}(\sigma_r) = H_j^{\pi}(\sigma_v) - a(r+1)$ (resp., $H_j^{\pi}(\sigma_r) = H_j^{\pi}(\sigma_v) + a(r)$). We see by (5.14) that $H_j^{\pi}(\sigma_v) < H_j^{\pi}(\sigma_r) < H_j^{\pi}(\sigma_v) + 1$ for all $r \in M$ such that $r \not\equiv v \bmod 2$. Thus we have shown part (2), thereby completing the proof of Lemma 5.13. □

Proposition 5.14. *Fix $n \geq 1$. Assume that $\pi \in \mathbb{B}(\lambda)$ satisfies the condition $C(m,n)$ for some $m \in \mathbb{Z}$. Let $i \in I$. If $\tilde{e}_i\pi \neq \mathbf{0}$, then $\tilde{e}_i\pi$ satisfies the condition $C(m,n)$ or $C(m-1,n)$. If $\tilde{f}_i\pi \neq \mathbf{0}$, then $\tilde{f}_i\pi$ satisfies the condition $C(m,n)$ or $C(m+1,n)$.*

Proof. For simplicity, we prove the assertion only for the case of $n = 2$. Define j and j' as (5.9).

Now, assume that $\tilde{f}_j\pi \neq \mathbf{0}$; we show that $\tilde{f}_j\pi$ satisfies the condition $C(m,2)$. For this, it suffices to show that $(\tilde{f}_j\pi)(t) = \pi(t)$ or $(\tilde{f}_j\pi)(t) = \pi(t) - \alpha_j$ for $t \in (\sigma_{v-3}, \sigma_{v+2}) = \{t \in \mathbb{R} \mid \sigma_{v-3} < t < \sigma_{v+2}\}$. By Lemma 5.13, we have $H_j^{\pi}(\sigma_{v-2}) = H_j^{\pi}(\sigma_v)$, $H_j^{\pi}(\sigma_{v-2}) < H_j^{\pi}(\sigma_{v-1}) < H_j^{\pi}(\sigma_{v-2}) + 1$, and $H_j^{\pi}(\sigma_v) < H_j^{\pi}(\sigma_{v+1}) < H_j^{\pi}(\sigma_v) + 1$. Note that there exists $\sigma_{v+1} < t' \leq \sigma_{v+2}$ such that $H_j^{\pi}(\sigma_{v-2}) = H_j^{\pi}(\sigma_v) = H_j^{\pi}(t')$ since $\langle \nu_{v+2}, \alpha_j^{\vee} \rangle < 0$ and $H_j^{\pi}(\sigma_{v+2}) \in \mathbb{Z}$ by Remark 5.9 and (2.7); see Figure 1. Let t_0 and t_1 be as (2.11) and (2.12), respectively; note that $t_0 = \sigma_{s'}$ for some $0 \leq s' \leq u$. By Lemma 5.13 and the definition of t_0, we obtain $t_0 < \sigma_{v-3}$ or $\sigma_{v+2} \leq t_0$. Let m_j^{π} be as (2.6). If $u = 5$, then we obtain $v = 3$ since $2 = n < v < u - n + 1 = 4$. Hence, we see that $m_j^{\pi} = H_j^{\pi}(\sigma_5)$, which contradicts the assumption that $\tilde{f}_j\pi \neq \mathbf{0}$. Therefore we obtain $u \geq 6$. If $\sigma_{v+2} \leq t_0$, then it is obvious from the definition of $\tilde{f}_j$ that $(\tilde{f}_j\pi)(t) = \pi(t)$ for $t \in (\sigma_{v-3}, \sigma_{v+2})$. If $t_0 < \sigma_{v-3}$, then $H_j^{\pi}(\sigma_{v-2}) > H_j^{\pi}(t_0) = m_j^{\pi} \in \mathbb{Z}$ by the definition of t_0. Note that $H_j^{\pi}(\sigma_{v-2}) \in \mathbb{Z}$ by (2.7), and hence $H_j^{\pi}(\sigma_{v-2}) \geq H_j^{\pi}(t_0) + 1 = m_j^{\pi} + 1$. Because $H_j^{\pi}(\sigma_{v-3}) > H_j^{\pi}(\sigma_{v-2}) \geq m_j^{\pi} + 1$, we see that $t_1 < \sigma_{v-3}$. Therefore we obtain $(\tilde{f}_j\pi)(t) = \pi(t) - \alpha_j$ for $t \in (\sigma_{v-3}, \sigma_{v+2})$ by the definition of $\tilde{f}_j$.

Assume that $\tilde{e}_j\pi \neq \mathbf{0}$; we show that $\tilde{e}_j\pi$ satisfies the condition $C(m,2)$ or $C(m-1,2)$. Take t_1 and t_0 as (2.8) and (2.9), respectively; note that $t_1 = \sigma_{s'}$ for some $0 \leq s' \leq u$. By the definition

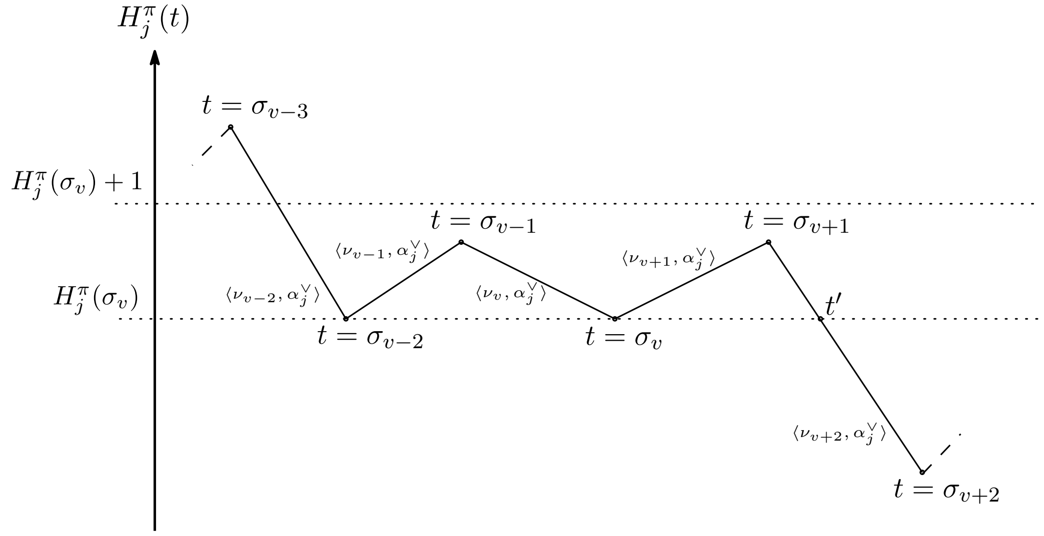

Figure 1.

of t_1 and Lemma 5.13, we obtain $t_1 < \sigma_{v-3}$, $\sigma_{v+2} \le t_1$, or $t_1 = \sigma_{v-2}$. If $t_1 < \sigma_{v-3}$, then it is obvious by the definition of $\tilde{e}_j$ that $(\tilde{e}_j\pi)(t) = \pi(t) + \alpha_j$ for $t \in (\sigma_{v-3}, \sigma_{v+2})$, and hence $\tilde{e}_j\pi$ satisfies the condition $C(m,2)$. If $\sigma_{v+2} < t_1$, then $H_j^\pi(\sigma_{v+2}) > H_j^\pi(t_1) = m_j^\pi \in \mathbb{Z}$ by the definition of t_1. Note that $H_j^\pi(\sigma_{v+2}) \in \mathbb{Z}$ by (2.7), and hence $H_j^\pi(\sigma_{v+2}) \ge H_j^\pi(t_1) + 1 = m_j^\pi + 1$. Because $H_j^\pi(\sigma_{v+3}) > H_j^\pi(\sigma_{v+2}) \ge m_j^\pi + 1$, we see that $\sigma_{v+3} < t_0$. Therefore we obtain $(\tilde{e}_j\pi)(t) = \pi(t)$ for $t \in (\sigma_{v-3}, \sigma_{v+2})$, and hence $\tilde{e}_j\pi$ satisfies the condition $C(m,2)$. Assume that $t_1 = \sigma_{v+2}$. Since $H_j^\pi(\sigma_v) = H_j^\pi(t') > H_j^\pi(t_1) = m_j^\pi \in \mathbb{Z}$ by the definition of t_1, we see that $H_j^\pi(t') \ge H_j^\pi(t_1) + 1$. Hence, we obtain $t' \le t_0$. Therefore, we have

$$\tilde{e}_j\pi = \begin{cases} (\nu_1, \ldots, \nu_u, w_{m-3}\lambda; \sigma_0, \ldots, \sigma_{v+1}, t_0, 1) & \text{if } v = u-2, \\ (\nu_1, \ldots, \nu_u; \sigma_0, \ldots, \sigma_{v+1}, t_0, \sigma_{v+3}, \ldots, \sigma_u) & \text{if } v < u-2, \end{cases}$$

which satisfies the condition $C(m,2)$. Assume that $t_1 = \sigma_{v-2}$. If $v = 3$, then it is obvious that $\sigma_0 = \sigma_{v-3} \le t_0$; note that $\sigma_{v-3} = t_0$ if and only if $H_j^\pi(\sigma_1) = -1$. If $v > 3$, then $H_j^\pi(\sigma_{v-4}) > H_j^\pi(t_1) = m_j^\pi \in \mathbb{Z}$ by the definition of t_1. Note that $H_j^\pi(\sigma_{v-4}) \in \mathbb{Z}$ by (2.7), and hence $H_j^\pi(\sigma_{v-4}) \ge H_j^\pi(t_1) + 1 = m_j^\pi + 1$. Because $H_j^\pi(\sigma_{v-3}) > H_j^\pi(\sigma_{v-4}) \ge m_j^\pi + 1$, we see that $\sigma_{v-3} < t_0$. Therefore we see that

$$\tilde{e}_j\pi = \begin{cases} (\nu_2, \nu_3, \ldots, \nu_u; \sigma_0, \sigma_2, \sigma_3, \ldots, \sigma_u) & \text{if } v = 3 \text{ and } H_j^\pi(\sigma_1) = -1, \\ (\nu_1, \ldots, \nu_u; \sigma_0, \ldots, \sigma_{v-3}, t_0, \sigma_{v-1}, \ldots, \sigma_u) & \text{otherwise.} \end{cases}$$

Also, we see by $H_j^\pi(\sigma_{v-2}) = H_j^\pi(t')$ and the definition of t_1 that $t' = \sigma_{v+2}$. Since $H_j^\pi(\sigma_v) = H_j^\pi(\sigma_{v+2})$, we have

$$a(v+1) = H_j^\pi(\sigma_{v+1}) - H_j^\pi(\sigma_{v+2}). \tag{5.15}$$

Here we can rewrite (5.15) as

$$p_m \left(\frac{q_{m-1}}{p_{m-1}} - \frac{q_m}{p_m} \right) = p_{m-2} \left(\sigma_{v+2} - \frac{q_{m-1}}{p_{m-1}} \right).$$

By (3.1), (3.2), and (5.7), we see that

$$\sigma_{v+2} = \frac{1}{p_{m-2}} \left(\frac{(p_m + p_{m-2})q_{m-1}}{p_{m-1}} - q_m \right) = \frac{q_{m-2}}{p_{m-2}}.$$

Since $\sigma_{v+2} = q_{m-2}/p_{m-2} < 1$ by Lemma 5.12, we obtain $v + 2 < u$. Write $\tilde{e}_j\pi$ as:

$$\tilde{e}_j\pi = (\nu'_1, \ldots, \nu'_{u'}; \sigma'_0, \ldots, \sigma'_{u'}).$$

If $v = 3$ and $H^{\pi}_{j}(\sigma_1) = -1$, then $u' = u - 1, \nu'_s = \nu_{s+1}$ for $s = 1, \ldots u'$, and $\sigma'_0 = 0$, $\sigma'_s = \sigma_{s+1}$ for $s = 1, \ldots u'$. We set $v' := v$. Then we obtain $2 < v' < u' - 1$, $\nu'_{v'} = w_{m-1}\lambda$, and $\sigma'_{v'+s} = q_{(m-1)-s}/p_{(m-1)-s}$ for $s = -2, -1, 0, 1$. Hence we see that $\tilde{e}_j\pi$ satisfies the condition $C(m-1, 2)$. If $v > 3$, or $v = 3$ and $H^{\pi}_{j}(\sigma_1) \neq -1$, then $u' = u, \nu'_s = \nu_s$ for $s = 1, \ldots u'$, and $\sigma'_{v-2} = t_0$, $\sigma'_s = \sigma_s$ for $s = 1, \ldots, v-3, v-1, \ldots, u'$. Hence, we see that $\tilde{e}_j\pi$ satisfies the condition $C(m-1, 2)$ with $v' := v+1$.

Similarly, we can show that if $\tilde{e}_{j'}\pi \neq \mathbf{0}$ (resp., $\tilde{f}_{j'}\pi \neq \mathbf{0}$), then it satisfies the condition $C(m, 2)$ (resp., $C(m, 2)$ or $C(m+1, 2)$). Thus we have proved Proposition 5.14. □

Now, for each $n \in \mathbb{Z}_{\geq 1}$, we define

$$\begin{aligned} &\underline{\nu^{(n)}} : w_n\lambda, \ldots, w_1\lambda, w_0\lambda, w_{-1}\lambda, \ldots, w_{-n}\lambda, \\ &\underline{\sigma^{(n)}} : 0 < \frac{q_n}{p_n} < \cdots < \frac{q_1}{p_1} < \frac{q_0}{p_0} < \frac{q_{-1}}{p_{-1}} < \cdots < \frac{q_{-n+1}}{p_{-n+1}} < 1; \end{aligned}$$

the inequalities in $\underline{\sigma^{(n)}}$ follow from Lemma 5.12. We see that $\pi^{(n)} := (\underline{\nu^{(n)}}, \underline{\sigma^{(n)}})$ is an LS path of shape λ satisfying the condition $C(0, n)$. We denote by $\mathbb{B}(\lambda; \pi^{(n)})$ the connected component of $\mathbb{B}(\lambda)$ containing $\pi^{(n)}$; note that an element of $\mathbb{B}(\lambda; \pi^{(n)})$ satisfies the condition $C(m, n)$ for some $m \in \mathbb{Z}$ by Proposition 5.14.

Corollary 5.15. *If $n \neq n'$, then $\mathbb{B}(\lambda; \pi^{(n)}) \cap \mathbb{B}(\lambda; \pi^{(n')}) = \emptyset$. In particular, the crystal graph of $\mathbb{B}(\lambda)$ has infinitely many connected components (which proves* Proposition 5.2*).*

Proof. We may assume that $n' < n$. Suppose, for a contradiction, that $\mathbb{B}(\lambda; \pi^{(n)}) = \mathbb{B}(\lambda; \pi^{(n')})$. For $\pi = (\nu_1, \ldots, \nu_u; \sigma_0, \ldots, \sigma_u) \in \mathbb{B}(\lambda)$, we define $\ell(\pi) := u$. By Proposition 5.14, we see that if $\pi \in \mathbb{B}(\lambda; \pi^{(n)})$, then $\ell(\pi) \geq 2n+1$. Hence, we have $\ell(\pi^{(n')}) \geq 2n+1$ since $\pi^{(n')} \in \mathbb{B}(\lambda; \pi^{(n)})$. However, we have $\ell(\pi^{(n')}) = 2n' + 1 < 2n + 1$ by the definition of $\pi^{(n')}$ and $n' < n$, which is a contradiction. Thus we have proved Corollary 5.15. □

5.4 Proof of Proposition 5.3.

Assume that k_1 and k_2 are not relatively prime. Let $d \geq 2$ be the greatest common divisor of k_1 and k_2. We set $k'_1 := k_1/d$, $k'_2 := k_2/d$, and $\lambda' := (1/d)\lambda = k'_1\Lambda_1 - k'_2\Lambda_2 \in P$. Define the sequence $\{p_m\}_{m\in\mathbb{Z}}$ by (3.1) and (3.2) for λ':

$$p_0 = k'_2, \quad p_1 = k'_1, \quad p_{m+2} = \begin{cases} a_2 p_{m+1} - p_m & \text{if } m \text{ is even,} \\ a_1 p_{m+1} - p_m & \text{if } m \text{ is odd.} \end{cases}$$

Then we see from Lemma 5.8 (1) that p_m and p_{m+1} are relatively prime for all $m \in \mathbb{Z}$. Note that

$$w_m\lambda = \begin{cases} dp_{m+1}\Lambda_1 - dp_m\Lambda_2 & \text{if } m \text{ is even,} \\ -dp_m\Lambda_1 + dp_{m+1}\Lambda_2 & \text{if } m \text{ is odd,} \end{cases} \tag{5.16}$$

for $m \in \mathbb{Z}$.

We first show that the proof of Proposition 5.3 is reduced to the case that $d = 2$, i.e., $\lambda = 2\lambda'$. For this purpose, we recall the concatenation of LS paths (in a general setting). Let $\mu \in P$ be an arbitrary integral weight, and $m \in \mathbb{Z}_{\geq 1}$. For $\pi_1, \pi_2, \ldots, \pi_m \in \mathbb{B}(\mu)$, we define a concatenation $\pi_1 * \pi_2 * \cdots * \pi_m$ of them by:

$$(\pi_1 * \pi_2 * \cdots * \pi_m)(t) = \sum_{l=1}^{k-1} \pi_l(1) + \pi_k(mt - k + 1)$$
$$\text{for } \frac{k-1}{m} \leq t \leq \frac{k}{m}, \ 1 \leq k \leq m,$$

and set

$$\mathbb{B}(\mu)^{*m} = \underbrace{\mathbb{B}(\mu) * \cdots * \mathbb{B}(\mu)}_{m \text{ times}} := \{\pi_1 * \cdots * \pi_m \mid \pi_k \in \mathbb{B}(\mu),\ 1 \le k \le m\}.$$

We endow $\mathbb{B}(\mu)^{*m}$ with a crystal structure as follows. Let $\pi = \pi_1 * \cdots * \pi_m \in \mathbb{B}(\mu)^{*m}$. First, we define $\mathrm{wt}(\pi) := \pi(1) = \pi_1(1) + \cdots + \pi_m(1)$; notice that $\pi(1) \in P$ since $\pi_k(1) \in P$ for all $1 \le k \le m$. Next, for $i \in I$, we define $\tilde{e}_i\pi$ and $\tilde{f}_i\pi$ in exactly the same way as for elements in $\mathbb{B}(\mu)$ (see §2.5); notice that the condition (2.7) holds for every element in $\mathbb{B}(\mu)^{*m}$. We deduce that if $\tilde{e}_i\pi \ne \mathbf{0}$, then $\tilde{e}_i\pi = \pi_1 * \cdots * \tilde{e}_i\pi_k * \cdots * \pi_m$ for some $1 \le k \le m$; the same holds also for $\tilde{f}_i$. Therefore the set $\mathbb{B}(\mu)^{*m} \cup \{\mathbf{0}\}$ is stable under the action of $\tilde{e}_i$ and $\tilde{f}_i$ for $i \in I$. Finally, for $i \in I$, we set $\varepsilon_i(\pi) := \max\{n \ge 0 \mid \tilde{e}_i^n\pi \ne \mathbf{0}\}$ and $\varphi_i(\pi) := \max\{n \ge 0 \mid \tilde{f}_i^n\pi \ne \mathbf{0}\}$. We know the following proposition from [10, §2].

Proposition 5.16. *Let $\mu \in P$ be an arbitrary integral weight, and $m \in \mathbb{Z}_{\ge 1}$. The set $\mathbb{B}(\mu)^{*m}$ together with the maps $\mathrm{wt} : \mathbb{B}(\mu)^{*m} \to P$, $\tilde{e}_i, \tilde{f}_i : \mathbb{B}(\mu)^{*m} \to \mathbb{B}(\mu)^{*m} \cup \{\mathbf{0}\}$, $i \in I$, and $\varepsilon_i, \varphi_i : \mathbb{B}(\mu)^{*m} \to \mathbb{Z}_{\ge 0}$, $i \in I$, is a crystal. Moreover, the map*

$$\mathbb{B}(\mu)^{*m} \to \mathbb{B}(\mu)^{\otimes m},\ \pi_1 * \cdots * \pi_m \mapsto \pi_1 \otimes \cdots \otimes \pi_m,$$

is an isomorphism of crystals.

Let $\pi = (\nu_1, \nu_2, \ldots, \nu_u; \sigma_0, \sigma_1, \ldots, \sigma_u) \in \mathbb{B}(m\mu)$. For each $1 \le k \le m$, let s, s' be such that $\sigma_{s-1} \le (k-1)/m < \sigma_s$ and $\sigma_{s'-1} < k/m \le \sigma_{s'}$, respectively. We set

$$\pi_k := \left(\frac{1}{m}\nu_s, \frac{1}{m}\nu_{s+1}, \ldots, \frac{1}{m}\nu_{s'}; 0, m\sigma_s - k + 1, m\sigma_{s+1} - k + 1, \ldots, m\sigma_{s'-1} - k + 1, 1 \right).$$

By the definition of LS paths and the assumption that $\pi \in \mathbb{B}(m\mu)$, we deduce that $\pi_k \in \mathbb{B}(\mu)$. Moreover, it is easy to check that $\pi = \pi_1 * \cdots * \pi_m$ since

$$\pi_k(t) = \pi\left(\frac{1}{m}t + \frac{k-1}{m}\right) - \pi\left(\frac{k-1}{m}\right) \text{ for } t \in [0,1].$$

Therefore, it follows that $\mathbb{B}(m\mu)$ is contained in $\mathbb{B}(\mu)^{*m}$, and hence is a subcrystal of $\mathbb{B}(\mu)^{*m} \cong \mathbb{B}(\mu)^{\otimes m}$ consisting of the elements $\pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_m$ such that $\kappa(\pi_p) \ge \iota(\pi_{p+1})$ for all $1 \le p \le m-1$.

Now, we return to the proof of Proposition 5.3. The map

$$\mathbb{B}(\lambda')^{\otimes d} \to \mathbb{B}(\lambda')^{\otimes 2},\ \pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_d \mapsto \pi_1 \otimes \pi_2,$$

induces a surjective map Φ from $\mathbb{B}(d\lambda') \subset \mathbb{B}(\lambda')^{*d} \cong \mathbb{B}(\lambda')^{\otimes d}$ to $\mathbb{B}(2\lambda') \subset \mathbb{B}(\lambda')^{*2} \cong \mathbb{B}(\lambda')^{\otimes 2}$; note that Φ is not necessarily a morphism of crystals. It follows that the inverse image $\Phi^{-1}(C) \subset \mathbb{B}(d\lambda')$ of a connected component C of $\mathbb{B}(2\lambda')$ is a subcrystal of $\mathbb{B}(d\lambda')$. This shows that if $\mathbb{B}(2\lambda')$ has infinitely many connected components, then so does $\mathbb{B}(d\lambda')$. Therefore our proof of Proposition 5.3 is reduced to the case that $d = 2$, i.e., $\lambda = 2\lambda'$. We can prove Proposition 5.3 with $d = 2$ in exactly the same way as the main result in [14], which proved that the crystal graph of $\mathbb{B}(2\Lambda_1 - 2\Lambda_2)$ has infinitely many connected components. However, since [14] is written in Japanese, we write the proof also here for completion.

Lemma 5.17. *Assume that $d = 2$. Let $m, n \in \mathbb{Z}$ be such that $m - n \ge 2$, and let $0 < \sigma < 1$ be a rational number. There exists a σ-chain for $(w_m\lambda, w_n\lambda)$ if and only if $\sigma = 1/2$.*

Proof. Assume that there exists a σ-chain $w_m\lambda = \nu_0 > \nu_1 > \cdots > \nu_u = w_n\lambda$ for $(w_m\lambda, w_n\lambda)$. We see from Proposition 5.4 that $\nu_v = w_{m-v}\lambda$ for $v = 0, 1, \ldots, m-n$. We set

$$i := \begin{cases} 2 & \text{if } m \text{ is even}, \\ 1 & \text{if } m \text{ is odd}, \end{cases} \qquad j := \begin{cases} 1 & \text{if } m \text{ is even}, \\ 2 & \text{if } m \text{ is odd}. \end{cases}$$

By (5.16), it follows that $\sigma\langle w_m\lambda, \alpha_i^\vee\rangle = -2\sigma p_m$ and $\sigma\langle w_{m-1}\lambda, \alpha_j^\vee\rangle = -2\sigma p_{m-1}$. By the definition of a σ-chain and the assumption that $m-n \geq 2$, both $-2\sigma p_m$ and $-2\sigma p_{m-1}$ are integers. Since p_m and p_{m-1} are relatively prime, we obtain $\sigma = 1/2$.

Conversely, we assume that $\sigma = 1/2$. By Proposition 5.4 and (5.16), it is obvious that the sequence $w_m\lambda, w_{m-1}\lambda, \ldots, w_{n+1}\lambda, w_n\lambda$ becomes a σ-chain for $(w_m\lambda, w_n\lambda)$. $\square$

Lemma 5.18. *Assume that $d = 2$. Then $\pi(1/2) \in P$ for every $\pi \in \mathbb{B}(\lambda)$.*

Proof. Let $\pi = (\nu_1, \ldots, \nu_{s-1}, \nu_s, \ldots, \nu_u; \sigma_0, \ldots, \sigma_{s-1}, \sigma_s, \ldots, \sigma_u) \in \mathbb{B}(\lambda)$, and assume that $\sigma_{s-1} < 1/2 \leq \sigma_s$. Then we compute

$$\begin{aligned}\pi\left(\frac{1}{2}\right) &= \sum_{v=1}^{s-1}(\sigma_v - \sigma_{v-1})\nu_v + \left(\frac{1}{2} - \sigma_{s-1}\right)\nu_s \\ &= \sum_{v=1}^{s-1}\sigma_v(\nu_v - \nu_{v+1}) + \frac{1}{2}\nu_s.\end{aligned}$$

Since there exists a σ_v-chain for (ν_v, ν_{v+1}) for each $1 \leq v \leq s-1$, we deduce that $\sum_{v=1}^{s-1}\sigma_v(\nu_v - \nu_{v+1}) \in P$ (see [10, §4]). Moreover, (5.16) implies $(1/2)\nu_s \in P$. Thus we obtain $\pi(1/2) \in P$, as desired. $\square$

For $r \geq 2$, we set

$$\mathbb{B}_r(\lambda) := \{\pi = (\nu_1, \ldots, \nu_u; \sigma_0, \ldots, \sigma_u) \in \mathbb{B}(\lambda) \mid \pi \text{ satisfies the condition (I) or (II)}\},$$

where

$$\begin{aligned}&\text{(I)} : \text{There exists } 1 \leq s \leq u-1 \text{ such that } z(\nu_s) - z(\nu_{s+1}) = 2r-2;\\ &\text{(II)} : \text{There exists } 1 \leq s \leq u-1 \text{ such that } z(\nu_s) - z(\nu_{s+1}) = 2r-1.\end{aligned}$$

Remark 5.19. Assume that $d = 2$. Let $(\nu_1, \ldots, \nu_u; \sigma_0, \ldots, \sigma_u) \in \mathbb{B}(\lambda)$. Assume that there exists $1 \leq s \leq u-1$ such that $z(\nu_s) - z(\nu_{s+1}) \geq 2$. We see by Lemma 5.17 that $\sigma_s = 1/2$ and $z(\nu_v) - z(\nu_{v+1}) = 1$ for each $v = 1, 2, \ldots, s-1, s+1, \ldots, u-1$. Hence, we obtain $\mathbb{B}(\lambda) = \bigsqcup_{r \in \mathbb{Z}_{\geq 1}} \mathbb{B}_r(\lambda)$.

Proposition 5.3 with $d = 2$ is a corollary of the following theorem and Remark 5.19.

Theorem 5.20. *Assume that $d = 2$. Let $r \in \mathbb{Z}_{\geq 1}$, $\pi \in \mathbb{B}_r(\lambda)$, and $i \in I$. If $\tilde{e}_i\pi \neq \mathbf{0}$, then $\tilde{e}_i\pi \in \mathbb{B}_r(\lambda)$. If $\tilde{f}_i\pi \neq \mathbf{0}$, then $\tilde{f}_i\pi \in \mathbb{B}_r(\lambda)$. Therefore, $\mathbb{B}_r(\lambda)$ is a subcrystal of $\mathbb{B}(\lambda)$ for each $r \in \mathbb{Z}_{\geq 1}$.*

Proof. The proof is divided into three cases.

Case 1. Assume that $r = 1$. We have shown the assertion in Proposition 5.6.

Case 2. Assume that $r \geq 2$, and $\pi = (\nu_1, \ldots, \nu_u; \sigma_0, \ldots, \sigma_u) \in \mathbb{B}_r(\lambda)$ satisfies the condition (I). We set $z(\nu_s) = m$, $z(\nu_{s+1}) = n$; note that $m - n = 2r - 2$. We set

$$j := \begin{cases} 2 & \text{if } m \text{ is even,} \\ 1 & \text{if } m \text{ is odd,} \end{cases} \qquad j' := \begin{cases} 1 & \text{if } m \text{ is even,} \\ 2 & \text{if } m \text{ is odd.} \end{cases}$$

Then we see by (5.16) that $\langle \nu_s, \alpha_j^\vee\rangle < 0$ and $\langle \nu_s, \alpha_{j'}^\vee\rangle > 0$. Moreover, we see that $\langle \nu_{s+1}, \alpha_j^\vee\rangle < 0$ and $\langle \nu_{s+1}, \alpha_{j'}^\vee\rangle > 0$ because $m - n \in 2\mathbb{Z}$.

First, let us show that $\tilde{e}_j\pi \in \mathbb{B}_r(\lambda)$ if $\tilde{e}_j\pi \neq \mathbf{0}$. Take t_1 and t_0 as (2.8) and (2.9), respectively (with i replaced by j); note that $t_1 = \sigma_v$ for some $0 \leq v \leq u$. Since $\langle \nu_s, \alpha_j^\vee\rangle < 0$ and $\langle \nu_{s+1}, \alpha_j^\vee\rangle < 0$ as seen above, the function $H_j^\pi(t)$ does not attain its minimum value at $t = \sigma_{s-1}, \sigma_s$ (see the left figure in Figure 2). Thus we obtain $t_1 \neq \sigma_{s-1}, \sigma_s$. If $t_1 < \sigma_{s-1}$, then the assertion is obvious by the definition of $\tilde{e}_j$ and Remark 5.19. Assume that $t_1 \geq \sigma_{s+1}$. Since $H_j^\pi(\sigma_s) > H_j^\pi(t_1) = m_j^\pi \in \mathbb{Z}$ by

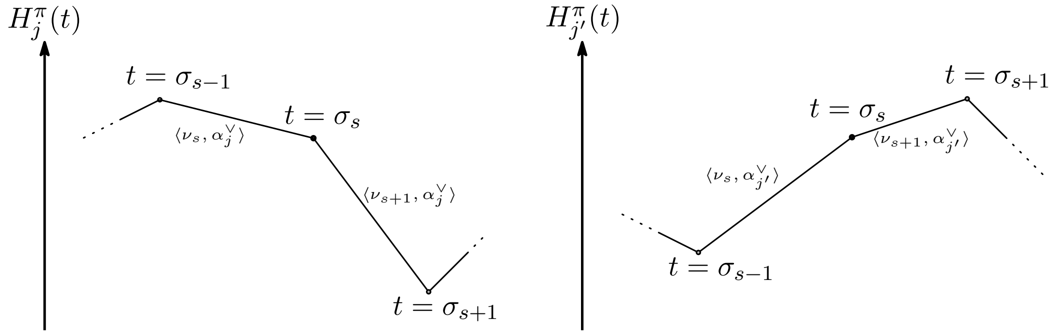

Figure 2.

the definition of t_1, and since $H_j^\pi(\sigma_s) = H_j^\pi(1/2) \in \mathbb{Z}$ by Lemma 5.18 and Remark 5.19, we see that $H_j^\pi(\sigma_s) \geq H_j^\pi(t_1) + 1$. Therefore, we have $t_0 \geq \sigma_s$ by the definition of t_0. If $t_0 > \sigma_s$, then it is obvious by the definition of $\tilde{e}_j$ and Remark 5.19 that $\tilde{e}_j\pi \in \mathbb{B}_r(\lambda)$. If $t_0 = \sigma_s$, then we deduce by the definition of $\tilde{e}_j$ that $\tilde{e}_j\pi$ is of the form

$$\tilde{e}_j\pi = \begin{cases} (\nu_1, \ldots, \nu_s, s_j\nu_{s+1}; \sigma_0, \ldots, \sigma_s, \sigma_u) & \text{if } s = u-1 \text{ or } u-2, \\ (\nu_1, \ldots, \nu_s, s_j\nu_{s+1}, \ldots, \nu_u; \sigma_0, \ldots, \sigma_s, \sigma_{s+2}, \ldots, \sigma_u) & \text{if } s \leq u-3. \end{cases}$$

Since $s_j\nu_{s+1} = s_jw_n\lambda = w_{n-1}\lambda$ and $m - (n-1) = 2r - 1$, we obtain $\tilde{e}_j\pi \in \mathbb{B}_r(\lambda)$.

Next, let us show that $\tilde{e}_{j'}\pi \in \mathbb{B}_r(\lambda)$ if $\tilde{e}_{j'}\pi \neq \mathbf{0}$. Take t_1 and t_0 as (2.8) and (2.9), respectively (with i replaced by j'). Since $\langle \nu_s, \alpha_{j'}^\vee \rangle > 0$ and $\langle \nu_{s+1}, \alpha_{j'}^\vee \rangle > 0$ as seen above, the function $H_{j'}^\pi(t)$ does not attain its minimum value at $t = \sigma_s, \sigma_{s+1}$ (see the right figure in Figure 2). Thus we obtain $t_1 \neq \sigma_s, \sigma_{s+1}$. If $t_1 \leq \sigma_{s-1}$, then the assertion is obvious by the definition of $\tilde{e}_{j'}$ and Remark 5.19. If $t_1 \geq \sigma_{s+2}$, then $H_{j'}^\pi(\sigma_{s-1}) > H_{j'}^\pi(t_1) = m_{j'}^\pi \in \mathbb{Z}$ by the definition of t_1. Notice that $H_{j'}^\pi(\sigma_{s-1}) \in \mathbb{Z}$ by (2.7), and hence $H_{j'}^\pi(\sigma_{s-1}) \geq H_{j'}^\pi(t_1) + 1 = m_{j'}^\pi + 1$. Because $H_{j'}^\pi(\sigma_{s+1}) > H_{j'}^\pi(\sigma_{s-1}) \geq m_{j'}^\pi + 1$, we see that $\sigma_{s+1} < t_0$. Therefore, $\tilde{e}_{j'}\pi \in \mathbb{B}_r(\lambda)$ by the definition of $\tilde{e}_{j'}$ and Remark 5.19.

Similarly, we can show (in Case 2) that if $\tilde{f}_i\pi \neq \mathbf{0}$ for $i \in I$, then $\tilde{f}_i\pi \in \mathbb{B}_r(\lambda)$.

Case 3. Assume that $r \geq 2$, and $\pi = (\nu_1, \ldots, \nu_u; \sigma_0, \ldots, \sigma_u) \in \mathbb{B}_r(\lambda)$ satisfies the condition (II). We set $z(\nu_s) = m$, $z(\nu_{s+1}) = n$; note that $m - n = 2r - 1$. We set

$$j := \begin{cases} 2 & \text{if } m \text{ is even}, \\ 1 & \text{if } m \text{ is odd}, \end{cases} \qquad j' := \begin{cases} 1 & \text{if } m \text{ is even}, \\ 2 & \text{if } m \text{ is odd}. \end{cases}$$

Then we see by (5.16) that $\langle \nu_s, \alpha_j^\vee \rangle < 0$ and $\langle \nu_s, \alpha_{j'}^\vee \rangle > 0$. Moreover, we see that $\langle \nu_{s+1}, \alpha_j^\vee \rangle > 0$ and $\langle \nu_{s+1}, \alpha_{j'}^\vee \rangle < 0$ because $m - n \in 2\mathbb{Z} + 1$.

First, let us show that $\tilde{e}_j\pi \in \mathbb{B}_r(\lambda)$ if $\tilde{e}_j\pi \neq \mathbf{0}$. Take t_1 and t_0 as (2.8) and (2.9), respectively (with i replaced by j). Since $\langle \nu_s, \alpha_j^\vee \rangle < 0$ and $\langle \nu_{s+1}, \alpha_j^\vee \rangle > 0$ as seen above, the function $H_j^\pi(t)$ does not attain its minimum value at $t = \sigma_{s-1}, \sigma_{s+1}$ (see the left figure in Figure 3). Thus we obtain $t_1 \neq \sigma_{s-1}, \sigma_{s+1}$. If $t_1 < \sigma_{s-1}$, then the assertion is obvious by the definition of $\tilde{e}_j$ and Remark 5.19. If $t_1 > \sigma_{s+1}$, then $H_j^\pi(\sigma_s) > H_j^\pi(t_1) = m_j^\pi \in \mathbb{Z}$ by the definition of t_1. Notice that $H_j^\pi(\sigma_s) \in \mathbb{Z}$ by (2.7), and hence $H_j^\pi(\sigma_s) \geq H_j^\pi(t_1) + 1 = m_j^\pi + 1$. Because $H_j^\pi(\sigma_{s+1}) > H_j^\pi(\sigma_s) \geq m_j^\pi + 1$, we see that $\sigma_{s+1} < t_0$. Therefore, $\tilde{e}_j\pi \in \mathbb{B}_r(\lambda)$ by the definition of $\tilde{e}_j$ and Remark 5.19. Assume that $t_1 = \sigma_s$. If $s = 1$, i.e., $\sigma_{s-1} = 0$, then it is obvious that $t_0 \geq \sigma_{s-1}$. If $s > 1$, then we see that $H_j^\pi(\sigma_{s-2}) > H_j^\pi(t_1) \in \mathbb{Z}$ by the definition of t_1. Notice that $H_j^\pi(\sigma_{s-2}) \in \mathbb{Z}$ by (2.7) and hence $H_j^\pi(\sigma_{s-2}) \geq H_j^\pi(t_1) + 1 = m_j^\pi + 1$. Since $H_j^\pi(\sigma_{s-1}) > H_j^\pi(\sigma_{s-2}) \geq m_j^\pi + 1$ by $\langle \nu_{s-1}, \alpha_j^\vee \rangle > 0$ (see (5.16) and Remark 5.19), we see that $\sigma_{s-1} < t_0$ by the definition of t_0. Then we deduce by the definition of $\tilde{e}_j$ that $\tilde{e}_j\pi$ is of the form

$$\tilde{e}_j\pi = \begin{cases} (s_j\nu_s, \nu_{s+1}, \ldots, \nu_u; \sigma_0, \ldots, \sigma_u) & \text{if } t_0 = \sigma_{s-1}, \\ (\nu_1, \ldots, \nu_s, s_j\nu_s, \nu_{s+1}, \ldots, \nu_u; \sigma_0, \ldots, \sigma_{s-1}, t_0, \sigma_{s+1}, \ldots \sigma_u) & \text{if } t_0 > \sigma_{s-1}. \end{cases}$$

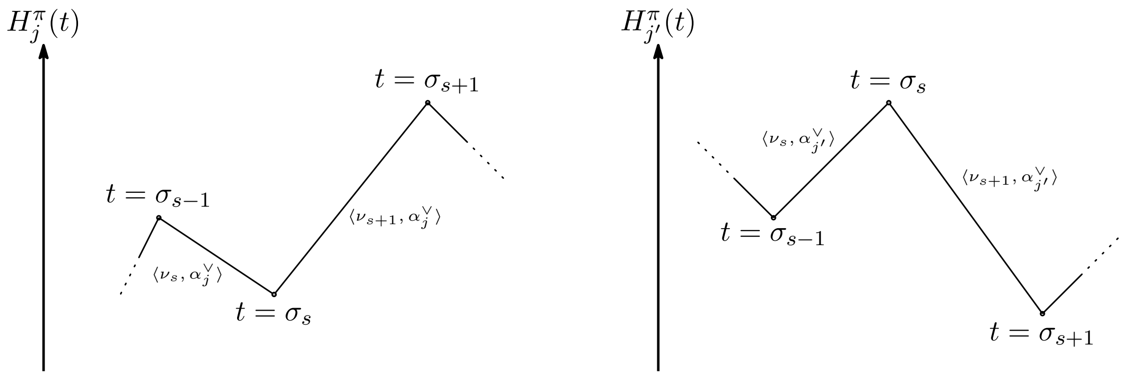

Figure 3.

Since $s_j \nu_s = s_j w_m \lambda = w_{m-1}\lambda$ and $(m-1) - n = 2r - 2$, we obtain $\tilde{e}_j \pi \in \mathbb{B}_r(\lambda)$.

Next, let us show that $\tilde{e}_{j'}\pi \in \mathbb{B}_r(\lambda)$ if $\tilde{e}_{j'}\pi \neq \mathbf{0}$. Take t_1 and t_0 as (2.8) and (2.9), respectively (with i replaced by j'). Since $\langle \nu_s, \alpha_{j'}^\vee \rangle > 0$ and $\langle \nu_{s+1}, \alpha_{j'}^\vee \rangle < 0$ as seen above, the function $H_{j'}^\pi(t)$ does not attain its minimum value at $t = \sigma_s$ (see the right figure in Figure 3). Thus we see that $t_1 \neq \sigma_s$. If $t_1 \leq \sigma_{s-1}$, the assertion is obvious by the definition of $\tilde{e}_{j'}$ and Remark 5.19. If $\sigma_{s+1} \leq t_1$, then $H_{j'}^\pi(\sigma_{s-1}) > H_{j'}^\pi(t_1) = m_{j'}^\pi \in \mathbb{Z}$ by the definition of t_1. Notice that $H_{j'}^\pi(\sigma_{s-1}) \in \mathbb{Z}$ by (2.7), and hence $H_{j'}^\pi(\sigma_{s-1}) \geq H_{j'}^\pi(t_1) + 1 = m_{j'}^\pi + 1$. Because $H_{j'}^\pi(\sigma_s) > H_{j'}^\pi(\sigma_{s-1}) \geq m_{j'}^\pi + 1$, we see that $\sigma_s < t_0$. Then $\tilde{e}_{j'}\pi \in \mathbb{B}_r(\lambda)$ by the definition of $\tilde{e}_{j'}$ and Remark 5.19.

Similarly, we can show (in Case 3) that if $\tilde{f}_i \pi \neq \mathbf{0}$ for $i \in I$, then $\tilde{f}_i \pi \in \mathbb{B}_r(\lambda)$.

Thus we have proved Theorem 5.20. □

6 Relationship between the crystal of LS paths and the crystal basis of extremal weight modules.

Throughout this section, let $\iota = (\iota^+, \iota^-)$ be as in §3, and $\lambda = k_1\Lambda_1 - k_2\Lambda_2 \in P$ as in Theorem 3.1.

6.1 Polyhedral realization of $\mathcal{B}(\pm\infty)$ in the rank 2 case.

We define the sequences $\{c_j\}_{j\geq 0}$ and $\{c'_j\}_{j\geq 0}$ of integers by the following recursive formulas: for $j \geq 0$,

$$c_0 \coloneqq 0, \quad c_1 \coloneqq 1, \quad c_{j+2} \coloneqq \begin{cases} a_1 c_{j+1} - c_j & \text{if } j \text{ is even,} \\ a_2 c_{j+1} - c_j & \text{if } j \text{ is odd;} \end{cases}$$

$$c'_0 \coloneqq 0, \quad c'_1 \coloneqq 1, \quad c'_{j+2} \coloneqq \begin{cases} a_2 c'_{j+1} - c'_j & \text{if } j \text{ is even,} \\ a_1 c'_{j+1} - c'_j & \text{if } j \text{ is odd;} \end{cases}$$

we can show by the same argument as Lemmas 4.4 and 4.7 that $c_j > 0$ and $c'_j > 0$ for all $j \geq 1$. By [2, Corollary 4.7] and the fact that $1/\beta = (a_1a_2 - \sqrt{a_1^2a_2^2 - 4a_1a_2})/2a_2$, we obtain the following lemma.

Lemma 6.1. *The following sequences are strictly decreasing, and converge to α and β, respectively:*

$$\frac{c_2}{c_1} > \frac{c'_3}{c'_2} > \frac{c_4}{c_3} > \frac{c'_5}{c'_4} > \cdots \to \alpha, \qquad \frac{c'_2}{c'_1} > \frac{c_3}{c_2} > \frac{c'_4}{c'_3} > \frac{c_5}{c_4} > \cdots \to \beta.$$

Applying [13, Theorem 4.1] to our rank 2 case, we obtain the following explicit descriptions of the images of the maps $\Psi^+_{\iota^+} : \mathcal{B}(\infty) \to \mathbb{Z}^{+\infty}_{\geq 0,\iota^+}$ and $\Psi^-_{\iota^-} : \mathcal{B}(-\infty) \to \mathbb{Z}^{-\infty}_{\leq 0,\iota^-}$.

Proposition 6.2. *It hold that*

$$\begin{aligned} \mathrm{Im}(\Psi^+_{\iota^+}) &= \{(\ldots, x_2, x_1) \in \mathbb{Z}^{+\infty}_{\geq 0} \mid c_j x_j - c_{j-1} x_{j+1} \geq 0 \text{ for } j \geq 1\}, \\ \mathrm{Im}(\Psi^-_{\iota^-}) &= \{(x_0, x_{-1}, \ldots) \in \mathbb{Z}^{-\infty}_{\leq 0} \mid c'_{-j+1} x_j - c'_{-j} x_{j-1} \leq 0 \text{ for } j \leq 0\}. \end{aligned}$$

Recall that the sequence $\{p_m\}_{m\in\mathbb{Z}}$ is defined by the recursive formulas (3.1) and (3.2). The following lemma will be needed in §6.2.

Lemma 6.3. *Let $m, n \in \mathbb{Z}$ be such that $n < m$, and let $q_{n+1}, q_{n+2}, \ldots, q_m \in \mathbb{Z}$ be such that $0 < q_j < p_j$ for $n+1 \leq j \leq m$, and $q_{j+1}/p_{j+1} < q_j/p_j$ for $n+1 \leq j \leq m-1$.*

(1) *If $0 < m$, then $(\ldots, 0, p_m, \ldots, p_2, p_1) \in \mathrm{Im}(\Psi^+_{\iota^+})$.*

(2) *If $0 < n < m$, then $(\ldots, 0, q_m, \ldots, q_{n+2}, q_{n+1}, p_n, \ldots, p_2, p_1) \in \mathrm{Im}(\Psi^+_{\iota^+})$.*

(3) *If $n = 0 < m$, then $(\ldots, 0, q_m, \ldots, q_2, q_1) \in \mathrm{Im}(\Psi^+_{\iota^-})$.*

(4) *If $n < 0 = m$, then $(q_0 - p_0, q_{-1} - p_{-1}, \ldots, q_{n+1} - p_{n+1}, 0, \ldots) \in \mathrm{Im}(\Psi^-_{\iota^-})$.*

(5) *If $n < m < 0$, then $(-p_0, -p_{-1}, \ldots, -p_{m+1}, q_m - p_m, \ldots, q_{n+1} - p_{n+1}, 0, \ldots) \in \mathrm{Im}(\Psi^-_{\iota^-})$.*

(6) *If $n < 0$, then $(-p_0, -p_{-1}, \ldots, -p_{n+1}, 0, \ldots) \in \mathrm{Im}(\Psi^-_{\iota^-})$.*

Proof. We give proofs only for parts (2) and (5); the proofs for the other cases are easier than these cases.

First, we show part (2). By Proposition 6.2, it suffices to show that

$$\begin{aligned} c_j p_j - c_{j-1} p_{j+1} &\geq 0 \quad \text{for } 1 \leq j \leq n-1, && (6.1) \\ c_j p_j - c_{j-1} q_{j+1} &\geq 0 \quad \text{for } j = n, && (6.2) \\ c_j q_j - c_{j-1} q_{j+1} &\geq 0 \quad \text{for } n+1 \leq j \leq m-1. && (6.3) \end{aligned}$$

We can easily see by induction on j that

$$c_j p_j - c_{j-1} p_{j+1} \geq 0 \quad \text{for } j \geq 1. \tag{6.4}$$

Thus we get (6.1). Since $q_{n+1}/p_{n+1} < 1$, we see that $c_n p_n - c_{n-1} q_{n+1} > c_n p_n - c_{n-1} p_{n+1}$. Combining this inequality and (6.4), we obtain (6.2). For $n+1 \leq j \leq m-1$, we see that $c_j q_j - c_{j-1} q_{j+1} > c_j (q_{j+1} p_j / p_{j+1}) - c_{j-1} q_{j+1} = (q_{j+1}/p_{j+1})(c_j p_j - c_{j-1} p_{j+1})$ since $q_{j+1}/p_{j+1} < q_j/p_j$. Combining this inequality and (6.4), we obtain (6.3). Thus we have proved part (2).

Next, we show part (5). It suffices to show that

$$c'_{-j}(-p_{j+1}) - c'_{-j-1}(-p_j) \leq 0 \qquad \text{for } m+1 \leq j \leq -1, \tag{6.5}$$

$$c'_{-j}(-p_{j+1}) - c'_{-j-1}(q_j - p_j) \leq 0 \qquad \text{for } j = m, \tag{6.6}$$

$$c'_{-j}(q_{j+1} - p_{j+1}) - c'_{-j-1}(q_j - p_j) \leq 0 \quad \text{for } n+1 \leq j \leq m-1. \tag{6.7}$$

We can easily see by induction on j that

$$c'_{-j}(-p_{j+1}) - c'_{-j-1}(-p_j) \leq 0 \quad \text{for } j \leq -1. \tag{6.8}$$

Thus we get (6.5). We see that $c'_{-m}(-p_{m+1}) - c'_{-m-1}(q_m - p_m) = c'_{-m}(-p_{m+1}) - c'_{-m-1}(-p_m) - c'_{-m-1} q_m$. Combining this equality and (6.8), we obtain (6.6). For $n+1 \leq j \leq m-1$, we see that

$$\begin{aligned} c'_{-j}(q_{j+1} - p_{j+1}) - c'_{-j-1}(q_j - p_j) &= c'_{-j} q_{j+1} - c'_{-j-1} q_j - c'_{-j} p_{j+1} + c'_{-j-1} p_j \\ &< c'_{-j}\left(\frac{q_j p_{j+1}}{p_j}\right) - c'_{-j-1} q_j + (-c'_{-j} p_{j+1} + c'_{-j-1} p_j) \\ &= \left(1 - \frac{q_j}{p_j}\right)(-c'_{-j} p_{j+1} + c'_{-j-1} p_j) \end{aligned}$$

since $q_{j+1}/p_{j+1} < q_j/p_j$. Combining this inequality and (6.8), we obtain (6.7). Thus we have proved part (5). □

6.2 Proof of Theorem 3.3.

We see by (4.3) (and the fact that $p_m > 0$ for all $m \in \mathbb{Z}$) that an element $\pi \in \mathbb{B}_1(\lambda)$ is of the form

$$\pi = \left(w_m \lambda, w_{m-1}\lambda, \ldots, w_n \lambda; 0, \frac{q_m}{p_m}, \frac{q_{m-1}}{p_{m-1}}, \ldots, \frac{q_{n+1}}{p_{n+1}}, 1\right),$$

where $n \leq m$, and $q_m, q_{m-1}, \ldots, q_{n+1}$ are integers satisfying

$$0 < q_j < p_j \text{ for } n+1 \leq j \leq m, \text{ and } \frac{q_{j+1}}{p_{j+1}} < \frac{q_j}{p_j} \text{ for } n+1 \leq j \leq m-1.$$

We define a map Φ_ι^λ from $\mathbb{B}_1(\lambda) \cup \{\mathbf{0}\}$ to $\mathrm{Im}(\Psi_\iota^\lambda) \cup \{\mathbf{0}\}$ as follows. First, we set $\Phi_\iota^\lambda(\mathbf{0}) := \mathbf{0}$. Let

$$\pi = \left(w_m \lambda, w_{m-1}\lambda, \ldots, w_n \lambda; 0, \frac{q_m}{p_m}, \frac{q_{m-1}}{p_{m-1}}, \ldots, \frac{q_{n+1}}{p_{n+1}}, 1\right) \in \mathbb{B}_1(\lambda), \tag{6.9}$$

where $n \leq m$, and $q_m, q_{m-1}, \ldots, q_{n+1}$ are integers satisfying $0 < q_j < p_j$ for $n+1 \leq j \leq m$ and $q_{j+1}/p_{j+1} < q_j/p_j$ for $n+1 \leq j \leq m-1$. We set

$$z_k = z_k(\pi) := \begin{cases} q_k & \text{if } 1 \leq k \text{ and } n+1 \leq k \leq m, \\ p_k & \text{if } 1 \leq k \text{ and } k \leq n, \\ q_k - p_k & \text{if } k \leq 0 \text{ and } n+1 \leq k \leq m, \\ -p_k & \text{if } k \leq 0 \text{ and } m+1 \leq k, \\ 0 & \text{otherwise} \end{cases} \tag{6.10}$$

for $k \in \mathbb{Z}$, and then define $\Phi_\iota^\lambda(\pi) := (\ldots, z_2, z_1) \otimes t_\lambda \otimes (z_0, z_{-1}, \ldots) \in \mathbb{Z}_\iota(\lambda)$.

Remark 6.4. More explicitly, we can describe $\Phi_\iota^\lambda(\pi)$ as follows:

(i) if $n = m = 0$, that is, $\pi = \pi_\lambda$, then $\Phi_\iota^\lambda(\pi) = (\dots, 0, 0) \otimes t_\lambda \otimes (0, 0, \dots)$;

(ii) if $0 < n = m$, then $\Phi_\iota^\lambda(\pi) = (\dots, 0, p_m, \dots, p_2, p_1) \otimes t_\lambda \otimes (0, 0, \dots)$;

(iii) if $0 < n < m$, then $\Phi_\iota^\lambda(\pi) = (\dots, 0, q_m, \dots, q_{n+2}, q_{n+1}, p_n, \dots, p_2, p_1) \otimes t_\lambda \otimes (0, 0, \dots)$;

(iv) if $n = 0 < m$, then $\Phi_\iota^\lambda(\pi) = (\dots, 0, q_m, \dots, q_2, q_1) \otimes t_\lambda \otimes (0, 0, \dots)$;

(v) if $n < 0 < m$, then $\Phi_\iota^\lambda(\pi) = (\dots, 0, q_m, \dots, q_2, q_1) \otimes t_\lambda \otimes (q_0 - p_0, q_{-1} - p_{-1}, \dots, q_{n+1} - p_{n+1}, 0, \dots)$;

(vi) if $n < m = 0$, then $\Phi_\iota^\lambda(\pi) = (\dots, 0, 0) \otimes t_\lambda \otimes (q_0 - p_0, q_{-1} - p_{-1}, \dots, q_{n+1} - p_{n+1}, 0, \dots)$;

(vii) if $n < m < 0$, then $\Phi_\iota^\lambda(\pi) = (\dots, 0, 0) \otimes t_\lambda \otimes (-p_0, -p_{-1}, \dots, -p_{m+1}, q_m - p_m, q_{m-1} - p_{m-1}, \dots, q_{n+1} - p_{n+1}, 0, \dots)$;

(viii) if $n = m < 0$, then $\Phi_\iota^\lambda(\pi) = (\dots, 0, 0) \otimes t_\lambda \otimes (-p_0, -p_{-1}, \dots, -p_{n+1}, 0, \dots)$.

Therefore, by Lemma 6.3, we deduce that $\Phi_\iota^\lambda(\pi) \in \mathrm{Im}(\Psi_\iota^\lambda)$ for $\pi \in \mathbb{B}_1(\lambda)$.

Theorem 6.5. *The map $\Phi_\iota^\lambda : \mathbb{B}_1(\lambda) \to \mathrm{Im}(\Psi_\iota^\lambda)$ is an embedding of crystals.*

Assuming that Theorem 6.5 is true, we give a proof of Theorem 3.3.

Proof of Theorem 3.3 . Let $Z(\lambda) := \{\vec{x} \in \mathrm{Im}(\Psi_\iota^\lambda) \mid \vec{x}^* \text{ is extremal}\}$. We know from Corollary 2.8 that there exists an isomorphism $\Sigma : Z(\lambda) \to \mathcal{B}(\lambda)$ of crystals which sends $z_\lambda := (\dots, 0, 0) \otimes t_\lambda \otimes (0, 0, \dots)$ to u_λ. Recall from Remark 5.7 that $\mathbb{B}_0(\lambda) \subset \mathbb{B}_1(\lambda)$. Because $\Phi_\iota^\lambda(\pi_\lambda) = z_\lambda \in Z(\lambda)$, we see that $\Phi_\iota^\lambda(\mathbb{B}_0(\lambda)) \subset Z(\lambda)$. Therefore it follows from Theorem 6.5 that $\Sigma \circ \Phi_\iota^\lambda|_{\mathbb{B}_0(\lambda)}$ is an isomorphism of crystals from $\mathbb{B}_0(\lambda)$ onto $\mathcal{B}_0(\lambda)$. Thus we have proved Theorem 3.3. $\square$

The rest of this subsection is devoted to a proof of Theorem 6.5.

Lemma 6.6. *For $k \le l$, it holds that*

$$w_k\lambda - w_l\lambda = \sum_{j=k+1}^{l} p_j \alpha_{i_j}.$$

Proof. We proceed by induction on l; recall that $l \ge k$. If $l = k$, then the assertion is obvious. Assume that $l > k$. By the induction hypothesis, we have $w_k\lambda - w_{l-1}\lambda = \sum_{j=k+1}^{l-1} p_j \alpha_{i_j}$. We see by (4.3) that $w_l\lambda = w_{l-1}\lambda - p_l\alpha_{i_l}$. Therefore, we obtain

$$w_k\lambda - w_l\lambda = w_k\lambda - w_{l-1}\lambda + p_l\alpha_{i_l} = \sum_{j=k+1}^{l-1} p_j\alpha_{i_j} + p_l\alpha_{i_l} = \sum_{j=k+1}^{l} p_j\alpha_{i_j},$$

as desired. $\square$

Proposition 6.7. *Let $\pi \in \mathbb{B}_1(\lambda)$ be as (6.9). Then,*

$$\mathrm{wt}(\pi) = \mathrm{wt}(\Phi_\iota^\lambda(\pi)) = w_n\lambda - \sum_{j=n+1}^{m} q_j\alpha_{i_j}.$$

Proof. First, we show by induction on m that $\mathrm{wt}(\pi) = w_n\lambda - \sum_{j=n+1}^{m} q_j\alpha_{i_j}$; recall that $m \geq n$. If $m = n$, then $\mathrm{wt}(\pi) = w_n\lambda$ since $\pi = (w_n\lambda; 0, 1)$. Hence the assertion is obvious. Assume that $m > n$. We see that

$$\pi' := \left(w_{m-1}\lambda, w_{m-2}\lambda, \dots, w_n\lambda; 0, \frac{q_{m-1}}{p_{m-1}}, \frac{q_{m-2}}{p_{m-2}}, \dots, \frac{q_{n+1}}{p_{n+1}}, 1\right)$$

is also an element of $\mathbb{B}_1(\lambda)$. By the induction hypothesis, we obtain $\mathrm{wt}(\pi') = w_n\lambda - \sum_{j=n+1}^{m-1} q_j\alpha_{i_j}$. We see by the definition of wt that

$$\begin{aligned}\mathrm{wt}(\pi) &= \mathrm{wt}(\pi') - \frac{q_{m-1}}{p_{m-1}}w_{m-1}\lambda + \frac{q_m}{p_m}w_m\lambda + \left(\frac{q_{m-1}}{p_{m-1}} - \frac{q_m}{p_m}\right)w_{m-1}\lambda \\ &= \mathrm{wt}(\pi') + \frac{q_m}{p_m}(w_m\lambda - w_{m-1}\lambda).\end{aligned}$$

By Lemma 6.6, we have $w_m\lambda - w_{m-1}\lambda = -p_m\alpha_{i_m}$. Therefore we deduce that $\mathrm{wt}(\pi) = w_n\lambda - \sum_{j=n+1}^{m-1} q_j\alpha_{i_j} + (q_m/p_m)(-p_m\alpha_{i_m}) = w_n\lambda - \sum_{j=n+1}^{m} q_j\alpha_{i_j}$.

Next, we show that $\mathrm{wt}(\Phi_\iota^\lambda(\pi)) = w_n\lambda - \sum_{j=n+1}^{m} q_j\alpha_{i_j}$. By the definition of wt, if $0 \leq n \leq m$, then

$$\mathrm{wt}(\Phi_\iota^\lambda(\pi)) = \lambda - \sum_{j=n+1}^{m} q_j\alpha_{i_j} - \sum_{j=1}^{n} p_j\alpha_{i_j};$$

if $n < 0 < m$, then

$$\mathrm{wt}(\Phi_\iota^\lambda(\pi)) = \lambda - \sum_{j=1}^{m} q_j\alpha_{i_j} - \sum_{j=n+1}^{0} (q_j - p_j)\alpha_{i_j} = \lambda - \sum_{j=n+1}^{m} q_j\alpha_{i_j} + \sum_{j=n+1}^{0} p_j\alpha_{i_j};$$

if $n \leq m \leq 0$, then

$$\mathrm{wt}(\Phi_\iota^\lambda(\pi)) = \lambda - \sum_{j=m+1}^{0} (-p_j)\alpha_{i_j} - \sum_{j=n+1}^{m} (q_j - p_j)\alpha_{i_j} = \lambda + \sum_{j=n+1}^{0} p_j\alpha_{i_j} - \sum_{j=n+1}^{m} q_j\alpha_{i_j}.$$

It follows from Lemma 6.6 that

$$w_n\lambda = \begin{cases}\lambda - \sum_{j=1}^{n} p_j\alpha_{i_j} & \text{if } n \geq 0, \\ \lambda + \sum_{j=n+1}^{0} p_j\alpha_{i_j} & \text{if } n \leq 0.\end{cases}$$

Therefore we obtain $\mathrm{wt}(\Phi_\iota^\lambda(\pi)) = w_n\lambda - \sum_{j=n+1}^{m} q_j\alpha_{i_j}$ for $n, m \in \mathbb{Z}$ such that $n \leq m$. Thus we have proved the proposition. □

Lemma 6.8. *Let $\pi \in \mathbb{B}_1(\lambda)$ be as* (6.9). *Then, for $k \in \mathbb{Z}$,*

$$\sigma_k(\Phi_\iota^\lambda(\pi)) = \begin{cases}0 & \text{if } m+1 \leq k, \\ q_k + \sum_{j=k+1}^{m}\langle\alpha_{i_j}, \alpha_{i_k}^\vee\rangle q_j & \text{if } n+1 \leq k \leq m, \\ -\langle\mathrm{wt}(\Phi_\iota^\lambda(\pi)), \alpha_{i_k}^\vee\rangle & \text{if } k \leq n.\end{cases}$$

Proof. First, we assume that $0 \leq n \leq m$. Then, $\Phi_\iota^\lambda(\pi) \in \mathrm{Im}(\Psi_\iota^\lambda)$ is of the form $\Phi_\iota^\lambda(\pi) = b_1 \otimes t_\lambda \otimes (0, 0, \dots)$ with

$$b_1 = (\dots, 0, q_m, \dots, q_{n+2}, q_{n+1}, p_n, \dots, p_2, p_1),$$

where we understand $b_1 = (\dots, 0, p_m, \dots, p_2, p_1)$ (resp., $(\dots, 0, q_m, \dots, q_2, q_1)$) if $0 < n = m$ (resp., $n = 0 < m$). If $n+1 \leq k$, then we have $\sigma_k(\Phi_\iota^\lambda(\pi)) = \sigma_k^+(b_1)$. Hence the assertion is obvious by the definition of σ_k^+. Assume that $1 \le k \le n$. By Proposition 6.7, it suffices to show that $\sigma_k(\Phi_\iota^\lambda(\pi)) = -\langle w_n\lambda, \alpha_{i_k}^\vee\rangle + \sum_{j=n+1}^m \langle \alpha_{i_j}, \alpha_{i_k}^\vee\rangle q_j$. We see by the definition of σ_k that

$$\sigma_k(\Phi_\iota^\lambda(\pi)) = \sigma_k^+(b_1) = p_k + \sum_{j=k+1}^{n} \langle \alpha_{i_j}, \alpha_{i_k}^\vee\rangle p_j + \sum_{j=n+1}^{m} \langle \alpha_{i_j}, \alpha_{i_k}^\vee\rangle q_j. \tag{6.11}$$

It follows from Lemma 6.6 that $w_k\lambda - w_n\lambda = \sum_{j=k+1}^n p_j\alpha_{i_j}$. Therefore,

$$-\langle w_n\lambda, \alpha_{i_k}^\vee\rangle = -\langle w_k\lambda, \alpha_{i_k}^\vee\rangle + \sum_{j=k+1}^{n} \langle \alpha_{i_j}, \alpha_{i_k}^\vee\rangle p_j = p_k + \sum_{j=k+1}^{n} \langle \alpha_{i_j}, \alpha_{i_k}^\vee\rangle p_j \tag{6.12}$$

since $\langle w_k\lambda, \alpha_{i_k}^\vee\rangle = -p_k$ by (4.3). Combining (6.11) and (6.12), we obtain the desired equality. If $k \le 0$, then we have $\sigma_k(\Phi_\iota^\lambda(\pi)) = \sigma_k^-((0,0,\ldots)) - \langle \mathrm{wt}(\Phi_\iota^\lambda(\pi)), \alpha_{i_k}^\vee\rangle = -\langle \mathrm{wt}(\Phi_\iota^\lambda(\pi)), \alpha_{i_k}^\vee\rangle$.

Next, we assume that $n < 0 < m$. Then, $\Phi_\iota^\lambda(\pi) \in \mathrm{Im}(\Psi_\iota^\lambda)$ is of the form $\Phi_\iota^\lambda(\pi) = b_1 \otimes t_\lambda \otimes b_2$ with $b_1 = (\ldots, 0, q_m, \ldots, q_2, q_1)$ and $b_2 = (q_0 - p_0, q_{-1} - p_{-1}, \ldots, q_{n+1} - p_{n+1}, 0, \ldots)$. If $1 \le k$, then $\sigma_k(\Phi_\iota^\lambda(\pi)) = \sigma_k^+(b_1)$. Hence the assertion is obvious by the definition of σ_k^+. Assume that $n+1 \le k \le 0$. We see by the definition of σ_k^- that

$$\begin{aligned}\sigma_k^-(b_2) &= -(q_k - p_k) - \sum_{j=n+1}^{k-1} \langle \alpha_{i_j}, \alpha_{i_k}^\vee\rangle (q_j - p_j) \\ &= -q_k + p_k - \sum_{j=n+1}^{k-1} \langle \alpha_{i_j}, \alpha_{i_k}^\vee\rangle q_j + \sum_{j=n+1}^{k-1} \langle \alpha_{i_j}, \alpha_{i_k}^\vee\rangle p_j.\end{aligned}$$

By Proposition 6.7, we have $\langle \mathrm{wt}(\Phi_\iota^\lambda(\pi)), \alpha_{i_k}^\vee\rangle = \langle w_n\lambda, \alpha_{i_k}^\vee\rangle - \sum_{j=n+1}^m \langle \alpha_{i_j}, \alpha_{i_k}^\vee\rangle q_j$. Hence,

$$\begin{aligned}\sigma_k(\Phi_\iota^\lambda(\pi)) &= \sigma_k^-(b_2) - \langle \mathrm{wt}(\Phi_\iota^\lambda(\pi)), \alpha_{i_k}^\vee\rangle \\ &= -q_k + \sum_{j=k}^{m} \langle \alpha_{i_j}, \alpha_{i_k}^\vee\rangle q_j + p_k + \sum_{j=n+1}^{k-1} \langle \alpha_{i_j}, \alpha_{i_k}^\vee\rangle p_j - \langle w_n\lambda, \alpha_{i_k}^\vee\rangle.\end{aligned} \tag{6.13}$$

Because $\langle \alpha_{i_k}, \alpha_{i_k}^\vee\rangle = 2$, we obtain

$$-q_k + \sum_{j=k}^{m} \langle \alpha_{i_j}, \alpha_{i_k}^\vee\rangle q_j = -q_k + \langle \alpha_{i_k}, \alpha_{i_k}^\vee\rangle q_k + \sum_{j=k+1}^{m} \langle \alpha_{i_j}, \alpha_{i_k}^\vee\rangle q_j = q_k + \sum_{j=k+1}^{m} \langle \alpha_{i_j}, \alpha_{i_k}^\vee\rangle q_j. \tag{6.14}$$

It follows from Lemma 6.6 that $-w_n\lambda + w_k\lambda + \sum_{j=n+1}^k p_j\alpha_{i_j} = 0$, and hence,

$$\begin{aligned}0 &= -\langle w_n\lambda, \alpha_{i_k}^\vee\rangle + \langle w_k\lambda, \alpha_{i_k}^\vee\rangle + \sum_{j=n+1}^{k} \langle \alpha_{i_j}, \alpha_{i_k}^\vee\rangle p_j \\ &= -\langle w_n\lambda, \alpha_{i_k}^\vee\rangle + p_k + \sum_{j=n+1}^{k-1} \langle \alpha_{i_j}, \alpha_{i_k}^\vee\rangle p_j\end{aligned} \tag{6.15}$$

since $\langle w_k\lambda, \alpha_{i_k}^\vee\rangle = -p_k$ and $\langle \alpha_{i_k}, \alpha_{i_k}^\vee\rangle = 2$. By (6.13)–(6.15), we obtain $\sigma_k(\Phi_\iota^\lambda(\pi)) = q_k + \sum_{j=k+1}^m \langle \alpha_{i_j}, \alpha_{i_k}^\vee\rangle q_j$, as desired. If $k \le n$, then $\sigma_k^-(b_2) = 0$, which implies that $\sigma_k(\Phi_\iota^\lambda(\pi)) = \sigma_k^-(b_2) - \langle \mathrm{wt}(\Phi_\iota^\lambda(\pi)), \alpha_{i_k}^\vee\rangle = -\langle \mathrm{wt}(\Phi_\iota^\lambda(\pi)), \alpha_{i_k}^\vee\rangle$.

Finally, we assume that $n \le m \le 0$. Then, $\Phi_\iota^\lambda(\pi) \in \mathrm{Im}(\Psi_\iota^\lambda)$ is of the form $\Phi_\iota^\lambda(\pi) = (\ldots, 0, 0) \otimes t_\lambda \otimes b_2$ with

$$b_2 = (-p_0, -p_{-1}, \ldots, -p_{m+1}, q_m - p_m, \ldots, q_{n+1} - p_{n+1}, 0, \ldots),$$

where we understand $b_2 = (q_0 - p_0, q_{-1} - p_{-1}, \dots, q_{n+1} - p_{n+1}, 0, \dots)$ (resp., $b_2 = (-p_0, \dots, -p_{n+1}, 0, \dots)$) if $n < m = 0$ (resp., $n = m < 0$). If $1 \le k$, then it is obvious that $\sigma_k(\Phi_\iota^\lambda(\pi)) = \sigma_k^+(b_1) = 0$. Assume that $m + 1 \le k \le 0$. We see that

$$\begin{aligned}\sigma_k^-(b_2) &= -(-p_k) - \sum_{j=m+1}^{k-1} \langle \alpha_{i_j}, \alpha_{i_k}^\vee \rangle (-p_j) - \sum_{j=n+1}^{m} \langle \alpha_{i_j}, \alpha_{i_k}^\vee \rangle (q_j - p_j) \\ &= p_k + \sum_{j=n+1}^{k-1} \langle \alpha_{i_j}, \alpha_{i_k}^\vee \rangle p_j - \sum_{j=n+1}^{m} \langle \alpha_{i_j}, \alpha_{i_k}^\vee \rangle q_j.\end{aligned}$$

By Proposition 6.7, we have $\langle \mathrm{wt}(\Phi_\iota^\lambda(\pi)), \alpha_{i_k}^\vee \rangle = \langle w_n\lambda, \alpha_{i_k}^\vee \rangle - \sum_{j=n+1}^{m} \langle \alpha_{i_j}, \alpha_{i_k}^\vee \rangle q_j$. Hence,

$$\sigma_k(\Phi_\iota^\lambda(\pi)) = \sigma_k^-(b_2) - \langle \mathrm{wt}(\Phi_\iota^\lambda(\pi)), \alpha_{i_k}^\vee \rangle = p_k + \sum_{j=n+1}^{k-1} \langle \alpha_{i_j}, \alpha_{i_k}^\vee \rangle p_j - \langle w_n\lambda, \alpha_{i_k}^\vee \rangle. \tag{6.16}$$

It follows from Lemma 6.6 that $-w_n\lambda + w_k\lambda + \sum_{j=n+1}^{k} p_j \alpha_{i_j} = 0$, and hence,

$$\begin{aligned}0 &= -\langle w_n\lambda, \alpha_{i_k}^\vee \rangle + \langle w_k\lambda, \alpha_{i_k}^\vee \rangle + \sum_{j=n+1}^{k} \langle \alpha_{i_j}, \alpha_{i_k}^\vee \rangle p_j \\ &= -\langle w_n\lambda, \alpha_{i_k}^\vee \rangle + p_k + \sum_{j=n+1}^{k-1} \langle \alpha_{i_j}, \alpha_{i_k}^\vee \rangle p_j \end{aligned} \tag{6.17}$$

since $\langle w_k\lambda, \alpha_{i_k}^\vee \rangle = -p_k$ and $\langle \alpha_{i_k}, \alpha_{i_k}^\vee \rangle = 2$. By (6.16) and (6.17), we obtain $\sigma_k(\Phi_\iota^\lambda(\pi)) = \sigma_k^-(b_2) - \langle \mathrm{wt}(\Phi_\iota^\lambda(\pi)), \alpha_{i_k}^\vee \rangle = 0$, as desired. If $k \le m$, then we can show the equality by the same argument as in the case that $n < 0 < m$. Thus we have proved the lemma. □

Now, we set

$$i(k) := i_k = \begin{cases} 2 & \text{if } k \text{ is even}, \\ 1 & \text{if } k \text{ is odd}, \end{cases} \qquad i'(k) := \begin{cases} 1 & \text{if } k \text{ is even}, \\ 2 & \text{if } k \text{ is odd} \end{cases}$$

for $k \in \mathbb{Z}$; note that

$$\langle w_k\lambda, \alpha_{i(k)}^\vee \rangle = -p_k < 0, \quad \langle w_k\lambda, \alpha_{i'(k)}^\vee \rangle = p_{k+1} > 0 \tag{6.18}$$

by (4.3). Let us write $\pi \in \mathbb{B}_1(\lambda)$ as (6.9). We see by (6.18) that

$$H_{i(m)}^\pi\left(\frac{q_m}{p_m}\right) < 0 = H_{i(m)}^\pi(0). \tag{6.19}$$

Moreover, if $m + n$ is odd, then we see that $\langle w_n\lambda, \alpha_{i(m)}^\vee \rangle > 0$, and hence

$$H_{i(m)}^\pi\left(\frac{q_{n+1}}{p_{n+1}}\right) < \langle \mathrm{wt}(\pi), \alpha_{i(m)}^\vee \rangle = H_{i(m)}^\pi(1). \tag{6.20}$$

If $m + n$ is even, then we see that $\langle w_n\lambda, \alpha_{i'(m)}^\vee \rangle > 0$, and hence

$$H_{i'(m)}^\pi\left(\frac{q_{n+1}}{p_{n+1}}\right) < \langle \mathrm{wt}(\pi), \alpha_{i'(m)}^\vee \rangle = H_{i'(m)}^\pi(1). \tag{6.21}$$

Lemma 6.9. *Let $\pi \in \mathbb{B}_1(\lambda)$ be as* (6.9)*. If $n+1 \le k \le m$, then*

$$-q_k - \sum_{j=k+1}^{m} \langle \alpha_{i_j}, \alpha_{i_k}^\vee \rangle q_j = \begin{cases} H_{i(m)}^\pi(q_k/p_k) & \text{if } k-m \in 2\mathbb{Z}, \\ H_{i'(m)}^\pi(q_k/p_k) & \text{if } k-m+1 \in 2\mathbb{Z}. \end{cases}$$

Proof. We set $q_{m+1} \coloneqq 0$ and $q_n \coloneqq p_n$ by convention. Assume that $k-m \in 2\mathbb{Z}$. Then we obtain

$$\begin{aligned} H_{i(m)}^\pi\left(\frac{q_k}{p_k}\right) &= \sum_{j=k}^{m} \left(\frac{q_j}{p_j} - \frac{q_{j+1}}{p_{j+1}}\right) \langle w_j\lambda, \alpha_{i(m)}^\vee \rangle \\ &= \sum_{j=k,k+2,\dots,m-2} \left(\left(\frac{q_j}{p_j} - \frac{q_{j+1}}{p_{j+1}}\right)(-p_j) + \left(\frac{q_{j+1}}{p_{j+1}} - \frac{q_{j+2}}{p_{j+2}}\right) p_{j+2} \right) + \frac{q_m}{p_m}(-p_m) \\ &= \sum_{j=k,k+2,\dots,m-2} (-q_j + a_{i(m)}q_{j+1} - q_{j+2}) - q_m \qquad \text{by (3.1) and (3.2)} \\ &= -q_k + \sum_{j=k,k+2,\dots,m-2} (a_{i(m)}q_{j+1} - 2q_{j+2}) \\ &= -q_k - \sum_{j=k+1}^{m} \langle \alpha_{i_j}, \alpha_{i(m)}^\vee \rangle q_j, \end{aligned}$$

as desired.

Assume that $k-m+1 \in 2\mathbb{Z}$. Then we obtain

$$\begin{aligned} H_{i'(m)}^\pi\left(\frac{q_k}{p_k}\right) &= \sum_{j=k}^{m} \left(\frac{q_j}{p_j} - \frac{q_{j+1}}{p_{j+1}}\right) \langle w_j\lambda, \alpha_{i'(m)}^\vee \rangle \\ &= \sum_{j=k,k+2,\dots,m-1} \left(\left(\frac{q_j}{p_j} - \frac{q_{j+1}}{p_{j+1}}\right)(-p_j) + \left(\frac{q_{j+1}}{p_{j+1}} - \frac{q_{j+2}}{p_{j+2}}\right) p_{j+2} \right) \\ &= \sum_{j=k,k+2,\dots,m-1} (-q_j + a_{i'(m)}q_{j+1} - q_{j+2}) \qquad \text{by (3.1) and (3.2)} \\ &= -q_k + \sum_{j=k,k+2,\dots,m-3} (a_{i'(m)}q_{j+1} - 2q_{j+2}) + a_{i'(m)}q_m - q_{m+1} \\ &= -q_k - \sum_{j=k+1}^{m} \langle \alpha_{i_j}, \alpha_{i'(m)}^\vee \rangle q_j, \end{aligned}$$

as desired. □

By Lemmas 6.8 and 6.9, we obtain the following proposition.

Proposition 6.10. *Let $\pi \in \mathbb{B}_1(\lambda)$ be as* (6.9)*. Then,*

$$-\sigma_k(\Phi_\iota^\lambda(\pi)) = \begin{cases} H_{i(m)}^\pi(0) & \text{if } k-m \in 2\mathbb{Z} \text{ and } m+1 \le k, \\ H_{i(m)}^\pi(q_k/p_k) & \text{if } k-m \in 2\mathbb{Z} \text{ and } n+1 \le k \le m, \\ H_{i(m)}^\pi(1) & \text{if } k-m \in 2\mathbb{Z} \text{ and } k \le n, \\ H_{i'(m)}^\pi(0) & \text{if } k-m+1 \in 2\mathbb{Z} \text{ and } m+1 \le k, \\ H_{i'(m)}^\pi(q_k/p_k) & \text{if } k-m+1 \in 2\mathbb{Z} \text{ and } n+1 \le k \le m, \\ H_{i'(m)}^\pi(1) & \text{if } k-m+1 \in 2\mathbb{Z} \text{ and } k \le n. \end{cases}$$

Proof of Theorem 6.5. By Remark 6.4, it is easy to check that the map Φ_ι^λ is injective. We show that Φ_ι^λ is a morphism of crystals. Let $\pi \in \mathbb{B}_1(\lambda)$. We have $\mathrm{wt}(\pi) = \mathrm{wt}(\Phi_\iota^\lambda(\pi))$ by Proposition 6.7. We show that $\varepsilon_i(\pi) = \varepsilon_i(\Phi_\iota^\lambda(\pi))$ and $\Phi_\iota^\lambda(\tilde{e}_i\pi) = \tilde{e}_i\Phi_\iota^\lambda(\pi)$ for $i \in I$. Let us write π as (6.9).

Case 1. Assume that $i = i(m)$ and $m+n$ is odd. Note that the function $H_i^\pi(t)$ attains a minimal value at $t = q_k/p_k$, $k = m, m-2, \dots, n+1$ (see Remark 5.5 and (6.18)). By (2.13),

$$\begin{aligned}\varepsilon_i(\pi) &= -\min\left\{ H_i^\pi(t) \,\middle|\, t \in \left\{\frac{q_m}{p_m}, \frac{q_{m-2}}{p_{m-2}}, \dots, \frac{q_{n+1}}{p_{n+1}}\right\}\right\} \\ &= \max\left\{ -H_i^\pi(t) \,\middle|\, t \in \left\{\frac{q_m}{p_m}, \frac{q_{m-2}}{p_{m-2}}, \dots, \frac{q_{n+1}}{p_{n+1}}\right\}\right\}. \end{aligned} \tag{6.22}$$

By the definition of $\varepsilon_i(\Phi_\iota^\lambda(\pi))$, we have

$$\varepsilon_i(\Phi_\iota^\lambda(\pi)) = \max_{k:i_k=i} \sigma_k(\Phi_\iota^\lambda(\pi)) = \max_{k-m\in 2\mathbb{Z}} \sigma_k(\Phi_\iota^\lambda(\pi)). \tag{6.23}$$

We see from Proposition 6.10 that

$$\begin{aligned}\max_{k-m\in 2\mathbb{Z}} \sigma_k(\Phi_\iota^\lambda(\pi)) &= \max\left\{ -H_i^\pi(t) \,\middle|\, t \in \left\{0, \frac{q_m}{p_m}, \frac{q_{m-2}}{p_{m-2}}, \dots, \frac{q_{n+1}}{p_{n+1}}, 1\right\}\right\} \\ &= \max\left\{ -H_i^\pi(t) \,\middle|\, t \in \left\{\frac{q_m}{p_m}, \frac{q_{m-2}}{p_{m-2}}, \dots, \frac{q_{n+1}}{p_{n+1}}\right\}\right\}, \end{aligned} \tag{6.24}$$

where the second equality follows from (6.19) and (6.20). By (6.22)–(6.24), we obtain $\varepsilon_i(\pi) = \varepsilon_i(\Phi_\iota^\lambda(\pi))$, as desired. Next, we show that $\Phi_\iota^\lambda(\tilde{e}_i\pi) = \tilde{e}_i\Phi_\iota^\lambda(\pi)$. Since both $\mathbb{B}_1(\lambda)$ and $\mathrm{Im}(\Psi_\iota^\lambda)$ are normal crystals, the equality $\varepsilon_i(\pi) = \varepsilon_i(\Phi_\iota^\lambda(\pi))$ and the injectivity of Φ_ι^λ imply that

$$\Phi_\iota^\lambda(\tilde{e}_i\pi) = \mathbf{0} \iff \tilde{e}_i\pi = \mathbf{0} \iff \varepsilon_i(\pi) = 0 \iff \varepsilon_i(\Phi_\iota^\lambda(\pi)) = 0 \iff \tilde{e}_i\Phi_\iota^\lambda(\pi) = \mathbf{0}.$$

Assume that $\tilde{e}_i\pi \neq \mathbf{0}$, or equivalently, $\tilde{e}_i\Phi_\iota^\lambda(\pi) \neq \mathbf{0}$. By the definition of Φ_ι^λ, we have $\Phi_\iota^\lambda(\pi) = (\dots, y_2, y_1) \otimes t_\lambda \otimes (y_0, y_{-1}, \dots)$, where $y_k = z_k(\pi)$ (see (6.10)). Let $M_{(i)}$ be as (2.3), and set $k' := \max M_{(i)}$. Namely, k' is the largest integer k such that $\sigma_{(i)}(\Phi_\iota^\lambda(\pi)) = \sigma_k(\Phi_\iota^\lambda(\pi))$ and $k-m \in 2\mathbb{Z}$. Then we see by the definition of $\tilde{e}_i$ that $\tilde{e}_i\Phi_\iota^\lambda(\pi) = (\dots, y_2', y_1') \otimes t_\lambda \otimes (y_0', y_{-1}', \dots)$, where $y_k' := y_k - \delta_{k,k'}$. Let t_1 and t_0 be as (2.8) and (2.9), respectively. By (6.22)–(6.24), we obtain $t_1 = q_{k'}/p_{k'}$. By (2.10) and Remark 5.5, we have $t_0 = t_1 - 1/(-\langle w_{k'}\lambda, \alpha_i^\vee\rangle) = (q_{k'}-1)/p_{k'}$. Assume that $k' < m$. By (2.10) and Remark 5.5, we have $q_{k'+1}/p_{k'+1} \le t_0$. Suppose, for a contradiction, that $q_{k'+1}/p_{k'+1} = t_0$, that is,

$$H_i^\pi\left(\frac{q_{k'+1}}{p_{k'+1}}\right) = H_i^\pi\left(\frac{q_{k'}}{p_{k'}}\right) + 1. \tag{6.25}$$

Then it follows from Remark 5.5 that $H_i^\pi(t)$ attains a minimal value at $t = q_{k'+2}/p_{k'+2}$, and hence $H_i^\pi(q_{k'+2}/p_{k'+2}) \in \mathbb{Z}$ by (2.7). By (6.25), we obtain $H_i^\pi(q_{k'+2}/p_{k'+2}) \le H_i^\pi(q_{k'}/p_{k'})$, which contradicts the definition of t_1. Therefore we obtain $q_{k'+1}/p_{k'+1} < t_0$ and

$$\tilde{e}_i\pi = \left(w_m\lambda, \dots, w_{k'}\lambda \dots, w_n\lambda; 0, \frac{q_m}{p_m} \dots, \frac{q_{k'+1}}{p_{k'+1}}, \frac{q_{k'}-1}{p_{k'}}, \frac{q_{k'-1}}{p_{k'-1}}, \dots, \frac{q_{n+1}}{p_{n+1}}, 1\right).$$

If $k' = m$, then

$$\tilde{e}_i\pi = \begin{cases} \left(w_m\lambda, \dots, w_n\lambda; 0, \dfrac{q_m-1}{p_m}, \dfrac{q_{m-1}}{p_{m-1}}, \dots, \dfrac{q_{n+1}}{p_{n+1}}, 1\right) & \text{if } q_m > 1, \\ \left(w_{m-1}\lambda, \dots, w_n\lambda; 0, \dfrac{q_{m-1}}{p_{m-1}}, \dots, \dfrac{q_{n+1}}{p_{n+1}}, 1\right) & \text{if } q_m = 1. \end{cases}$$

Hence we see that

$$
\begin{aligned}
z_k(\tilde{e}_i\pi) &= \begin{cases} q_{k'}-1 & \text{if } 1 \le k = k', \\ (q_{k'}-1)-p_{k'} & \text{if } k = k' \le 0, \\ q_k & \text{if } k \ne k',\ 1 \le k, \text{ and } n+1 \le k \le m, \\ p_k & \text{if } k \ne k',\ 1 \le k, \text{ and } k \le n, \\ q_k - p_k & \text{if } k \ne k',\ k \le 0, \text{ and } n+1 \le k \le m, \\ -p_k & \text{if } k \ne k',\ k \le 0, \text{ and } m+1 \le k, \\ 0 & \text{otherwise}, \end{cases} \\
&= z_k(\pi) - \delta_{k,k'},
\end{aligned}
$$

which implies that $\Phi_\iota^\lambda(\tilde{e}_i\pi) = \tilde{e}_i\Phi_\iota^\lambda(\pi)$.

Case 2. Assume that $i = i'(m)$ and $m+n$ is even. Note that the function $H_i^\pi(t)$ attains a minimal value at $t = 0$ and $t = q_k/p_k$, $k = m-1, m-3, \ldots, n+1$. As in Case 1, we deduce by Proposition 6.10 and (6.21) that

$$
\begin{aligned}
\varepsilon_i(\pi) &= -\min\left\{ H_i^\pi(t) \,\middle|\, t \in \left\{0, \frac{q_{m-1}}{p_{m-1}}, \frac{q_{m-3}}{p_{m-3}}, \ldots, \frac{q_{n+1}}{p_{n+1}}\right\}\right\} \\
&= \max\left\{ -H_i^\pi(t) \,\middle|\, t \in \left\{0, \frac{q_{m-1}}{p_{m-1}}, \frac{q_{m-3}}{p_{m-3}}, \ldots, \frac{q_{n+1}}{p_{n+1}}\right\}\right\} \\
&= \max_{k-m+1 \in 2\mathbb{Z}} \sigma_k(\Phi_\iota^\lambda(\pi)) = \varepsilon_i(\Phi_\iota^\lambda(\pi)).
\end{aligned}
$$

We can show that $\Phi_\iota^\lambda(\tilde{e}_i\pi) = \tilde{e}_i\Phi_\iota^\lambda(\pi)$ in exactly the same way as Case 1.

Case 3. Assume that $i = i(m)$ and $m+n$ is even. Note that the function $H_i^\pi(t)$ attains a minimal value at $t = q_k/p_k$, $k = m, m-2, \ldots, n+2$ and $t = 1$. As in Case 1, we deduce by Proposition 6.10 and (6.19) that

$$
\begin{aligned}
\varepsilon_i(\pi) &= -\min\left\{ H_i^\pi(t) \,\middle|\, t \in \left\{\frac{q_m}{p_m}, \frac{q_{m-2}}{p_{m-2}}, \ldots, \frac{q_{n+2}}{p_{n+2}}, 1\right\}\right\} \\
&= \max\left\{ -H_i^\pi(t) \,\middle|\, t \in \left\{\frac{q_m}{p_m}, \frac{q_{m-2}}{p_{m-2}}, \ldots, \frac{q_{n+2}}{p_{n+2}}, 1\right\}\right\} \\
&= \max_{k-m \in 2\mathbb{Z}} \sigma_k(\Phi_\iota^\lambda(\pi)) = \varepsilon_i(\Phi_\iota^\lambda(\pi)).
\end{aligned} \tag{6.26}
$$

We show that $\Phi_\iota^\lambda(\tilde{e}_i\pi) = \tilde{e}_i\Phi_\iota^\lambda(\pi)$. If $m = n$, then $\pi = (w_n\lambda; 0, 1)$. We see by definition of Φ_ι^λ that

$$
\Phi_\iota^\lambda(\pi) = \begin{cases} (\ldots, 0, p_n, \ldots, p_2, p_1) \otimes t_\lambda \otimes (0, 0, \ldots) & \text{if } n > 0, \\ (\ldots, 0, 0) \otimes t_\lambda \otimes (0, 0, \ldots) & \text{if } n = 0, \\ (\ldots, 0, 0) \otimes t_\lambda \otimes (-p_0, -p_{-1}, \ldots, -p_{n+1}, 0, \ldots) & \text{if } n < 0. \end{cases}
$$

Also, we see that

$$
\tilde{e}_i\pi = \begin{cases} (w_n\lambda, w_{n-1}\lambda; 0, (p_n-1)/p_n, 1) & \text{if } p_n > 1, \\ (w_{n-1}\lambda; 0, 1) & \text{if } p_n = 1. \end{cases}
$$

Thus it is easy to verify that $\Phi_\iota^\lambda(\tilde{e}_i\pi) = \tilde{e}_i\Phi_\iota^\lambda(\pi)$ in this case. Assume that $m > n$; by the assumption that $m+n$ is even, we have $m \ge n+2$. Let $M_{(i)}$ be as (2.3), and set $k' := \max M_{(i)}$. If $k' \in \{m, m-2, \ldots, n+2\}$, then we can show in exactly the same way as Case 1 that $\Phi_\iota^\lambda(\tilde{e}_i(\pi)) = \tilde{e}_i\Phi_\iota^\lambda(\pi)$. Otherwise, we see by Proposition 6.10 and (6.26) that $k' = n$. Let $\Phi_\iota^\lambda(\pi) = (\ldots, y_2, y_1) \otimes t_\lambda \otimes (y_0, y_{-1}, \ldots)$, where

$y_k = z_k(\pi)$. Then we see by the definition of $\tilde{e}_i$ that $\tilde{e}_i\Phi_\iota^\lambda(\pi) = (\ldots, y'_2, y'_1) \otimes t_\lambda \otimes (y'_0, y'_{-1}, \ldots)$, where $y'_k = y_k - \delta_{k,n}$. Let t_1 and t_0 be as (2.8) and (2.9), respectively. We see that $t_1 = 1$ and $t_0 = 1 - 1/(-\langle w_n\lambda, \alpha_i^\vee\rangle) = 1 - 1/p_{n-1}$. By (2.10) and Remark 5.5, we have $q_{n+1}/p_{n+1} \leq t_0$. Suppose, for a contradiction, that $q_{n+1}/p_{n+1} = t_0$; note that $H_i^\pi(q_{n+1}/p_{n+1}) = H_i^\pi(1) + 1$. It follows from Remark 5.5 that $H_i^\pi(t)$ attains a minimal value at $t = q_{n+2}/p_{n+2}$, and hence $H_i^\pi(q_{n+2}/p_{n+2}) \in \mathbb{Z}$ by (2.7). Therefore, we obtain $H_i^\pi(q_{n+2}/p_{n+2}) \leq H_i^\pi(1)$, which contradicts the definition of t_1. Therefore we obtain $q_{n+1}/p_{n+1} < t_0$, which implies that

$$\tilde{e}_i\pi = \left(w_m\lambda, \ldots, w_n\lambda, w_{n-1}\lambda; 0, \frac{q_m}{p_m} \ldots, \ldots, \frac{q_{n+1}}{p_{n+1}}, \frac{p_n - 1}{p_n}, 1\right).$$

Therefore we see that

$$\begin{aligned} z_k(\tilde{e}_i\pi) &= \begin{cases} p_n - 1 & \text{if } 1 \leq k = n, \\ (p_n - 1) - p_n & \text{if } k = n \leq 0, \\ q_k & \text{if } k \neq n,\ 1 \leq k,\ \text{and } n+1 \leq k \leq m, \\ p_k & \text{if } k \neq n,\ 1 \leq k,\ \text{and } k \leq n, \\ q_k - p_k & \text{if } k \neq n,\ k \leq 0,\ \text{and } n+1 \leq k \leq m, \\ -p_k & \text{if } k \neq n,\ k \leq 0,\ \text{and } m+1 \leq k, \\ 0 & \text{otherwise,} \end{cases} \\ &= z_k(\pi) - \delta_{k,n}. \end{aligned}$$

Hence we obtain $\tilde{e}_i(\pi) = \tilde{e}_i\Phi_\iota^\lambda(\pi)$, as desired.

Case 4. Assume that $i = i'(m)$ and $m+n$ is odd. Note that the function $H_i^\pi(t)$ attains a minimal value at $t = 0$, $t = 1$, and $t = q_k/p_k$, $k = m-1, m-3, \ldots, n+2$. By Proposition 6.10, we get

$$\begin{aligned} \varepsilon_i(\pi) &= -\min\left\{ H_i^\pi(t) \,\middle|\, t \in \left\{0, \frac{q_{m-1}}{p_{m-1}}, \frac{q_{m-3}}{p_{m-3}}, \ldots, \frac{q_{n+2}}{p_{n+2}}, 1\right\}\right\} \\ &= \max\left\{ -H_i^\pi(t) \,\middle|\, t \in \left\{0, \frac{q_{m-1}}{p_{m-1}}, \frac{q_{m-3}}{p_{m-3}}, \ldots, \frac{q_{n+2}}{p_{n+2}}, 1\right\}\right\} \\ &= \varepsilon_i(\Phi_\iota^\lambda(\pi)). \end{aligned}$$

We can show in exactly the same way as Case 3 that $\Phi_\iota^\lambda(\tilde{e}_i\pi) = \tilde{e}_i\Phi_\iota^\lambda(\pi)$.

Let $\pi \in \mathbb{B}_1(\lambda)$, and $i \in I$. Because $\mathrm{wt}(\pi) = \mathrm{wt}(\Phi_\iota^\lambda(\pi))$ and $\varepsilon_i(\pi) = \varepsilon_i(\Phi_\iota^\lambda(\pi))$, we have $\varphi_i(\pi) = \varphi_i(\Phi_\iota^\lambda(\pi))$. Also, since both $\mathbb{B}_1(\lambda)$ and $\mathrm{Im}(\Psi_\iota^\lambda)$ are normal crystals, and since $\Phi_\iota^\lambda(\tilde{e}_i\pi) = \tilde{e}_i\Phi_\iota^\lambda(\pi)$, we see that $\Phi_\iota^\lambda(\tilde{f}_i\pi) = \tilde{f}_i\Phi_\iota^\lambda(\pi)$. This completes the proof of Theorem 6.5. □

6.3 Proof of Theorem 3.4.

We can prove Theorem 3.4 in exactly the same way as [15, Theorem 3.2]; we give only a sketch of the proof. In the following, we assume that $k_2 = 1$; the proof for the case that $k_1 = 1$ is similar. Let us identify $\mathcal{B}(\lambda)$ with $\{b \in \mathcal{B}(\infty) \otimes \mathcal{T}_\mu \otimes \mathcal{B}(-\infty) \mid b^* \text{ is extremal}\}$ by Theorem 2.4.

Lemma 6.11 (cf. [15, Lemmas 3.7 and 3.8])**.**

(1) *Let $i \in I$ and $b \in \mathcal{B}(\lambda)$ be such that $\tilde{e}_i b \neq \mathbf{0}$. If b is of the form $b = b_1 \otimes t_\lambda \otimes u_{-\infty}$ with $b_1 \neq u_\infty$, then $\tilde{e}_i b = \tilde{e}_i b_1 \otimes t_\lambda \otimes u_{-\infty}$.*

(2) *Let $i \in I$ and $b \in \mathcal{B}(\lambda)$ be such that $\tilde{f}_i b \neq \mathbf{0}$. If b is of the form $b = u_\infty \otimes t_\lambda \otimes b_2$ with $b_2 \neq u_{-\infty}$, then $\tilde{f}_i b = u_\infty \otimes t_\lambda \otimes \tilde{f}_i b_2$.*

Proof. We give a proof only for part (1). Suppose, for a contradiction, that $\tilde{e}_i b = b_1 \otimes t_\lambda \otimes \tilde{e}_i u_{-\infty}$. We see by (2.2) that $(\tilde{e}_i b)^* = b_1^* \otimes t_{-\lambda-\mathrm{wt}(b_1)-\alpha_i} \otimes \tilde{e}_i u_{-\infty}$. Since $\varphi_i((\tilde{e}_i b)^*) \geq \varphi_i(\tilde{e}_i u_{-\infty}) = 1$, it follows from the tensor product rule of crystals that $\tilde{f}_i(\tilde{e}_i b)^* \neq \mathbf{0}$. Because $\tilde{e}_i b \in \mathcal{B}(\lambda)$, we see that $(\tilde{e}_i b)^*$ is an extremal element of weight $-\lambda$. Since $\langle \mathrm{wt}(S_{\mathrm{id}}(\tilde{e}_i b)^*), \alpha_1^\vee \rangle = \langle -\lambda, \alpha_1^\vee \rangle = -k_1 \leq 0$, we obtain $\tilde{f}_1(\tilde{e}_i b)^* = \mathbf{0}$. Therefore we have $i = 2$ and $(\tilde{e}_2 b)^* = b_1^* \otimes t_{-\lambda-\mathrm{wt}(b_1)-\alpha_2} \otimes \tilde{e}_2 u_{-\infty}$. Because $\langle \mathrm{wt}(S_{\mathrm{id}}(\tilde{e}_2 b)^*), \alpha_2^\vee \rangle = \langle -\lambda, \alpha_2^\vee \rangle = 1 \geq 0$, and $(\tilde{e}_2 b)^*$ is an extremal element of extremal weight $-\lambda$, we see that $\tilde{e}_2(\tilde{e}_2 b)^* = \mathbf{0}$, and hence $\varepsilon_2((\tilde{e}_2 b)^*) = 0$. Since $\varepsilon_2((\tilde{e}_2 b)^*) \geq \varepsilon_2(b_1^*)$, we have $\varepsilon_2(b_1^*) = 0$, which implies $\varepsilon_1(b_1^*) \geq 1$ because $b_1 \neq u_\infty$. Hence,

$$\begin{aligned}
\varphi_2(b_1^* \otimes t_{-\lambda-\mathrm{wt}(b_1)-\alpha_2}) &= \varphi_2(b_1^*) + \langle -\lambda - \mathrm{wt}(b_1) - \alpha_2, \alpha_2^\vee \rangle \\
&= (\varepsilon_2(b_1^*) + \langle \mathrm{wt}(b_1^*), \alpha_2^\vee \rangle) + \langle -\lambda - \mathrm{wt}(b_1) - \alpha_2, \alpha_2^\vee \rangle \\
&= \varepsilon_2(b_1^*) + \langle -\lambda - \alpha_2, \alpha_2^\vee \rangle = -1.
\end{aligned}$$

By this equality and $\varepsilon_2(\tilde{e}_2 u_{-\infty}) = \varphi_2(\tilde{e}_2 u_{-\infty}) - \langle \mathrm{wt}(\tilde{e}_2 u_{-\infty}), \alpha_2^\vee \rangle = -1$, it follows from the tensor product rule of crystals that $S_2(\tilde{e}_2 b)^* = \tilde{f}_2(\tilde{e}_2 b)^* = b_1^* \otimes t_{-\lambda-\mathrm{wt}(b_1)-\alpha_2} \otimes u_{-\infty}$. Since $\varepsilon_1(b_1^*) \geq 1$, we obtain $\tilde{e}_1 b_1^* \neq \mathbf{0}$. Therefore it follows from the tensor product rule of crystals that $\varepsilon_1(S_2(\tilde{e}_2 b)^*) \geq \varepsilon_1(b_1^*) \geq 1$, that is, $\tilde{e}_1 S_2(\tilde{e}_2 b)^* \neq \mathbf{0}$. However, since $(\tilde{e}_2 b)^*$ is an extremal element of weight $-\lambda$ and $\langle \mathrm{wt}(S_2(\tilde{e}_2 b)^*), \alpha_1^\vee \rangle = \langle s_2(-\lambda), \alpha_1^\vee \rangle \geq 0$, we see that $\tilde{e}_1 S_2(\tilde{e}_2 b)^* = \mathbf{0}$, which is a contradiction. □

Lemma 6.11 implies the following proposition (see [15, Proposition 3.9]).

Proposition 6.12. *It holds that $\mathcal{B}(\lambda) \subset (\mathcal{B}(\infty) \otimes t_\lambda \otimes u_{-\infty}) \cup (u_\infty \otimes t_\lambda \otimes \mathcal{B}(-\infty))$.*

Here, we set $|\alpha| := \sum_{i \in I} |c_i|$ for $\alpha = \sum_{i \in I} c_i \alpha_i \in \bigoplus_{i \in I} \mathbb{Z}\alpha_i$. By Proposition 6.12, we see that $b \in \mathcal{B}(\lambda)$ is of the form either $b = b_1 \otimes t_\lambda \otimes u_{-\infty}$ with some $b_1 \in \mathcal{B}(\infty)$ or $b = u_\infty \otimes t_\lambda \otimes b_2$ with some $b_2 \in \mathcal{B}(-\infty)$. We deduce by induction on $|\mathrm{wt}(b_1)|$ (resp., $|\mathrm{wt}(b_2)|$) that if b is of the form $b_1 \otimes t_\lambda \otimes u_{-\infty}$ (resp., $b = u_\infty \otimes t_\lambda \otimes b_2$), then $b = \tilde{f}_{i_r} \cdots \tilde{f}_{i_1} u_\lambda$ (resp., $b = \tilde{e}_{i_r} \cdots \tilde{e}_{i_1} u_\lambda$) for some $i_1, \ldots, i_r$ (see [15, Proof of Theorem 3.2]). Thus we have proved Theorem 3.4.

7 Polyhedral realization of $\mathcal{B}(\lambda)$.

Throughout this section, let $\iota = (\iota^+, \iota^-)$ be as in §3, and $\lambda = k_1\Lambda_1 - k_2\Lambda_2 \in P$ as in Theorem 3.1.

7.1 Some propositions and corollary.

Let $\hat{x} = (\ldots, x_2, x_1) \in \mathrm{Im}(\Psi^+_{\iota^+})$ be such that $x_m \le p_m$ for all $m \in \mathbb{Z}_{\ge 1}$. For $l \ge 1$, we set

$$\begin{aligned} z_1(\hat{x}, l) &:= (\ldots, x_2, x_1, p_0, p_{-1}, \ldots, p_{-2l+2}, p_{-2l+1}) \in \mathbb{Z}^{+\infty}_{\ge 0, \iota^+}, \\ z_2(\hat{x}, l) &:= (x_{2l} - p_{2l}, \ldots, x_2 - p_2, x_1 - p_1, 0, 0 \ldots) \in \mathbb{Z}^{-\infty}_{\le 0, \iota^-}. \end{aligned}$$

Proposition 7.1. *Let $\hat{x} = (\ldots, x_2, x_1) \in \mathrm{Im}(\Psi^+_{\iota^+})$ be such that $x_m \le p_m$ for all $m \in \mathbb{Z}_{\ge 1}$, and let $l \ge 1$. The following are equivalent:*

(1) $z_1(\hat{x}, l) \in \mathrm{Im}(\Psi^+_{\iota^+})$;

(2) $c_j x_{j-2l} - c_{j-1} x_{j-2l+1} \ge 0$ *for* $j \ge 2l + 1$;

(3) $0 = \varepsilon_{i_j}(\tilde{f}^{p_{j+1}}_{i_{j+1}} \cdots \tilde{f}^{p_{-1}}_{i_{-1}} \tilde{f}^{p_0}_{i_0} \hat{x}^*)$ *for* $-2l + 1 \le j \le 0$.

Proof. (1) $\Leftrightarrow$ (2): By Proposition 6.2, we see that $z_1(\hat{x}, l) \in \mathrm{Im}(\Psi^+_{\iota^+})$ if and only if

$$\begin{aligned} &c_j p_{j-2l} - c_{j-1} p_{j-2l+1} \ge 0 \quad \text{for } 1 \le j \le 2l - 1, && (7.1) \\ &c_j p_{j-2l} - c_{j-1} x_{j-2l+1} \ge 0 \quad \text{for } j = 2l, && (7.2) \\ &c_j x_{j-2l} - c_{j-1} x_{j-2l+1} \ge 0 \quad \text{for } 2l + 1 \le j. \end{aligned}$$

Therefore it is obvious that (1) implies (2). Assume that (2) holds; we need to show that (7.1) and (7.2). We can easily see by induction on j that

$$c_j p_{j-2l} - c_{j-1} p_{j-2l+1} \ge 0 \quad \text{for } j \ge 1. \tag{7.3}$$

In particular, we get (7.1). Since $x_1 \le p_1$, we see that $c_{2l} p_0 - c_{2l-1} x_1 \ge c_{2l} p_0 - c_{2l-1} p_1$. Combining this inequality and (7.3), we obtain (7.2).

(1) $\Leftrightarrow$ (3): By Proposition 2.6, together with the fact that $i_s = i_t$ if $s \equiv t \bmod 2$, we see that $z_1(\hat{x}, l) \in \mathrm{Im}(\Psi^+_{\iota^+})$ if and only if

$$\begin{aligned} &0 = \varepsilon_{i_j}(\tilde{f}^{p_{j+1}}_{i_{j+1}} \tilde{f}^{p_{j+2}}_{i_{j+2}} \cdots \tilde{f}^{p_{-1}}_{i_{-1}} \tilde{f}^{p_0}_{i_0} \tilde{f}^{x_1}_{i_1} \tilde{f}^{x_2}_{i_2} \cdots z_\infty) \quad \text{for } -2l + 1 \le j \le 0, \\ &0 = \varepsilon_{i_j}(\tilde{f}^{x_{j+1}}_{i_{j+1}} \tilde{f}^{x_{j+2}}_{i_{j+2}} \cdots z_\infty) \quad \text{for } j \ge 1. \end{aligned}$$

Since $\hat{x} \in \mathrm{Im}(\Psi^+_{\iota^+})$, we have $\hat{x}^* = \tilde{f}^{x_1}_{i_1} \tilde{f}^{x_2}_{i_2} \cdots z_\infty$ and $0 = \varepsilon_{i_j}(\tilde{f}^{x_{j+1}}_{i_{j+1}} \tilde{f}^{x_{j+2}}_{i_{j+2}} \cdots z_\infty)$ for $j \ge 1$. Therefore (1) is equivalent to (3).

Thus we have proved the proposition. $\square$

By using Propositions 2.6 and 6.2, together with the fact that $i_s = i_t$ if $s \equiv t \bmod 2$, we can prove the following proposition in exactly the same way as Proposition 7.1.

Proposition 7.2. *Let $\hat{x} = (\ldots, x_2, x_1) \in \mathrm{Im}(\Psi^+_{\iota^+})$ be such that $x_m \le p_m$ for all $m \in \mathbb{Z}_{\ge 1}$, and let $l \ge 1$. The following are equivalent:*

(1) $z_2(\hat{x}, l) \in \mathrm{Im}(\Psi^-_{\iota^-})$;

(2) $c'_{-j+1}(x_{j+2l} - p_{j+2l}) - c'_{-j}(x_{j+2l-1} - p_{j+2l-1}) \le 0$ *for* $-2l + 2 \le j \le -1$;

(3) $0 = \varphi_{i_j}(\tilde{e}^{p_{j-1} - x_{j-1}}_{i_{j-1}} \cdots \tilde{e}^{p_2 - x_2}_{i_2} \tilde{e}^{p_1 - x_1}_{i_1} z_{-\infty})$ *for* $1 \le j \le 2l$.

Proposition 7.3. *Let $\hat{x} = (\ldots, x_2, x_1) \in \mathrm{Im}(\Psi^+_{\iota^+})$ be such that $x_m \leq p_m$ for all $m \in \mathbb{Z}_{\geq 1}$, and let $k \geq 1$. The following are equivalent:*

(1) $c_{k+2l}x_k - c_{k+2l-1}x_{k+1} \geq 0$ *for* $l \geq 1$;

(2) $\gamma_k x_k - x_{k+1} \geq 0$.

Proof. Assume that (2) holds. By Lemma 6.1, together with (3.3), we have $c_{k+2l} > \gamma_k c_{k+2l-1}$ for $l \geq 1$. Hence, $c_{k+2l}x_k - c_{k+2l-1}x_{k+1} \geq c_{k+2l-1}(\gamma_k x_k - x_{k+1})$. By the assumption, we obtain (1).

Assume that (1) holds; note that $x_k \geq 0$. If $x_k = 0$, then we have $-c_{k+2l-1}x_{k+1} \geq 0$. Since $c_{k+2l-1} > 0$, we see that $x_{k+1} = 0$, which gives $\gamma_k x_k - x_{k+1} = 0$. Assume that $x_k > 0$. By the assumption, we obtain $c_{k+2l}/c_{k+2l-1} \geq x_{k+1}/x_k$ for $l \geq 1$. Since the sequence $\{c_{k+l}/c_{k+2l-1}\}_{l \geq 1}$ is strictly decreasing, and converges to γ_k by Lemma 6.1, we see that $x_{k+1}/x_k \leq \gamma_k$, which is equivalent to (2). □

Proposition 7.4. *Let $\hat{x} = (\ldots, x_2, x_1) \in \mathrm{Im}(\Psi^+_{\iota^+})$ be such that $x_m \leq p_m$ for all $m \in \mathbb{Z}_{\geq 1}$, and let $k \geq 1$. The following are equivalent:*

(1) $c'_{2l+i_k}(x_{k+1} - p_{k+1}) - c'_{2l+i_k-1}(x_k - p_k) \leq 0$ *for* $l \geq 1$;

(2) $\gamma_{k+1}p_{k+1} - p_k + x_k - \gamma_{k+1}x_{k+1} \geq 0$.

Proof. Assume that (2) holds. By Lemma 6.1, we have $c'_{2l+i_k} > \gamma_{k+1}c'_{2l+i_k-1}$ for $l \geq 1$. Since $x_{k+1} - p_{k+1} \leq 0$, we see that

$$c'_{2l+i_k}(x_{k+1} - p_{k+1}) - c'_{2l+i_k-1}(x_k - p_k) \leq \underbrace{c'_{2l+i_k-1}}_{>0}(\underbrace{\gamma_{k+1}x_{k+1} - \gamma_{k+1}p_{k+1} - x_k + p_k}_{\leq 0 \text{ by assumption}}) \leq 0.$$

Assume that (1) holds; note that $(0 \leq)\, x_{k+1} \leq p_{k+1}$. If $x_{k+1} = p_{k+1}$, then we have $-c'_{2l+i_k-1}(x_k - p_k) \leq 0$. Since $c'_{2l+i_k-1} > 0$, we obtain $x_k = p_k$, which gives $\gamma_{k+1}p_{k+1} - p_k + x_k - \gamma_{k+1}x_{k+1} = 0$. Assume that $x_{k+1} < p_{k+1}$. By the assumption, we obtain $c'_{2l+i_k}/c'_{2l+i_k-1} \geq (x_k - p_k)/(x_{k+1} - p_{k+1})$. Since the sequence $\{c'_{2l+i_k}/c'_{2l+i_k-1}\}_{l \geq 1}$ is decreasing and converges to γ_{k+1} by Lemma 6.1, we obtain $(x_k - p_k)/(x_{k+1} - p_{k+1}) \leq \gamma_{k+1}$, which is equivalent to (2). □

By Propositions 7.1 – 7.4, we obtain the following corollary.

Corollary 7.5. *Let $\hat{x} = (\ldots, x_2, x_1) \in \mathrm{Im}(\Psi^+_{\iota^+})$ be such that $x_m \leq p_m$ for all $m \in \mathbb{Z}_{\geq 1}$.*

(1) $\gamma_k x_k - x_{k+1} \geq 0$ *for all* $k \geq 1$ *if and only if*

$$\varepsilon_{i_j}(\tilde{f}_{i_{j+1}}^{p_{j+1}} \cdots \tilde{f}_{i_{-1}}^{p_{-1}} \tilde{f}_{i_0}^{p_0} \hat{x}^*) = 0$$

for all $j \leq 0$.

(2) $\gamma_{k+1}p_{k+1} - p_k + x_k - \gamma_{k+1}x_{k+1} \geq 0$ *for all* $k \geq 1$ *if and only if*

$$\varphi_{i_j}(\tilde{e}_{i_{j-1}}^{p_{j-1}-x_{j-1}} \cdots \tilde{e}_{i_2}^{p_2-x_2} \tilde{e}_{i_1}^{p_1-x_1} z_{-\infty}) = 0$$

for all $j \geq 1$.

7.2 Proof of Theorem 3.6.

Lemma 7.6. *It holds that* $\Sigma_\iota(\lambda) \subset \mathrm{Im}(\Psi_\iota^\lambda)$.

Proof. Let $\vec{x} = (\ldots, x_2, x_1) \otimes t_\lambda \otimes (x_0, x_{-1}, \ldots) \in \Sigma_\iota(\lambda)$. By Proposition 6.2, it suffices to show that

$$c_j x_j - c_{j-1} x_{j+1} \geq 0 \qquad \text{for } j \geq 1, \tag{7.4}$$

$$c'_{-j+1} x_j - c'_{-j} x_{j-1} \leq 0 \qquad \text{for } j \leq 0. \tag{7.5}$$

First, we verify (7.4). If $j = 1$, then the assertion is obvious because $c_j = 1$ and $c_{j-1} = 0$. Assume that $j > 1$; note that $c_{j-1} > 0$. It follows from Lemma 6.1 that $\gamma_j < c_j / c_{j-1}$. Also, we have $(\gamma_j \zeta_j - \zeta_{j+1})(\vec{x}) = \gamma_j x_j - x_{j+1} \geq 0$ by the definition of $\Sigma_\iota(\lambda)$. Hence,

$$c_j x_j - c_{j-1} x_{j+1} = c_{j-1} \left(\frac{c_j}{c_{j-1}} x_j - x_{j+1} \right) \geq c_{j-1}(\gamma_j x_j - x_{j+1}) \geq 0.$$

Next, we verify (7.5). If $j = 0$, then the assertion is obvious because $c'_{-j+1} = 1$ and $c'_{-j} = 0$. Assume that $j < 0$; note that $c'_{-j} > 0$. It follows from Lemma 6.1 that $\gamma_j < c'_{-j+1}/c'_{-j}$. Also, we have $(\zeta_{j-1} - \gamma_j \zeta_j)(\vec{x}) = x_{j-1} - \gamma_j x_j \geq 0$ by the definition of $\Sigma_\iota(\lambda)$. Hence,

$$c'_{-j+1} x_j - c'_{-j} x_{j-1} = c'_{-j} \left(\frac{c'_{-j+1}}{c'_{-j}} x_j - x_{j-1} \right) \leq c'_{-j}(\gamma_j x_j - x_{j-1}) \leq 0.$$

Thus we have proved the lemma. □

For $k \in \mathbb{Z}$, we set $k^{(+)} := k + 2$ and $k^{(-)} := k - 2$. Also, we define the function $\bar{\beta}_k \colon \mathbb{R}^\infty \to \mathbb{R}$ by

$$\bar{\beta}_k = \begin{cases} -\langle \lambda, \alpha_{i_k}^\vee \rangle + \zeta_k - a_{i_k} \zeta_{k+1} + \zeta_{k+2} & \text{if } k = -1, 0, \\ \zeta_k - a_{i_k} \zeta_{k+1} + \zeta_{k+2} & \text{otherwise;} \end{cases}$$

note that $\bar{\beta}_k(\vec{x}) = \sigma_k(\vec{x}) - \sigma_{k^{(+)}}(\vec{x})$. Moreover, for $k \in \mathbb{Z}$, we define the operator F_k on $\{c + \sum_{l \in \mathbb{Z}} \phi_l \zeta_l \mid c, \phi_l \in \mathbb{R}\}$ as follows: for $\phi = c + \sum_{l \in \mathbb{Z}} \phi_l \zeta_l$ with $c, \phi_l \in \mathbb{R}$, we set

$$F_k(\phi) := \begin{cases} \phi - \phi_k \bar{\beta}_{k^{(+)}} & \text{if } \phi_k \geq 0, \\ \phi - \phi_k \bar{\beta}_{k^{(-)}} & \text{if } \phi_k < 0; \end{cases}$$

note that $F_k(\phi) = \phi$ if $\phi_k = 0$.

Lemma 7.7. *Let* Ξ *be a subset of* $\{c + \sum_{l \in \mathbb{Z}} \phi_l \zeta_l \mid c, \phi_l \in \mathbb{R}\}$. *Assume that*

$$F_k(\phi) \in \sum_{j \geq 1} \mathbb{R}_{\geq 0} \zeta_j + \sum_{j \leq 0} \mathbb{R}_{\geq 0}(-\zeta_j) + \sum_{\psi \in \Xi} \mathbb{R}_{\geq 0} \psi \tag{7.6}$$

for all $\phi \in \Xi$ *and* $k \in \mathbb{Z}$. *Then,* $\Sigma = \{\vec{x} \in \mathbb{Z}_\iota(\lambda) \mid \phi(\vec{x}) \geq 0$ *for all* $\phi \in \Xi\}$ *is a subcrystal of* $\mathbb{Z}_\iota(\lambda)$.

Proof. This lemma can be shown similarly to [3, Lemma 4.3]. Let $\vec{x} \in \Sigma$. We show that if $\tilde{f}_i \vec{x} \neq \mathbf{0}$, then $\tilde{f}_i \vec{x} \in \Sigma$, that is, $\phi(\tilde{f}_i \vec{x}) \geq 0$ for all $\phi \in \Xi$. Let us write $\phi = c + \sum_{l \in \mathbb{Z}} \phi_l \zeta_l$ with $c, \phi_l \in \mathbb{R}$. Define $M_{(i)} = M_{(i)}(\vec{x})$ as (2.3), and set $k := \min M_{(i)}$. We see by (2.4) that $\phi(\tilde{f}_i \vec{x}) = \phi(\vec{x}) + \phi_k$. If $\phi_k \geq 0$, then the assertion is obvious because $\phi(\tilde{f}_i \vec{x}) = \phi(\vec{x}) + \phi_k \geq \phi(\vec{x}) \geq 0$. Assume that $\phi_k < 0$. By the definition of $M_{(i)}$ and the fact that $i_k = i_m$ if $k \equiv m \bmod 2$, we have $\sigma_k(\vec{x}) > \sigma_{k-2n}(\vec{x})$ for all $n \in \mathbb{Z}_{\geq 1}$. In particular, $\sigma_k(\vec{x}) > \sigma_{k^{(-)}}(\vec{x})$. Since $\bar{\beta}_{k^{(-)}}(\vec{x}) = \sigma_{k^{(-)}}(\vec{x}) - \sigma_k(\vec{x}) \in \mathbb{Z}$, we deduce that $\bar{\beta}_{k^{(-)}}(\vec{x}) \leq -1$. It follows that

$$\phi(\tilde{f}_i \vec{x}) = \phi(\vec{x}) + \phi_k \geq \phi(\vec{x}) - \phi_k \bar{\beta}_{k^{(-)}}(\vec{x}) = (F_k(\phi))(\vec{x}).$$

By assumption (7.6), we see that $F_k(\phi)$ is of the form $F_k(\phi) = \sum_{j\geq 1} t_j\zeta_j + \sum_{j\leq 0} t_j(-\zeta_j) + \sum_{\psi\in\Xi} t_\psi \psi$, where $t_j, t_\psi \in \mathbb{R}_{\geq 0}$. Since $\vec{x} \in \Sigma$, we have $\psi(\vec{x}) \geq 0$ for any $\psi \in \Xi$. Therefore we see that

$$\phi(\tilde{f}_i\vec{x}) \geq (F_k(\phi))(\vec{x}) = \sum_{j\geq 1} t_j \underbrace{x_j}_{\geq 0} + \sum_{j\leq 0} t_j \underbrace{(-x_j)}_{\geq 0} + \sum_{\psi\in\Xi} t_\psi \underbrace{\psi(\vec{x})}_{\geq 0} \geq 0,$$

and hence $\tilde{f}_i\vec{x} \in \Sigma$. Similarly, we can show that $\tilde{e}_i\vec{x} \in \Sigma$ if $\tilde{e}_i\vec{x} \neq \mathbf{0}$. Thus we have proved the lemma. $\square$

Proof of Theorem 3.6. By Lemmas 7.6 and 7.7, it suffices to show that

$$F_k(\phi) \in \sum_{j\geq 1} \mathbb{R}_{\geq 0}\zeta_j + \sum_{j\leq 0} \mathbb{R}_{\geq 0}(-\zeta_j) + \sum_{\psi\in\Xi_\iota[\lambda]} \mathbb{R}_{\geq 0}\psi \tag{7.7}$$

for all $k \in \mathbb{Z}$ and $\phi \in \Xi_\iota[\lambda]$. Here we verify (7.7) for the case that $\phi = \gamma_0 p_0 + \gamma_0\zeta_0 - \zeta_1$; for the other cases, see Appendix A. If $k \neq 0, 1$, then the assertion is trivial since $F_k(\phi) = \phi$. Assume that $k = 0$. We compute

$$\begin{aligned} F_0(\phi) &= (\gamma_0 p_0 + \gamma_0\zeta_0 - \zeta_1) - \gamma_0\bar{\beta}_0 \\ &= (\gamma_0 p_0 + \gamma_0\zeta_0 - \zeta_1) - \gamma_0(p_0 + \zeta_0 - a_2\zeta_1 + \zeta_2) \\ &= \gamma_0\left(\left(a_{i_0} - \frac{1}{\gamma_0}\right)\zeta_1 - \zeta_2\right) \\ &= \gamma_0\underbrace{(\gamma_1\zeta_1 - \zeta_2)}_{\in\Xi_\iota[\lambda]} \quad \text{by (3.4).} \end{aligned}$$

Assume that $k = 1$. We compute

$$\begin{aligned} F_1(\phi) &= (\gamma_0 p_0 + \gamma_0\zeta_0 - \zeta_1) - (-1)\bar{\beta}_{-1} \\ &= (\gamma_0 p_0 + \gamma_0\zeta_0 - \zeta_1) + (-p_1 + \zeta_{-1} - a_{-1}\zeta_0 + \zeta_1) \\ &= \gamma_0 p_0 - p_1 + \zeta_{-1} + (\gamma_0 - a_{-1})\zeta_0 \\ &= \gamma_0 p_0 - p_1 + \zeta_{-1} - \frac{1}{\gamma_{-1}}\zeta_0 \quad \text{by (3.4)} \\ &= -p_{-1} + \left(\gamma_0 + \frac{1}{\gamma_1}\right)p_0 - p_1 + \frac{1}{\gamma_{-1}}(\gamma_{-1}p_{-1} - p_0 + \gamma_{-1}\zeta_{-1} - \zeta_0) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{note that } \gamma_{-1} = \gamma_1) \\ &= -p_{-1} + a_1 p_0 - p_1 + \frac{1}{\gamma_{-1}}(\gamma_{-1}p_{-1} - p_0 + \gamma_{-1}\zeta_{-1} - \zeta_0) \quad \text{by (3.4)} \\ &= 0 + \frac{1}{\gamma_{-1}}\underbrace{(\gamma_{-1}p_{-1} - p_0 + \gamma_{-1}\zeta_{-1} - \zeta_0)}_{\in\Xi_\iota[\lambda]} \quad \text{by (3.2).} \end{aligned}$$

Thus we have proved Theorem 3.6. $\square$

7.3 Proof of Theorem 3.7.

Let $\Sigma_\iota(\lambda)'$ be the subset of $\Sigma_\iota(\lambda)$ consisting of the elements of the form $\hat{x}\otimes t_\lambda\otimes z_{-\infty}$ with $\hat{x} \in \mathrm{Im}(\Psi^+_{\iota^+})$.

Proposition 7.8. *For $\vec{x} \in \Sigma_\iota(\lambda)'$, the element $\vec{x}^*$ is extremal.*

Proposition 7.9. *Let $\vec{y} \in \mathrm{Im}(\Psi^\lambda_\iota)$. If $\vec{y}^*$ is extremal, then there exist $i_1, \ldots, i_l \in I$ and $\vec{x} \in \Sigma_\iota(\lambda)'$ such that $\vec{x} = \tilde{f}_{i_l}\cdots\tilde{f}_{i_1}\vec{y}$.*

Assuming that Propositions 7.8 and 7.9 are true, we give a proof of Theorem 3.7.

Proof of Theorem 3.7. Set $B := \{\vec{x} \in \mathrm{Im}(\Psi_\iota^\lambda) \mid \vec{x}^* \text{ is extremal}\}$. First, we show that $\Sigma_\iota(\lambda) \subset B$. Let $\vec{x} = \hat{x}_1 \otimes t_\lambda \otimes \hat{x}_2 \in \Sigma_\iota(\lambda)$ with $\hat{x}_1 \in \mathbb{Z}^{+\infty}_{\geq 0,\iota^+}$ and $\hat{x}_2 \in \mathbb{Z}^{-\infty}_{\leq 0,\iota^-}$. By Theorem 3.6, we have $\hat{x}_2 \in \mathrm{Im}(\Psi_{\iota^-}^-)$. Since $\mathrm{Im}(\Psi_{\iota^-}^-) \cong \mathcal{B}(-\infty)$ as crystals, there exist $i_1, \ldots, i_l$ such that $\tilde{f}_{i_l}^{\max} \cdots \tilde{f}_{i_1}^{\max} \hat{x}_2 = z_{-\infty}$. Then we see by the tensor product rule of crystals that $\vec{y} := \tilde{f}_{i_l}^{\max} \cdots \tilde{f}_{i_1}^{\max} \vec{x}$ is an element of $\Sigma_\iota(\lambda)'$. Since $\vec{x} \in \Sigma_\iota(\lambda) \subset \mathrm{Im}(\Psi_\iota^\lambda)$, we see that $\vec{y} \in \mathrm{Im}(\Psi_\iota^\lambda)$. Also, it follows from Proposition 7.8 that $\vec{y}^*$ is extremal. Thus we obtain $\vec{y} \in B$. Since B is a subcrystal by Corollary 2.8, we obtain $\vec{x} \in B$.

Next, we show that $\Sigma_\iota(\lambda) \supset B$. Let $\vec{y} \in \mathrm{Im}(\Psi_\iota^\lambda)$ be such that $\vec{y}^*$ is extremal. By Proposition 7.9, there exist $i_1, \ldots, i_l \in I$ such that $\tilde{f}_{i_l} \cdots \tilde{f}_{i_1} \vec{y} \in \Sigma_\iota(\lambda)' \subset \Sigma_\iota(\lambda)$. Therefore, by Theorem 3.6, we obtain $\vec{y} \in \Sigma_\iota(\lambda)$. This completes the proof of Theorem 3.7. □

First, we prove Proposition 7.8. Let $\vec{z} = z_1 \otimes t_\lambda \otimes z_2 \in \mathrm{Im}(\Psi_\iota^\lambda)$. By the tensor product rule of crystals (see also [8, Appendix B]), we see that

$$\varepsilon_i(\vec{z}) = \max\{\varepsilon_i(z_1), \varphi_i(z_2) - \langle \mathrm{wt}(\vec{z}), \alpha_i^\vee \rangle\}, \tag{7.8}$$

$$\varphi_i(\vec{z}) = \max\{\varepsilon_i(z_1) + \langle \mathrm{wt}(\vec{z}), \alpha_i^\vee \rangle, \varphi_i(z_2)\}. \tag{7.9}$$

Moreover,

$$\tilde{e}_i^{\varepsilon_i(\vec{z})} \vec{z} = \tilde{e}_i^{\varepsilon_i(z_1)} z_1 \otimes t_\lambda \otimes \tilde{e}_i^c z_2, \tag{7.10}$$

where $c = \max\{-\varepsilon_i(z_1) + \varphi_i(z_2) - \langle \mathrm{wt}(\vec{z}), \alpha_i^\vee \rangle, 0\}$. Since $\mathrm{wt}(S_{w_k}\vec{z}^*) = w_k \mathrm{wt}(\vec{z}^*) = -w_k\lambda$, we see that

$$\langle \mathrm{wt}(S_{w_k}\vec{z}^*), \alpha_i^\vee \rangle = \begin{cases} p_k & \text{if } i = i_k, \\ -p_{k+1} & \text{if } i = i_{k+1}, \end{cases} \tag{7.11}$$

and hence

$$S_{w_k}\vec{z}^* = \tilde{e}_{i_k}^{p_k} S_{w_{k-1}} \vec{z}^* = \tilde{f}_{i_{k+1}}^{p_{k+1}} S_{w_{k+1}} \vec{z}^*. \tag{7.12}$$

Proposition 7.10. *Let $\vec{x} = \hat{x} \otimes t_\lambda \otimes z_{-\infty} \in \Sigma_\iota(\lambda)'$ with $\hat{x} = (\ldots, x_2, x_1)$. Then,*

$$S_{w_k}\vec{x}^* = \begin{cases} \tilde{e}_{i_k}^{x_k} \cdots \tilde{e}_{i_2}^{x_2} \tilde{e}_{i_1}^{x_1} \hat{x}^* \otimes t_\mu \otimes \tilde{e}_{i_k}^{p_k - x_k} \cdots \tilde{e}_{i_2}^{p_2 - x_2} \tilde{e}_{i_1}^{p_1 - x_1} z_{-\infty} & \text{if } k \geq 0, \\ \tilde{f}_{i_{k+1}}^{p_{k+1}} \cdots \tilde{f}_{i_{-1}}^{p_{-1}} \tilde{f}_{i_0}^{p_0} \hat{x}^* \otimes t_\mu \otimes z_{-\infty} & \text{if } k \leq 0, \end{cases}$$

where $\mu := -\lambda - \mathrm{wt}(\hat{x})$.

Proof. Since $\hat{x} \in \mathrm{Im}(\Psi_{\iota^+}^+)$ by Lemma 7.6, it follows from Proposition 2.6 that

$$x_j = \varepsilon_{i_j}(\tilde{e}_{i_{j-1}}^{x_{j-1}} \cdots \tilde{e}_{i_2}^{x_2} \tilde{e}_{i_1}^{x_1} \hat{x}^*) \quad \text{for } j \geq 1. \tag{7.13}$$

By the definition of $\Sigma_\iota(\lambda)'$, we have $p_k - x_k \geq 0$, $\gamma_k x_k - x_{k+1} \geq 0$, and $\gamma_{k+1} p_{k+1} - p_k + x_k - \gamma_{k+1} x_{k+1} \geq 0$ for all $k \geq 1$. By Corollary 7.5, we see that

$$\varepsilon_{i_j}(\tilde{f}_{i_{j+1}}^{p_{j+1}} \cdots \tilde{f}_{i_{-1}}^{p_{-1}} \tilde{f}_{i_0}^{p_0} \hat{x}^*) = 0 \quad \text{for } j \leq 0, \tag{7.14}$$

$$\varphi_{i_j}(\tilde{e}_{i_{j-1}}^{p_{j-1} - x_{j-1}} \cdots \tilde{e}_2^{p_2 - x_2} \tilde{e}_1^{p_1 - x_1} z_{-\infty}) = 0 \quad \text{for } j \geq 1. \tag{7.15}$$

Now, we show the assertion by induction on $|k|$. If $k = 0$, then the assertion is obvious by (2.5). Assume that $k \geq 1$. By the induction hypothesis, we obtain

$$S_{w_{k-1}}\vec{x}^* = \tilde{e}_{i_{k-1}}^{x_{k-1}} \cdots \tilde{e}_{i_2}^{x_2} \tilde{e}_{i_1}^{x_1} \hat{x}^* \otimes t_\mu \otimes \tilde{e}_{i_{k-1}}^{p_{k-1} - x_{k-1}} \cdots \tilde{e}_{i_2}^{p_2 - x_2} \tilde{e}_{i_1}^{p_1 - x_1} z_{-\infty}.$$

We have $\langle \mathrm{wt}(S_{w_{k-1}}\vec{x}^*), \alpha_{i_k}^\vee \rangle = -p_k \leq 0$ by (7.11), $\varepsilon_{i_k}(\tilde{e}_{i_{k-1}}^{x_{k-1}} \cdots \tilde{e}_{i_2}^{x_2} \tilde{e}_{i_1}^{x_1} \hat{x}^*) = x_k$ by (7.13), and

$$\varphi_{i_k}(\tilde{e}_{i_{k-1}}^{p_{k-1} - x_{k-1}} \cdots \tilde{e}_{i_2}^{p_2 - x_2} \tilde{e}_{i_1}^{p_1 - x_1} z_{-\infty}) = 0$$

by (7.15). Since $x_k \le p_k$ as seen above, we see by (7.10) and (7.12) that

$$\begin{aligned} S_{w_k}\vec{x}^* &= \tilde{e}_{i_k}^{p_k}(\tilde{e}_{i_{k-1}}^{x_{k-1}} \cdots \tilde{e}_{i_2}^{x_2}\tilde{e}_{i_1}^{x_1}\hat{x}^* \otimes t_\mu \otimes \tilde{e}_{i_{k-1}}^{p_{k-1}-x_{k-1}} \cdots \tilde{e}_{i_2}^{p_2-x_2}\tilde{e}_{i_1}^{p_1-x_1} z_{-\infty}) \\ &= \tilde{e}_{i_k}^{x_k}\tilde{e}_{i_{k-1}}^{x_{k-1}} \cdots \tilde{e}_{i_2}^{x_2}\tilde{e}_{i_1}^{x_1}\hat{x}^* \otimes t_\mu \otimes \tilde{e}_{i_k}^{p_k-x_k}\tilde{e}_{i_{k-1}}^{p_{k-1}-x_{k-1}} \cdots \tilde{e}_{i_2}^{p_2-x_2}\tilde{e}_{i_1}^{p_1-x_1} z_{-\infty}. \end{aligned}$$

Assume that $k \le -1$. By the induction hypothesis, we obtain

$$S_{w_{k+1}}\vec{x}^* = \tilde{f}_{i_{k+2}}^{p_{k+2}} \cdots \tilde{f}_{i_{-1}}^{p_{-1}}\tilde{f}_{i_0}^{p_0}\hat{x}^* \otimes t_\mu \otimes z_{-\infty}.$$

Since $S_{w_k}\vec{x}^* \ne \mathbf{0}$, we see by (7.12) that

$$S_{w_k}\vec{x}^* = \tilde{f}_{i_{k+1}}^{p_{k+1}}(\tilde{f}_{i_{k+2}}^{p_{k+2}} \cdots \tilde{f}_{i_{-1}}^{p_{-1}}\tilde{f}_{i_0}^{p_0}\hat{x}^* \otimes t_\mu \otimes z_{-\infty}) = \tilde{f}_{i_{k+1}}^{p_{k+1}}\tilde{f}_{i_{k+2}}^{p_{k+2}} \cdots \tilde{f}_{i_{-1}}^{p_{-1}}\tilde{f}_{i_0}^{p_0}\hat{x}^* \otimes t_\mu \otimes z_{-\infty}.$$

Thus we have proved the proposition. □

Proof of Proposition 7.8. Keep the notation and setting in Proposition 7.10. We show that $\vec{x}^*$ is extremal; by (7.11), it suffices to show that $\varepsilon_{i_k}(S_{w_k}\vec{x}^*) = 0$ and $\varphi_{i_{k+1}}(S_{w_k}\vec{x}^*) = 0$ for all $k \in \mathbb{Z}$.

Step 1. Assume that $k \ge 0$. We show that $\varphi_{i_{k+1}}(S_{w_k}\vec{x}^*) = 0$. We know from Proposition 7.10 that

$$S_{w_k}\vec{x}^* = \tilde{e}_{i_k}^{x_k} \cdots \tilde{e}_{i_2}^{x_2}\tilde{e}_{i_1}^{x_1}\hat{x}^* \otimes t_\mu \otimes \tilde{e}_{i_k}^{p_k-x_k} \cdots \tilde{e}_{i_2}^{p_2-x_2}\tilde{e}_{i_1}^{p_1-x_1} z_{-\infty}.$$

By the same argument as in the proof of Proposition 7.10, we see that $\langle \mathrm{wt}(S_{w_k}\vec{x}^*), \alpha_{i_{k+1}}^\vee \rangle = -p_{k+1} \le 0$, $\varepsilon_{i_{k+1}}(\tilde{e}_{i_k}^{x_k} \cdots \tilde{e}_{i_2}^{x_2}\tilde{e}_{i_1}^{x_1}\hat{x}^*) = x_{k+1}$, $\varphi_{i_{k+1}}(\tilde{e}_{i_k}^{p_k-x_k} \cdots \tilde{e}_{i_2}^{p_2-x_2}\tilde{e}_{i_1}^{p_1-x_1} z_{-\infty}) = 0$, and $x_{k+1} \le p_{k+1}$. Thus, by (7.9), $\varphi_{i_{k+1}}(S_{w_k}\vec{x}^*) = \max\{x_{k+1} + (-p_{k+1}), 0\} = 0$.

Step 2. Assume that $k > 0$. We show that $\varepsilon_{i_k}(S_{w_k}\vec{x}^*) = 0$. We have

$$\begin{aligned} \varepsilon_{i_k}(S_{w_k}\vec{x}^*) &= \varepsilon_{i_k}(\tilde{e}_{i_k}^{p_k} S_{w_{k-1}}\vec{x}^*) = \varepsilon_{i_k}(S_{w_{k-1}}\vec{x}^*) - p_k \\ &= \varphi_{i_k}(S_{w_{k-1}}\vec{x}^*) - \underbrace{\langle \mathrm{wt}(S_{w_{k-1}}\vec{x}^*), \alpha_{i_k}^\vee \rangle}_{=-p_k \text{ by } (7.11)} - p_k = \varphi_{i_k}(S_{w_{k-1}}\vec{x}^*). \end{aligned}$$

Since $\varphi_{i_k}(S_{w_{k-1}}\vec{x}^*) = 0$ by Step 1, we obtain $\varepsilon_{i_k}(S_{w_k}\vec{x}^*) = 0$.

Step 3. Assume that $k \le 0$. We show that $\varepsilon_{i_k}(S_{w_k}\vec{x}^*) = 0$. We know from Proposition 7.10 that

$$S_{w_k}\vec{x}^* = \tilde{f}_{i_{k+1}}^{p_{k+1}} \cdots \tilde{f}_{i_{-1}}^{p_{-1}}\tilde{f}_{i_0}^{p_0}\hat{x}^* \otimes t_\mu \otimes z_{-\infty}.$$

We have $\langle \mathrm{wt}(S_{w_k}\vec{x}^*), \alpha_{i_k}^\vee \rangle = p_k$ by (7.11) and $\varepsilon_{i_k}(\tilde{f}_{i_{k+1}}^{p_{k+1}} \cdots \tilde{f}_{i_{-1}}^{p_{-1}}\tilde{f}_{i_0}^{p_0}\hat{x}^*) = 0$ by (7.14). Since $\varphi_{i_k}(z_{-\infty}) = 0$, we see by (7.8) that $\varepsilon_{i_k}(S_{w_k}\vec{x}^*) = \max\{0, 0 - p_k\} = 0$.

Step 4. Assume that $k < 0$. We show that $\varphi_{i_{k+1}}(S_{w_k}\vec{x}^*) = 0$. We have

$$\begin{aligned} \varphi_{i_{k+1}}(S_{w_k}\vec{x}^*) &= \varphi_{i_{k+1}}(\tilde{f}_{i_{k+1}}^{p_{k+1}} S_{w_{k+1}}\vec{x}^*) = \varphi_{i_{k+1}}(S_{w_{k+1}}\vec{x}^*) - p_{k+1} \\ &= \varepsilon_{i_{k+1}}(S_{w_{k+1}}\vec{x}^*) + \underbrace{\langle \mathrm{wt}(S_{w_{k+1}}\vec{x}^*), \alpha_{i_{k+1}}^\vee \rangle}_{=p_{k+1} \text{ by } (7.11)} - p_{k+1} = \varepsilon_{i_{k+1}}(S_{w_{k+1}}\vec{x}^*). \end{aligned}$$

Since $\varepsilon_{i_{k+1}}(S_{w_{k+1}}\vec{x}^*) = 0$ by Step 3, we obtain $\varphi_{i_{k+1}}(S_{w_k}\vec{x}^*) = 0$.

This completes the proof of Proposition 7.8. □

Next, we prove Proposition 7.9. Let $\vec{y} = \hat{y}_1 \otimes t_\lambda \otimes \hat{y}_2 \in \mathrm{Im}(\Psi_\iota^\lambda)$ with $\hat{y}_1 \in \mathrm{Im}(\Psi_{\iota^+}^+)$ and $\hat{y}_2 \in \mathrm{Im}(\Psi_{\iota^-}^-)$, and assume that $\vec{y}^*$ is extremal. Since $\mathrm{Im}(\Psi_{\iota^-}^-) \cong \mathcal{B}(-\infty)$ as crystals, there exist $i_1, \dots, i_l$ such that $\tilde{f}_{i_l}^{\max} \cdots \tilde{f}_{i_1}^{\max}\hat{y}_2 = z_{-\infty}$. By the tensor product rule of crystals, if we set $\vec{x} := \tilde{f}_{i_l}^{\max} \cdots \tilde{f}_{i_1}^{\max}\vec{y}$, then $\vec{x}$ is of the form $\vec{x} = \hat{x} \otimes t_\lambda \otimes z_{-\infty}$ with $\hat{x} \in \mathrm{Im}(\Psi_{\iota^+}^+)$; in order to prove Proposition 7.9, it suffices to show that $\vec{x} \in \Sigma_\iota(\lambda)$. Let us write $\hat{x} = (\ldots, x_2, x_1)$. By the definition of $\Sigma_\iota(\lambda)$, we deduce that $\vec{x} = (\ldots, x_2, x_1) \otimes t_\lambda \otimes z_{-\infty} \in \Sigma_\iota(\lambda)$ if and only if

$$
\begin{aligned}
& p_k - x_k \geq 0 && \text{for } k \geq 1; && (7.16)\\
& \gamma_k x_k - x_{k+1} \geq 0 && \text{for } k \geq 1; && (7.17)\\
& \gamma_{k+1} p_{k+1} - p_k + x_k - \gamma_{k+1} x_{k+1} \geq 0 && \text{for } k \geq 1; && (7.18)\\
& \gamma_0 p_0 + \gamma_0 \cdot 0 - x_1 \geq 0; && && (7.19)\\
& \gamma_1 p_1 + 0 - \gamma_1 x_1 \geq 0. && && (7.20)
\end{aligned}
$$

Assume that (7.16) holds. Then it is obvious that (7.20) holds. Moreover, we obtain $\gamma_0 p_0 + \gamma_0 \cdot 0 - x_1 \geq \gamma_0 p_0 - p_1$. Recall that $a_1 a_2 > 4$. Thus we obtain $\sqrt{a_1^2 a_2^2 - 4a_1 a_2} > a_1 a_2 - 3$, and hence

$$\gamma_0 = \alpha = \frac{a_1 a_2 + \sqrt{a_1^2 a_2^2 - 4a_1 a_2}}{2a_2} > \frac{2a_1 a_2 - 3}{2a_2} = a_1 - \frac{3}{2a_2}.$$

Assume that $a_1, a_2 \geq 2$. Then $a_1 - 3/2a_2 > a_1 - 1 > 0$. By the definition of λ, either $p_0 \leq p_1 < (a_1 - 1)p_0$ or $p_1 < p_0 \leq (a_2 - 1)p_1$ holds. In both cases, we deduce that $\gamma_0 p_0 - p_1 \geq 0$. Assume that $a_1 = 1$ (resp., $a_2 = 1$). Then $a_1 - 3/2a_2 > 1/2$ (resp., $a_1 - 3/2a_2 > a_1 - 2$). By the definition of λ, we have $2p_1 \leq p_0 \leq (a_2 - 2)p_1$ (resp., $2p_0 \leq p_1 \leq (a_1 - 2)p_0$). Hence we deduce that $\gamma_0 p_0 - p_1 \geq 0$. Thus we get (7.19). Therefore, it remains to show that (7.16), (7.17), and (7.18).

Now, since $\{\vec{z} \in \mathrm{Im}(\Psi_\iota^\lambda) \mid \vec{z}^* \text{ is extremal}\}$ is a subcrystal of $\mathrm{Im}(\Psi_\iota^\lambda)$, it follows that $\vec{x} \in \{\vec{z} \in \mathrm{Im}(\Psi_\iota^\lambda) \mid \vec{z}^* \text{ is extremal}\}$. Also, by Proposition 2.6, we have

$$x_j = \varepsilon_{i_j}(\tilde{e}_{i_{j-1}}^{x_{j-1}} \cdots \tilde{e}_{i_2}^{x_2} \tilde{e}_{i_1}^{x_1} \hat{x}^*) \text{ for } j \geq 1. \tag{7.21}$$

Proposition 7.11 (proof of (7.16))**.** *Let $\vec{x} = \hat{x} \otimes t_\lambda \otimes z_{-\infty} \in \mathrm{Im}(\Psi_\iota^\lambda)$, and write $\hat{x} = (\ldots, x_2, x_1)$. If $\vec{x}^*$ is extremal, then $p_k - x_k \geq 0$, and*

$$S_{w_k} \vec{x}^* = \tilde{e}_{i_k}^{x_k} \cdots \tilde{e}_{i_2}^{x_2} \tilde{e}_{i_1}^{x_1} \hat{x}^* \otimes t_\mu \otimes \tilde{e}_{i_k}^{p_k - x_k} \cdots \tilde{e}_{i_2}^{p_2 - x_2} \tilde{e}_{i_1}^{p_1 - x_1} z_{-\infty}$$

for $k \geq 1$, where $\mu := -\lambda - \mathrm{wt}(\hat{x})$.

Proof. We proceed by induction on k. Assume that $k = 1$. Since $\vec{x}^* = \hat{x}^* \otimes t_\mu \otimes z_{-\infty}$, and $\langle \mathrm{wt}(\vec{x}^*), \alpha_{i_1}^\vee \rangle = \langle -\lambda, \alpha_{i_1}^\vee \rangle = -p_1$, we see by (7.8) and (7.21) that

$$\varepsilon_1(\vec{x}^*) = \max\{x_1, 0 - (-p_1)\} = \max\{x_1, p_1\}. \tag{7.22}$$

Because $\vec{x}^*$ is extremal, the inequality $\langle \mathrm{wt}(\vec{x}^*), \alpha_{i_1}^\vee \rangle = -p_1 \leq 0$ implies that $\varepsilon_1(\vec{x}^*) = p_1$. By (7.22), we obtain $p_1 = \max\{x_1, p_1\}$, and hence $x_1 \leq p_1$. Also we see by (7.10) that $S_{w_1} \vec{x}^* = \tilde{e}_{i_1}^{p_1} \vec{x}^* = \tilde{e}_{i_1}^{x_1} \hat{x}^* \otimes t_\mu \otimes \tilde{e}_{i_1}^{p_1 - x_1} z_{-\infty}$.

Let $k \geq 2$. By the induction hypothesis, we have

$$S_{w_{k-1}} \vec{x}^* = \tilde{e}_{i_{k-1}}^{x_{k-1}} \cdots \tilde{e}_{i_2}^{x_2} \tilde{e}_{i_1}^{x_1} \hat{x}^* \otimes t_\mu \otimes \tilde{e}_{i_{k-1}}^{p_{k-1} - x_{k-1}} \cdots \tilde{e}_{i_2}^{p_2 - x_2} \tilde{e}_{i_1}^{p_1 - x_1} z_{-\infty}.$$

Hence we see by (7.21) that

$$
\begin{aligned}
\varepsilon_{i_k}(S_{w_{k-1}} \vec{x}^*) &= \max\{\varepsilon_{i_k}(\tilde{e}_{i_{k-1}}^{x_{k-1}} \cdots \tilde{e}_{i_1}^{x_1} \hat{x}^*), \underbrace{\varphi_{i_k}(\tilde{e}_{i_{k-1}}^{p_{k-1} - x_{k-1}} \cdots \tilde{e}_{i_1}^{p_1 - x_1} z_{-\infty}) - \langle \mathrm{wt}(S_{w_{k-1}} \vec{x}^*), \alpha_{i_k}^\vee \rangle}_{=: m_k}\} \\
&= \max\{x_k, m_k\}.
\end{aligned}
$$

By (7.11), we have $\langle \mathrm{wt}(S_{w_{k-1}}\vec{x}^*), \alpha_{i_k}^\vee \rangle = -p_k$. Since $\vec{x}^*$ is extremal, the inequality $\langle \mathrm{wt}(S_{w_{k-1}}\vec{x}^*), \alpha_{i_k}^\vee \rangle = -p_k \leq 0$ implies that $\varepsilon_{i_k}(S_{w_{k-1}}\vec{x}^*) = p_k$. Hence we obtain $p_k = \max\{x_k, m_k\}$, which implies $x_k \leq p_k$. Therefore we see by (7.10) and (7.12) that

$$\begin{aligned}
S_{w_k}\vec{x}^* &= \tilde{e}_{i_k}^{p_k} S_{w_{k-1}}\vec{x}^* \\
&= \tilde{e}_{i_k}^{p_k} \tilde{e}_{i_{k-1}}^{x_{k-1}} \cdots \tilde{e}_{i_2}^{x_2} \tilde{e}_{i_1}^{x_1} \hat{x}^* \otimes t_\mu \otimes \tilde{e}_{i_{k-1}}^{p_{k-1}-x_{k-1}} \cdots \tilde{e}_{i_2}^{p_2-x_2} \tilde{e}_{i_1}^{p_1-x_1} z_{-\infty} \\
&= \tilde{e}_{i_k}^{x_k} \cdots \tilde{e}_{i_2}^{x_2} \tilde{e}_{i_1}^{x_1} \hat{x}^* \otimes t_\mu \otimes \tilde{e}_{i_k}^{p_k-x_k} \cdots \tilde{e}_{i_2}^{p_2-x_2} \tilde{e}_{i_1}^{p_1-x_1} z_{-\infty}.
\end{aligned}$$

Thus we have proved the proposition. □

Proposition 7.12 (proof of (7.18))**.** *Let $\vec{x} = \hat{x} \otimes t_\lambda \otimes z_{-\infty} \in \mathrm{Im}(\Psi_\iota^\lambda)$, and write $\hat{x} = (\ldots, x_2, x_1)$. If $\vec{x}^*$ is extremal, then $\gamma_{k+1}p_{k+1} - p_k + x_k - \gamma_{k+1}x_{k+1} \geq 0$ for $k \geq 1$.*

Proof. By Corollary 7.5, it suffices to show that $\varphi_{i_j}(\tilde{e}_{i_{j-1}}^{p_{j-1}-x_{j-1}} \cdots \tilde{e}_{i_2}^{p_2-x_2} \tilde{e}_{i_1}^{p_1-x_1} z_{-\infty}) = 0$ for all $j \geq 1$. Let $j \geq 1$. Since $\vec{x}^*$ is extremal, and since $\langle \mathrm{wt}(S_{w_{j-1}}\vec{x}^*), \alpha_{i_j}^\vee \rangle = -p_j \leq 0$ by (7.11), we see that $\varphi_{i_j}(S_{w_{j-1}}\vec{x}^*) = 0$. We know from Proposition 7.11 that

$$S_{w_{j-1}}\vec{x}^* = \tilde{e}_{i_{j-1}}^{x_{j-1}} \cdots \tilde{e}_{i_2}^{x_2} \tilde{e}_{i_1}^{x_1} \hat{x}^* \otimes t_\mu \otimes \tilde{e}_{i_{j-1}}^{p_{j-1}-x_{j-1}} \cdots \tilde{e}_{i_2}^{p_2-x_2} \tilde{e}_{i_1}^{p_1-x_1} z_{-\infty}.$$

We see by (7.9) that

$$\begin{aligned}
0 &= \varphi_{i_j}(S_{w_{j-1}}\vec{x}^*) \\
&= \max\{\varepsilon_{i_j}(e_{i_{j-1}}^{x_{j-1}} \cdots \tilde{e}_{i_1}^{x_1} \hat{x}^*) + \langle \mathrm{wt}(S_{w_{j-1}}\vec{x}^*), \alpha_{i_j}^\vee \rangle, \varphi_{i_{j-1}}(\tilde{e}_{i_{j-1}}^{p_{j-1}-x_{j-1}} \cdots \tilde{e}_{i_1}^{p_1-x_1} z_{-\infty})\}.
\end{aligned}$$

Hence we obtain $0 \geq \varphi_{i_j}(\tilde{e}_{i_{j-1}}^{p_{j-1}-x_{j-1}} \cdots \tilde{e}_{i_2}^{p_2-x_2} \tilde{e}_{i_1}^{p_1-x_1} z_{-\infty})$. Because $\varphi_i(\hat{z}) \geq 0$ for all $i \in I$ and $\hat{z} \in \mathbb{Z}_{\leq 0, \iota^-}^{-\infty}$, we conclude that $0 = \varphi_{i_j}(\tilde{e}_{i_{j-1}}^{p_{j-1}-x_{j-1}} \cdots \tilde{e}_{i_2}^{p_2-x_2} \tilde{e}_{i_1}^{p_1-x_1} z_{-\infty})$. Thus we have proved the proposition. □

Proposition 7.13 (proof of (7.17))**.** *Let $\vec{x} = \hat{x} \otimes t_\lambda \otimes z_{-\infty} \in \mathrm{Im}(\Psi_\iota^\lambda)$, and write $\hat{x} = (\ldots, x_2, x_1)$. If $\vec{x}^*$ is extremal, then $\gamma_k x_k - x_{k+1} \geq 0$ for $k \geq 1$.*

Proof. By Corollary 7.5, it suffices to show that $\varepsilon_{i_j}(\tilde{f}_{i_{j+1}}^{p_{j+1}} \tilde{f}_{i_{j+2}}^{p_{j+2}} \cdots \tilde{f}_{i_0}^{p_0} \hat{x}^*) = 0$ for all $j \leq 0$. Let $j \leq 0$. Since $\vec{x}^*$ is extremal, and since $\langle \mathrm{wt}(S_{w_j}\vec{x}^*), \alpha_{i_j}^\vee \rangle = p_j \geq 0$ by (7.11), we see that $\varepsilon_{i_j}(S_{w_j}\vec{x}^*) = 0$. We see by (7.12) that

$$S_{w_j}\vec{x}^* = \tilde{f}_{i_{j+1}}^{p_{j+1}} \cdots \tilde{f}_{i_{-1}}^{p_{-1}} \tilde{f}_{i_0}^{p_0} S_{w_0}\vec{x}^* = \tilde{f}_{i_{j+1}}^{p_{j+1}} \cdots \tilde{f}_{i_{-1}}^{p_{-1}} \tilde{f}_{i_0}^{p_0} (\hat{x}^* \otimes t_\mu \otimes z_{-\infty}).$$

Since $S_{w_j}\vec{x}^* \neq \mathbf{0}$, and since $\tilde{f}_i z_{-\infty} = \mathbf{0}$ for all $i \in I$, we see that

$$S_{w_j}\vec{x}^* = \tilde{f}_{i_{j+1}}^{p_{j+1}} \cdots \tilde{f}_{i_{-1}}^{p_{-1}} \tilde{f}_{i_0}^{p_0} \hat{x}^* \otimes t_\mu \otimes z_{-\infty}.$$

It follows from (7.8) that $0 = \varepsilon_{i_j}(S_{w_j}\vec{x}^*) = \max\{\varepsilon_{i_j}(\tilde{f}_{i_{j+1}}^{p_{j+1}} \cdots \tilde{f}_{i_{-1}}^{p_{-1}} \tilde{f}_{i_0}^{p_0} \hat{x}^*), 0 - p_j\}$. Since $-p_j < 0$, and since $\varepsilon_i(\hat{z}) \geq 0$ for all $i \in I$ and $\hat{z} \in \mathbb{Z}_{\geq 0, \iota^+}^{+\infty}$, we obtain

$$\varepsilon_{i_k}(\tilde{f}_{i_{j+1}}^{p_{j+1}} \tilde{f}_{i_{j+2}}^{p_{j+2}} \cdots \tilde{f}_{i_{-1}}^{p_{-1}} \hat{x}^*) = 0.$$

Thus we have proved the proposition. □

Appendix.

A Action of F_k on $\Xi_\iota[\lambda]$.

In this appendix, we compute $F_k(\phi)$, $k \in \mathbb{Z}$, for $\phi = c + \sum_{l \in \mathbb{Z}} \phi_l \zeta_l \in \Xi_\iota[\lambda]$; recall that $F_k(\phi) = \phi$ for $k \in \mathbb{Z}$ such that $\phi_k = 0$.

$$\begin{aligned}
F_0(\gamma_0 p_0 + \gamma_0 \zeta_0 - \zeta_1) &= \alpha(\gamma_1 \zeta_1 - \zeta_2).\\
F_1(\gamma_0 p_0 + \gamma_0 \zeta_0 - \zeta_1) &= \frac{1}{\beta}(\gamma_{-1} p_{-1} - p_0 + \gamma_{-1}\zeta_{-1} - \zeta_0).\\
F_0(\gamma_1 p_1 + \zeta_0 - \gamma_1 \zeta_1) &= \frac{1}{\alpha}(\gamma_2 p_2 - p_1 + \zeta_1 - \gamma_2 \zeta_2).\\
F_1(\gamma_1 p_1 + \zeta_0 - \gamma_1 \zeta_1) &= \beta(\zeta_{-1} - \gamma_0 \zeta_0).
\end{aligned}$$

For $k \geq 1$,

$$F_k(p_k - \zeta_k) = \begin{cases} (\zeta_{-1} - \gamma_0\zeta_0) + \dfrac{1}{\beta}(-\zeta_0) & \text{if } k = 1,\\ \dfrac{1}{\alpha}(p_1 - \zeta_1) + (\gamma_1 p_1 + \zeta_0 - \gamma_1\zeta_1) & \text{if } k = 2,\\ \dfrac{1}{\gamma_k}(p_{k-1} - \zeta_{k-1}) + (\gamma_{k-1}p_{k-1} - p_{k-2} + \zeta_{k-2} - \gamma_{k-1}\zeta_{k-1}) & \text{if } k \geq 3. \end{cases}$$

$$F_k(\gamma_k\zeta_k - \zeta_{k+1}) = \gamma_k(\gamma_{k+1}\zeta_{k+1} - \zeta_{k+2}).$$

$$F_{k+1}(\gamma_k\zeta_k - \zeta_{k+1}) = \begin{cases} \dfrac{1}{\alpha}(\gamma_0 p_0 + \gamma_0\zeta_0 - \zeta_1) & \text{if } k = 1,\\ \dfrac{1}{\gamma_{k-1}}(\gamma_{k-1}p_{k-1} - \zeta_k) & \text{if } k \geq 2. \end{cases}$$

$$F_k(\gamma_{k+1}p_{k+1} - p_k + \zeta_k - \gamma_{k+1}\zeta_{k+1}) = \frac{1}{\gamma_{k+2}}(\gamma_{k+2}p_{k+2} - p_{k+1} + \zeta_{k+1} - \gamma_{k+2}\zeta_{k+2}).$$

$$F_{k+1}(\gamma_{k+1}p_{k+1} - p_k + \zeta_k - \gamma_{k+1}\zeta_{k+1}) = \begin{cases} \alpha(\gamma_1 p_1 + \zeta_0 - \gamma_1\zeta_1) & \text{if } k = 1,\\ \gamma_{k+1}(\gamma_k p_k - p_{k-1} + \zeta_{k+1} - \gamma_k\zeta_k) & \text{if } k \geq 2. \end{cases}$$

For $k \leq 0$,

$$F_k(p_k + \zeta_k) = \begin{cases} \dfrac{1}{\alpha}(-\zeta_1) + (\gamma_1\zeta_1 - \zeta_2) & \text{if } k = 0,\\ \dfrac{1}{\beta}(p_0 + \zeta_0) + (\gamma_0 p_0 + \gamma_0\zeta_0 - \zeta_1) & \text{if } k = -1,\\ \dfrac{1}{\gamma_k}(p_{k+1} + \zeta_{k+1}) + (\gamma_{k+1}p_{k+1} - p_{k+2} + \gamma_{k+1}\zeta_{k+1} - \zeta_{k+2}) & \text{if } k \leq -2. \end{cases}$$

$$F_{k-1}(\zeta_{k-1} - \gamma_k\zeta_k) = \begin{cases} \dfrac{1}{\beta}(\gamma_1 p_1 + \zeta_0 - \gamma_1\zeta_1) & \text{if } k = 0,\\ \dfrac{1}{\gamma_{k+1}}(\zeta_k - \gamma_{k+1}\zeta_{k+1}) & \text{if } k \leq -1. \end{cases}$$

$$F_k(\zeta_{k-1} - \gamma_k\zeta_k) = \gamma_k(\zeta_{k-2} - \gamma_{k-1}\zeta_{k-1}).$$

$$F_{k-1}(\gamma_{k-1}p_{k-1} + \gamma_{k-1}\zeta_{k-1} - p_k + \zeta_k) = \begin{cases} \beta(\gamma_0 p_0 + \gamma_0\zeta_0 - \zeta_1) & \text{if } k = 0,\\ \gamma_{k-1}(\gamma_k p_k - p_{k+1} + \gamma_k\zeta_k - \zeta_{k+1}) & \text{if } k \leq -1. \end{cases}$$

$$F_k(\gamma_{k-1}p_{k-1} + \gamma_{k-1}\zeta_{k-1} - p_k + \zeta_k) = \frac{1}{\gamma_{k-2}}(\gamma_{k-2}p_{k-2} - p_{k-1} + \gamma_{k-2}\zeta_{k-2} - \zeta_{k-1}).$$

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