# MAGNETIC CURVES IN QUASI-SASAKIAN 3-MANIFOLDS

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Abstract. We study magnetic trajectories corresponding to contact magnetic fields in 3-dimensional quasi-Sasakian manifolds. We show that they are slant curves, that is their contact angles are constant. We prove that such magnetic curves are geodesics for a certain linear connection for which all four structure tensor fields are parallel.

## Introduction

From mathematical point of view, static magnetic fields on oriented Euclidean 3-space $\mathbb{E}^3$ are regarded as closed 2-forms. Based on this fundamental fact, we can introduce the notion of magnetic field on arbitrary Riemannian manifolds. A *magnetic field* $F$ on a Riemannian manifold $(M, g)$ is a closed 2-form. Denote by $\phi$ the endomorphism field metrically equivalent to the magnetic field $F$, then the *Lorentz equation* (called sometimes also Newton equation) is defined as $\nabla_{\gamma'}\gamma' = q\phi\gamma'$. Here $\nabla$ is the Levi-Civita connection of $(M, g)$ and $q$ is a constant. Solutions to Lorentz equation are called *magnetic curves* with strength $q$. Thus, a magnetic curve $\gamma$ is a mathematical model of trajectory of a charged particle moving under the action of the Lorentz force derived from the magnetic field $F$. Note that when $F = 0$, *i.e.*, the Lorentz force is null, then the trajectories are geodesics. In this manner we may regard the magnetic curves as generalizations of geodesics.

Return to the original 3-dimensional situation, magnetic fields are identified with divergence free vector fields. More precisely, on every oriented Riemannian 3-manifold $(M, g)$, the space $\Lambda^2(M)$ of all smooth 2-forms is identified with the space $\mathfrak{X}(M)$ of all smooth vector fields via the Hodge star operator and the volume form $dv_g$. Under this identification, magnetic fields are considered as divergence-free vector fields.

Moreover, if a divergence free vector field $\xi$ is a *unit vector field*, then one can see that $(\phi, \xi, \eta)$ is an *almost contact structure* on $M$ compatible with the metric $g$. Here $\eta$ is the 1-form dual to $\xi$. Hence, an oriented Riemannian 3-manifold $(M, g)$ together with a magnetic field $F$ whose corresponding divergence free vector field is of unit length can be regarded as an almost contact metric manifold with closed fundamental 2-form.

This observation motivates us to study Lorentz equations in almost contact metric manifolds of arbitrary odd-dimension with *closed* fundamental 2-form.

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*Date*: June 3, 2017.

2010 *Mathematics Subject Classification.* 53C15, 53C25, 53C80.

*Key words and phrases.* magnetic curve; slant curve; quasi-Sasakian manifold; geodesic.

The first named author is partially supported by Kakenhi 15K04834. The third named author is supported by CNCS-UEFISCDI (Romania) PN-II-RU-TE-2014-4-0004.

In our previous works, we have studied magnetic fields in Sasakian manifolds [4, 10] and cosymplectic manifolds [5, 13, 15], respectively. Both Sasakian and cosymplectic manifolds are particular cases of quasi-Sasakian manifolds. In two other papers [11, 14] we have studied magnetic curves in odd-dimensional Cartesian space $\mathbb{R}^{2n+1}$ equipped with *non-Sasakian* quasi-Sasakian structure with $n > 1$. As we have seen before, 3-dimensional magnetic theory is exceptional among odd-dimensional magnetic theory, we study magnetic curves in 3-dimensional quasi-Sasakian manifolds in this paper. From another point of view, the class of quasi-Sasakian 3-manifolds contains important Riemannian 3-manifolds. In fact, all *model spaces of Thurston geometry* except the space $\mathrm{Sol}_3$ and hyperbolic 3-space $\mathbb{H}^3$ admit homogeneous almost contact structures compatible with the corresponding metric. The resulting homogeneous almost contact metric 3-manifolds are quasi-Sasakian.

This paper is organized as follows. After recalling, in Section 1, prerequisite knowledge on almost contact geometry, we show, in Section 2, that every contact magnetic curve is a *slant curve*, that is, a curve making constant angle with the trajectories of the characteristic vector field. In Section 3, we study contact magnetic curves in quasi-Sasakian 3-manifolds, in detail. As it is well known, magnetic trajectories in the Euclidean 3-space $\mathbb{E}^3$ are helices. Furthermore, magnetic curves in Sasakian and cosymplectic manifolds of arbitrary dimension are also helices. Remarkably, the magnetic curves in quasi-Sasakian manifolds are not, in general, helices. We exhibit explicit examples of contact magnetic curves in a quasi-Sasakian space which are not helices. In Section 4, we study magnetic curves under pseudo-conformal deformations. In the last section, we define a family linear connections with respect to which the four structure tensor fields are parallel and we call them the *Okumura type connections*. We give some reinterpretation of magnetic curves, namely we prove that they are geodesic for the Okumura type connections.

## 1. Almost contact metric manifolds

1.1. **General settings.** Let $M$ be an odd-dimensional manifold. An *almost contact structure* on $M$ is a triplet of tensor fields $(\varphi, \xi, \eta)$ where $\varphi$ is an endomorphism field, $\xi$ is a vector field, $\eta$ is a one form, respectively, such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1.$$

An $(2n+1)$-dimensional manifold together with an almost contact structure is called an *almost contact manifold*. A Riemannian metric $g$ on an almost contact manifold $(M, \varphi, \xi, \eta)$ is said to be a *compatible metric* if it satisfies

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{1.1}$$

An almost contact structure together with a compatible metric is called an *almost contact metric structure*. An $(2n+1)$-dimensional manifold together with an almost contact metric structure is called an *almost contact metric manifold*.

On an almost contact manifold $M$, we define a hyperplane field $D$ by

$$D = \{X \in TM \mid \eta(X) = 0\}.$$

Here $TM$ denotes the tangent bundle of $M$. A diffeomorphism $f$ on an almost contact metric manifold $M$ is said to be a *pseudo-conformal transformation* if $f^*g$ is a Riemannian metric on $M$ and represented as

$$f^*g = ag + b\eta \otimes \eta,$$

for some positive smooth function $a$ and smooth function $b$ such that $a + b > 0$. Clearly, pseudo-conformal transformations are conformal on $D$, and hence, they preserve $D$.

In particular, if a diffeomorphism $f$ satisfies

$$f^*g = ag + a(a-1)\eta \otimes \eta,$$

for some positive constant $a$, then $f$ is a *$D$-homothetic transformation* in the sense of Tanno [22].

One can see that on an almost contact metric 3-manifold $(M, \varphi, \xi, \eta, g)$, another Riemannian metric $\tilde{g}$ is compatible to $(\varphi, \xi, \eta)$ if and only if there exists a smooth positive function $\sigma$ such that $\tilde{g} = \sigma g + (1-\sigma)\eta \otimes \eta$ (see [18]).

A plane section at a point $p$ of an almost contact metric manifold is called a *holomorphic plane* at $p$ if it is invariant under $\varphi_p$. The sectional curvature function of holomorphic planes is called the *$\varphi$-holomorphic sectional curvature*.

On the other hand, a plane section at $p$ is said to be a *$\xi$-section* at $p$ if it contains $\xi_p$.

The *fundamental 2-form* $\Phi$ of an almost contact metric manifold $M$ is defined by

$$\Phi(X, Y) := g(X, \varphi Y), \quad X, Y \in \mathfrak{X}(M).$$

An almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is called a *contact metric manifold* if

$$\Phi = d\eta. \tag{1.2}$$

The formula (1.2) implies that the one-form $\eta$ is actually a *contact form*, namely $\eta$ satisfies $\eta \wedge (d\eta)^n \neq 0$.

It should be remarked that every almost contact metric manifold is *orientable*. In fact, the volume element of the associated metric coincides with $(-1)^n \eta \wedge \Phi^n/(2^n n!)$.

An almost contact manifold $M$ is said to be of *rank* $r = 2s$, $(s > 0)$ if $(d\eta)^s \neq 0$ and $\eta \wedge (d\eta)^s = 0$, respectively of *rank* $r = 2s+1$ if $\eta \wedge (d\eta)^s \neq 0$ and $(d\eta)^{s+1} = 0$. Thus, contact metric manifolds are of rank $2n+1$.

An almost contact manifold $M$ is said to be *normal* if its normality tensor vanishes, that is if $[\varphi, \varphi] + 2d\eta \otimes \xi = 0$, where $[\varphi, \varphi]$ is the Nijenhuis torsion of $\varphi$ defined by

$$[\varphi, \varphi](X, Y) = [\varphi X, \varphi Y] - \varphi[X, \varphi Y] - \varphi[\varphi X, Y] + \varphi^2[X, Y],$$

for all $X$, $Y \in \mathfrak{X}(M)$.

A normal almost contact metric manifold is said to be a *quasi-Sasakian manifold* if its fundamental 2-form $\Phi$ is closed [1]. In particular, a contact metric manifold is called a *Sasakian manifold* if it is normal. By definition, Sasakian manifolds are quasi-Sasakian manifolds of rank $2n+1$.

**Assumption.** Through the rest of the paper we suppose that the manifold $M$ has dimension 3, even that some formulas are valid for arbitrary dimension.

1.2. **Normal almost contact metric structures of type** $(\alpha, \beta)$**.** For an arbitrary almost contact metric 3-manifold $M$, we have:

$$(\nabla_X \varphi)Y = g(\varphi \nabla_X \xi, Y)\xi - \eta(Y)\varphi \nabla_X \xi, \tag{1.3}$$

where $\nabla$ is the Levi-Civita connection on $M$.

Olszak showed in [18] that an almost contact metric 3-manifold $M$ is normal if and only if $\nabla\xi \circ \varphi = \varphi \circ \nabla\xi$ or, equivalently,

$$\nabla_X \xi = -\alpha\varphi X + \beta(X - \eta(X)\xi), \quad X \in \mathfrak{X}(M), \tag{1.4}$$

where $\alpha$ and $\beta$ are the functions defined by

$$\alpha = \frac{1}{2}\mathrm{trace}\,(\varphi\nabla\xi), \quad \beta = \frac{1}{2}\mathrm{trace}\,(\nabla\xi) = \mathrm{div}\,\xi. \tag{1.5}$$

See also [12]. We call the pair $(\alpha, \beta)$ the *type* of a normal almost contact metric 3-manifold $M$.

We note that the functions $\alpha$ and $\beta$ defined by (1.5) are interchanged compared to the original work of Olszak, but we have adopted the notations from Blair's book [2].

Using (1.3) and (1.4) we note that the covariant derivative $\nabla\varphi$ of a normal almost contact metric 3-manifold is given by

$$(\nabla_X \varphi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\varphi X, Y)\xi - \eta(Y)\varphi X). \tag{1.6}$$

Moreover, the functions $\alpha$ and $\beta$ satisfy

$$2\alpha\beta + \xi(\alpha) = 0.$$

This implies that, if $\alpha$ is a nonzero constant, then $\beta = 0$. In particular, a normal almost contact metric 3-manifold is said to be

- *cosymplectic* (or *coKähler*) *manifold* if $\alpha = \beta = 0$,
- *$\alpha$-Sasakian manifold* if $\alpha$ is a nonzero constant and $\beta = 0$,
- *$\beta$-Kenmotsu manifold* if $\alpha = 0$ and $\beta$ is a nonzero constant.

1-Sasakian manifolds and 1-Kenmotsu manifolds are simply called *Sasakian manifolds* and *Kenmotsu manifolds*, respectively. Sasakian manifolds of constant $\varphi$-holomorphic sectional curvature are called *Sasakian space forms*. Note that an almost contact metric manifold of dimension $2n + 1 \geq 3$ is said to be a *trans-Sasakian manifold* if it satisfies (1.6).

1.3. **Quasi-Sasakian 3-manifolds.** Let $M$ be a quasi-Sasakian 3-manifold. The following statements hold true:

- rank $M = 1$ if and only if $M$ is cosymplectic.
- There are no quasi-Sasakian 3-manifolds with rank $M = 2$ (*cf.* [1]).
- rank $M = 3$ if and only if $\eta$ is a contact form on $M$.

Typical examples of cosymplectic 3-manifolds are the Euclidean 3-space $\mathbb{E}^3$ and the product manifolds $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$. Magnetic curves in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ have been studied in [13] and [15], respectively. In other news, for contact magnetic curves in Sasakian space forms and cosymplectic manifolds of arbitrary odd-dimension, we refer to [4, 10] and [5], respectively.

**Remark 1.1** (Thurston geometry)**.** It should be remarked that *all* the eight model spaces of Thurston geometry admit homogeneous almost contact structure naturally associated to the metric. In particular, other than the model space $\mathrm{Sol}_3$ of solvegeometry, the naturally associated almost contact structures are normal. As we have mentioned above, the space form $\mathbb{E}^3$ and the product spaces $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ are cosymplectic. The unit 3-sphere $\mathbb{S}^3$, the Heisenberg group $\mathrm{Nil}_3$ and the universal covering $\widetilde{\mathrm{SL}}_2\mathbb{R}$ of the special linear group $\mathrm{SL}_2\mathbb{R}$ equipped with the compatible normal contact metric structure are Sasakian space forms. In particular $\mathrm{Nil}_3$ is identified with the Sasakian space form $\mathbb{R}^3(-3)$. The hyperbolic 3-space $\mathbb{H}^3$ equipped with the compatible normal contact metric structure is a Kenmotsu manifold. The space $\mathrm{Sol}_3$ equipped with a naturally associated almost contact structure is a non-Sasakian contact metric 3-manifold. Thus, the six model spaces $\mathbb{E}^3$, $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\mathrm{Nil}_3$, $\widetilde{\mathrm{SL}}_2\mathbb{R}$ are quasi-Sasakian.

The following result is due to Olszak.

**Proposition 1.1.** ([18]) *Let $M$ be an almost contact metric 3-manifold. Then $M$ is quasi-Sasakian if and only if $M$ satisfies*

$$(\nabla_X \varphi)Y = \alpha\big(g(X,Y)\xi - \eta(Y)X\big),$$

*for some function $\alpha$ satisfying $d\alpha(\xi) = 0$.*

Compare this formula with (1.6). Thus quasi-Sasakian 3-manifolds are characterized as normal almost contact metric manifolds of type $(\alpha, 0)$ with $\xi(\alpha) = 0$.

On a quasi-Sasakian 3-manifold, we have

$$\nabla_X \xi = -\alpha\varphi X.$$

Note that on a quasi-Sasakian manifold of arbitrary odd dimension, $\xi$ is a Killing vector field, especially, $\nabla_\xi \xi = 0$.

Olszak studied quasi-Sasakian 3-manifolds and obtained the following fundamental facts.

**Proposition 1.2** ([17, 23])**.** *Let $M$ be a quasi-Sasakian 3-manifold satisfying $\nabla\xi = -\alpha\varphi$ and $\sigma$ a positive function on $M$ satisfying $d\sigma(\xi) = 0$. Then $M$ equipped with a new structure $(\varphi^\sigma, \xi^\sigma, \eta^\sigma, g^\sigma)$ defined by*

$$\varphi^\sigma := \varphi,\ \xi^\sigma := \varepsilon\xi,\ \eta^\sigma = \varepsilon\eta,\ g^\sigma := \sigma g + (1-\sigma)\eta \otimes \eta, \quad \varepsilon = \pm 1,$$

*is a quasi-Sasakian 3-manifold. The Levi-Civita connection $\nabla^\sigma$ of $g^\sigma$ satisfies*

$$\nabla^\sigma \xi^\sigma = -\alpha^\sigma \varphi^\sigma,$$

*with $\alpha^\sigma = \varepsilon\alpha/\sigma$. In particular, when $M$ is Sasakian, the new quasi-Sasakian structure is always of rank 3.*

**Proposition 1.3.** ([17]) *Let $M$ be a quasi-Sasakian 3-manifold of rank 3 which satisfies $\nabla\xi = -\alpha\varphi$. Assume that $\alpha$ has constant sign $\varepsilon = \pm 1$. Then the new structure $(\widetilde{\varphi}, \widetilde{\xi}, \widetilde{\eta}, \widetilde{g})$ defined by*

$$\widetilde{\varphi} := \varphi,\ \widetilde{\xi} := \varepsilon\xi,\ \widetilde{\eta} := \varepsilon\eta,\ \widetilde{g} := \sigma g + (1-\sigma)\eta \otimes \eta, \quad \sigma := \varepsilon\alpha > 0,$$

*is Sasakian.*

Therefore, every quasi-Sasakian 3-manifold of rank 3 is locally pseudo-conformal to a Sasakian 3-manifold.

## 2. Slant curves

**2.1.** Let $\gamma : I \to (M^3, g)$ be a Frenet curve parametrized by arc-length in a Riemannian 3-manifold $M^3$ with Frenet frame field $(T, N, B)$. Here $T$, $N$ and $B$ are the tangent, principal normal and binormal vector fields, respectively. Denote by $\nabla$ the Levi-Civita connection of $(M^3, g)$. Then the Frenet frame satisfies the following *Frenet-Serret* equations:

$$\nabla_T T = \kappa N, \quad \nabla_T N = -\kappa T + \tau B, \quad \nabla_T B = -\tau N,$$

where $\kappa = |\nabla_T T|$ and $\tau$ are the *curvature* and *torsion* of $\gamma$, respectively.

**2.2.** Let $M = (M, \varphi, \xi, \eta, g)$ be an almost contact metric 3-manifold and $\gamma(s)$ a smooth curve in $M$ parametrized by arclength. The *contact angle* of $\gamma$ is defined as the angle $\theta(s) \in [0, \pi]$ made by $\gamma$ with the trajectories of $\xi$, that is we have

$$\cos\theta(s) = g(\gamma'(s), \xi).$$

The curve $\gamma(s)$ in $M$ is said to be a *slant curve* if the contact angle $\theta$ is constant. Slant curves of contact angle $\pi/2$ are called (almost) *Legendre curves* or *almost contact curves*.

Now let $M$ be a quasi-Sasakian 3-manifold. Then we have

**Proposition 2.1.** *A non-geodesic Frenet curve $\gamma$ is a slant curve on a quasi-Sasakian 3-manifold if and only if $\gamma$ satisfies*

$$\eta(N) = 0,$$

*where $N$ is the principal normal to $\gamma$.*

*Proof.* Direct computations lead to

$$\frac{d}{ds}\eta(\gamma') = g(\nabla_{\gamma'}\gamma', \xi) + g(\gamma', \nabla_{\gamma'}\xi) = g(\kappa N, \xi) + g(\gamma', \alpha\varphi\gamma') = \kappa\eta(N).$$

This shows the required result. □

We suppose that $\gamma$ is non-geodesic; then $\gamma$ can not be an integral curve of $\xi$. Using (1.1) we find an orthonormal frame field on the normal almost contact metric 3-manifold $M$ along $\gamma$

$$e_1 = T = \gamma', \quad e_2 = \frac{\varphi\gamma'}{\sin\theta}, \quad e_3 = \frac{\xi - \cos\theta\gamma'}{\sin\theta}.$$

Hence, the characteristic vector field $\xi$ decomposes as $\xi = \cos\theta e_1 + \sin\theta e_3$.

Then for a slant curve $\gamma$ in a quasi-Sasakian 3-manifold $M$ we have

$$\begin{cases} \nabla_{\gamma'} e_1 = \delta \sin\theta \, e_2, \\ \nabla_{\gamma'} e_2 = -\,\delta\sin\theta \, e_1 + (\alpha + \delta\cos\theta)\, e_3, \\ \nabla_{\gamma'} e_3 = -\,(\alpha + \delta\cos\theta)\, e_2, \end{cases}$$

where $\delta = g(\nabla_{\gamma'}\gamma', \varphi\gamma')/\sin^2\theta$. Moreover, we also deduce that

$$\nabla_{\gamma'}\xi = -\alpha\sin\theta\, e_2, \quad \kappa = |\delta|\sin\theta, \quad \tau = \alpha + \delta\cos\theta.$$

For more information on slant curves, we refer to [9].

## 3. Magnetic curves

**3.1.** Let $(M, g)$ be a Riemannian manifold equipped with a *closed* 2-form $F$. The 2-form $F$ is referred as a *magnetic field* on $M$. The *Lorentz force* is an endomorphism field $\phi$ on $M$ associated to $F$ via the metric $g$, *i.e.*,

$$g(\phi X, Y) = F(X, Y),$$

for all $X$ and $Y \in \mathfrak{X}(M)$. A regular curve $\gamma$ is said to be a *magnetic curve* with respect to $F$ if it satisfies the Lorentz equation (also called the Newton equation):

$$\nabla_{\gamma'}\gamma' = q\phi\gamma'.$$

**3.2.** Now we consider an almost contact metric manifold $M$ with *closed* fundamental 2-form. Then we can consider magnetic curves with respect to the magnetic field $-\Phi$. This magnetic field $-\Phi$ is referred as to the *contact magnetic field.* The associated endomorphism field $\phi$ is $\varphi$.

As we have mentioned in Introduction, 3-dimensional electromagnetic theory on manifolds is rather special. On every oriented Riemannian 3-manifold $(M, g)$ with volume form $dv_g$, the space of all closed 2-forms is identified with the space of divergence free vector fields, via the Hodge star operator. Let us denote by $V$ the divergence free vector field corresponding to a magnetic field $F$ on $M$. Assume that $V$ is unitary; we define a quadruple of tensor fields $(\varphi, \xi, \eta, g)$ by $\varphi = \phi$, $\xi = V$, $\eta = g(V, \cdot)$; then one can see that $(\varphi, \xi, \eta, g)$ is an almost contact metric structure whose fundamental 2-form is closed. The contact magnetic field is the original magnetic field $F$. This fact means that in the 3-dimensional electromagnetic theory on manifolds, the contact magnetic fields constitute a nice class of magnetic fields. Typical examples of almost contact metric 3-manifolds with closed fundamental 2-form are quasi-Sasakian 3-manifolds.

**3.3.** In the following we investigate contact magnetic curves on quasi-Sasakian 3-dimensional manifolds.

Let $\gamma$ be a normal magnetic trajectory in a quasi-Sasakian 3-manifold $M$ with respect to the Lorentz force $q\varphi$. Namely, $\gamma$ is parametrized by the arclength and it satisfies

$$\nabla_{\gamma'}\gamma' = q\ \varphi\gamma'. \tag{3.1}$$

The first fundamental result is the following one.

**Proposition 3.1.** *Every normal contact magnetic curve on a quasi-Sasakian 3-manifold is a slant curve.*

*Proof.* The contact angle $\theta$ is constant along $\gamma$. In fact,

$$\begin{aligned}\frac{d}{ds}\cos\theta = \frac{d}{ds}g(\gamma', \xi) &= g(\nabla_{\gamma'}\gamma', \xi) + g(\gamma', \nabla_{\gamma'}\xi)\\ &= g(q\varphi\gamma', \xi) + g(\gamma', -\alpha\varphi\gamma') = 0.\end{aligned}$$

□

**Remark 3.1.** The previous result can be proved in a general setting; we just need $\xi$ to be a Killing vector field.

*Proof.* Because of (3.1), then along $\gamma$ the following holds:

$$\frac{d}{ds}g(\gamma',\xi)=g(\nabla_{\gamma'}\gamma',\xi)+g(\gamma',\nabla_{\gamma'}\xi)=qg(\varphi\gamma',\xi)=0.$$

It follows that $\gamma$ is a slant curve. □

**3.4.** On an arbitrary oriented Riemannian 3-manifold one can canonically define a *cross product* $\times$ of two vector fields $X,Y\in\mathfrak{X}(M)$ as follows:

$$g(X\times Y,Z)=dv_g(X,Y,Z),\ \text{for any } Z\in\mathfrak{X}(M),$$

where $dv_g$ denotes the volume form defined by $g$. When $M$ is an almost contact metric 3-manifold, the cross product is given by the formula

$$X\times Y=g(\varphi X,Y)\xi-\eta(Y)\varphi X+\eta(X)\varphi Y.$$

Note that for a unitary vector field $X$ orthogonal to $\xi$, the basis $\{X,\varphi X,\xi\}$ is considered to be positively oriented. Then we have

$$\xi\times\gamma'=\varphi\gamma'.$$

Take the Frenet frame field $(T,N,B)$ along $\gamma$. By definition $T=\gamma'$. Hence, the magnetic equation is written as

$$\nabla_{\gamma'}\gamma'=q\xi\times\gamma'=\kappa N. \tag{3.2}$$

Consequently, we get

$$\kappa^2=q^2g(\xi\times\gamma',\xi\times\gamma')=q^2\big[g(\xi,\xi)g(\gamma',\gamma')-g(\gamma',\xi)^2\big]=q^2\sin^2\theta.$$

Thus $\gamma$ has constant curvature $\kappa=|q|\sin\theta$. Assume that $\gamma$ is a non-geodesic normal magnetic curve; then from (3.2) we have

$$N=\frac{q}{\kappa}\varphi\gamma'. \tag{3.3}$$

Next, the binormal vector field $B$ is obtained from the formula

$$B=\gamma'\times N=\gamma'\times\left\{\frac{q}{\kappa}(\xi\times\gamma')\right\}=\frac{q}{\kappa}(\xi-\cos\theta\gamma'). \tag{3.4}$$

The covariant derivative of the binormal may be computed as

$$\nabla_{\gamma'}B=\frac{q}{\kappa}\nabla_{\gamma'}(\xi-\cos\theta\gamma')=-\frac{q}{\kappa}(\alpha+q\cos\theta)\varphi\gamma'.$$

Comparing this with

$$\nabla_{\gamma'}B=-\tau N=-\frac{\tau q}{\kappa}\varphi\gamma',$$

we obtain the expression of the torsion of $\gamma$, that is

$$\tau=\alpha+q\cos\theta.$$

**3.5.** Recall that a magnetic curve in a Sasakian (respectively a cosymplectic) manifold is a helix. Unlike these situations, a magnetic curve on a quasi-Sasakian 3-manifold is *not*, in general, a helix.

In order to sustain this remark, let us consider the following quasi-Sasakian 3-manifold introduced by Wełyczko in [24]. Let $M = \{(x, y, z) \in \mathbb{R}^3 \mid x > 0\}$ be a half space. We equip $M$ with a Riemannian metric $g$ defined by

$$g = x^2(dx^2 + dy^2) + \eta \otimes \eta, \text{ where } \eta = dz + 2xdy.$$

Then we can take a global orthonormal frame field

$$e_1 = \frac{1}{x}\frac{\partial}{\partial x}, \ e_2 = \frac{1}{x}\frac{\partial}{\partial y} - 2\frac{\partial}{\partial z}, \ e_3 = \frac{\partial}{\partial z}.$$

The Lie brackets satisfy

$$[e_1, e_2] = -\frac{1}{x^2}e_2 - \frac{2}{x^2}e_3, \ [e_2, e_3] = [e_3, e_1] = 0.$$

Define an endomorphism field $\varphi$ by

$$\varphi e_1 = e_2, \ \varphi e_2 = -e_1, \ \varphi e_3 = 0$$

and put $\xi = e_3$. Then $(\varphi, \xi, \eta, g)$ is an almost contact metric structure on $M$. One can check that the following relation holds on $(M, \varphi, \xi, \eta, g)$:

$$\nabla_X \xi = \frac{1}{x^2}\ \varphi X,$$

where $\nabla$ is the Levi-Civita connection of $g$. Thus, $M$ is a quasi-Sasakian manifold with $\alpha = -1/x^2 < 0$. The eigenvalues of the Ricci operator are

$$-\frac{1}{x^4}, \ -\frac{2}{x^4}, \ \frac{3}{x^4}.$$

Hence, $M$ is scalar flat.

Let us study the magnetic curves in Wełyczko's space. The magnetic equation (3.1) of Wełyczko's space is a system of three second order differential equations, that is

$$\begin{cases} \dfrac{\dot{x}^2}{x} - \dfrac{5\dot{y}^2}{x} - \dfrac{2\dot{y}\dot{z}}{x^2} + \ddot{x} & = \quad -q\dot{y}, \\ \dfrac{6\dot{x}\dot{y}}{x} + \dfrac{2\dot{x}\dot{z}}{x^2} + \ddot{y} & = \quad q\dot{x}, \\ -10\dot{x}\dot{y} - \dfrac{4\dot{x}\dot{z}}{x} + \ddot{x} & = \quad -2qx\dot{x}. \end{cases} \tag{3.5}$$

We denoted by dot $(\cdot)$ the derivative with respect to the arclength parameter $s$.

From Proposition 3.1 we know that $\eta(\dot{\gamma}) = \cos\theta$, where $\theta$ is the constant contact angle. Hence, we have

$$\dot{z} + 2x\dot{y} = \cos\theta. \tag{3.6}$$

As the curve $\gamma$ is parametrized by arclength, we also have

$$x^2(\dot{x}^2 + \dot{y}^2) = \sin^2\theta.$$

Therefore, there exists a (smooth) function $u$ (depending on $s$) such that

$$\begin{cases} \dot{x} = \frac{1}{x}\sin\theta\cos u(s), \\ \dot{y} = \frac{1}{x}\sin\theta\sin u(s). \end{cases}$$

Hence, when $u$ is known, the $x$-coordinate may be found from the equation

$$x(s)^2 = c_0 + 2\sin\theta\int_0^s \cos u(t)dt, \tag{3.7}$$

where $c_0$ is a positive constant. Then, from (3.6), we get

$$z(s) = z_0 + s\cos\theta - 2\sin\theta\int_0^s \sin u(t)dt,\ \ z_0\in\mathbb{R}.$$

Finally, we compute $y$:

$$y(s) = y_0 + \sin\theta\int_0^s \frac{\sin u(t)}{x(t)}dt,\ \ y_0\in\mathbb{R}.$$

The key point is to obtain $u$.

From (3.5), when $\sin\theta\neq 0$, we have

$$\sin u(s)\big[2\cos\theta + \sin\theta\sin u(s) + x^2(s)\big(-q+\dot{u}(s)\big)\big] = 0,$$
$$\cos u(s)\big[2\cos\theta + \sin\theta\sin u(s) + x^2(s)\big(-q+\dot{u}(s)\big)\big] = 0.$$

Combining with (3.7) we deduce that $u$ is a solution of the following integro-differential equation:

$$2\cos\theta + \sin\theta\sin u(s) + \big(-q+\dot{u}(s)\big)\Big[c_0 + 2\sin\theta\int_0^s\cos u(t)dt\Big] = 0.$$

Thus, in general, normal magnetic curves in Wełyczko space are not helices. In fact, the torsion $\tau = -1/x(s)^2 + q\cos\theta$ is non-constant.

In the sequel we investigate a particular example $u = u_0$ (constant).

On one hand we have

$$x(s)^2 = c_0 + 2s\sin\theta\cos u_0.$$

On the other hand, the following equation must be satisfied:

$$2\cos\theta + \sin\theta\sin u_0 - q(c_0 + 2s\sin\theta\cos u_0) = 0,\ \text{for all } s.$$

As $q\sin\theta\neq 0$, we should have $\cos u_0 = 0$ and therefore

$$2\cos\theta + \epsilon\sin\theta - qc_0 = 0,\quad \epsilon = \pm 1.$$

This implies $x(s) = \sqrt{c_0}$, $y(s) = y_0 + \epsilon\frac{\sin\theta}{\sqrt{c_0}}\ s$, $z(s) = z_0 + (\cos\theta - 2\epsilon\sin\theta)s$. Thus, along this magnetic curve, $\alpha$ is constant. Hence, this magnetic curve is a helix.

For $\theta = \frac{\pi}{2}$, that is $\gamma$ is a Legendre magnetic curve, and for $\epsilon = 1$, we obtain

$$\gamma(s) = \left(\sqrt{c_0}, y_0 + \frac{s}{\sqrt{c_0}}, z_0 - 2s\right).$$

Its strength is $q = \frac{1}{c_0}$. This magnetic curve is a helix with $\kappa = \tau = 1/c_0 > 0$.

For more examples of non-Sasakian quasi-Sasakian 3-manifolds, see [19]. In addition, in [19], Olszak constructed explicit examples of conformally flat quasi-Sasakian 3-manifolds.

## 4. Pseudo-conformal deformations

Let $M = (M, \varphi, \xi, \eta, g)$ be a quasi-Sasakian 3-manifold. Consider the pseudo-conformally deformed structure

$$\tilde{\varphi} := \varphi,\ \tilde{\xi} := \varepsilon\, \xi,\ \tilde{\eta} := \varepsilon\eta,\ \tilde{g} = \sigma g + (1-\sigma)\eta \otimes \eta,\ \varepsilon = \pm 1,$$

where $\sigma$ is a positive smooth function satisfying $d\sigma(\xi) = 0$. The resulting almost contact metric 3-manifold is still a quasi-Sasakian 3-manifold.

Take an arclength parametrized curve $\gamma(s)$ in $(M, g)$. Then its velocity vector field $\gamma'(s)$ satisfies

$$\tilde{g}(\gamma'(s), \gamma'(s)) = \cos^2\theta(s) + \sigma(\gamma(s))\sin^2\theta(s),$$

where $\theta$ is the contact angle, that is $\cos\theta = \eta(\gamma'(s))$. Obviously, the property "arclength parametrized" is not preserved under the pseudo-conformal deformation, while the property to be "Legendre" is.

In the following we study the behavior of magnetic curves under pseudo-conformal deformation.

The Levi-Civita connection $\widetilde{\nabla}$ of $\tilde{g}$ is related to the Levi-Civita connection $\nabla$ of $g$ by

$$\begin{aligned}\widetilde{\nabla}_X Y = \nabla_X Y &+ \frac{1}{2\sigma}\big\{d\sigma(X)(Y - \eta(Y)\xi) + d\sigma(Y)(X - \eta(X)\xi)\big\} \\ &- \frac{1}{2\sigma}\big\{g(X,Y) - \eta(X)\eta(Y)\big\}\operatorname{grad}\sigma \\ &- \frac{\alpha(1-\sigma)}{\sigma}\left\{\eta(X)\varphi Y + \eta(Y)\varphi X\right\}.\end{aligned} \tag{4.1}$$

**Remark 4.1.** From this relation we get

$$\widetilde{\nabla}_X \xi = -\frac{\alpha}{\sigma}\varphi X.$$

When $\alpha$ has constant sign, if we choose $\sigma = \varepsilon\alpha$, we have $\widetilde{\nabla}_X \tilde{\xi} = -\varphi X$. So, the new structure $(\varphi, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is really Sasakian.

Take a normal contact magnetic curve $\gamma(s)$ in $(M, g)$ satisfying $\nabla_{\gamma'}\gamma' = q\varphi\gamma'$. Then we have

$$\widetilde{\nabla}_{\gamma'}\gamma' = q\varphi\gamma' + \frac{\sigma'}{\sigma}\,(\gamma' - \cos\theta\xi) - \frac{\sin^2\theta}{2\sigma}\operatorname{grad}\sigma\big|_{\gamma} - \frac{2\alpha(1-\sigma)}{\sigma}\cos\theta\ \varphi\gamma'. \tag{4.2}$$

Here $\sigma'$ denotes the derivative $\frac{d}{ds}\sigma(\gamma(s))$. Thus, "contact magnetic" is not preserved. Even if every quasi-Sasakian 3-manifold of rank 3 are locally pseudo-conformal to Sasakian 3-manifolds, contact magnetic curves are not invariant under the deformation. Thus, the study of contact magnetic curves in quasi-Sasakian 3-manifolds does *not* reduce to that of Sasakian 3-manifolds. In other words, study of contact magnetic curves in quasi-Sasakian 3-manifolds has its own interest.

Assume that $\gamma$ is *non-geodesic*, *i.e.*, $\kappa \neq 0$ and $q \neq 0$, then from (3.3) and (3.4), the unit normal $N$ and the binormal $B$ are related to $\varphi\gamma'$ and $\xi$ by

$$\varphi\gamma' = \frac{\kappa}{q}N, \quad \xi = \frac{\kappa}{q}B + \cos\theta T.$$

We rewrite the formula (4.2) as

$$\widetilde{\nabla}_{\gamma'}\gamma' = \frac{\sigma' \sin^2\theta}{\sigma}\ T + \left\{ q - \frac{2\alpha(1-\sigma)\ \cos\theta}{\sigma} \right\} \varphi\gamma' - \frac{\kappa\sigma'\cos\theta}{q\sigma}\ B - \frac{\sin^2\theta}{2\sigma}\ \mathrm{grad}\ \sigma\big|_{\gamma}.$$

Next we have

$$g(\mathrm{grad}\,\sigma, T) = \sigma',$$
$$\kappa g(\mathrm{grad}\,\sigma, B) = q g(\mathrm{grad}\,\sigma, \xi - \cos\theta\gamma') = -q\cos\theta\sigma'.$$

Hence, if both the $\sigma$ and $\alpha$ are *constant along* $\gamma$, then $\gamma$ is also magnetic with respect to the new metric $\tilde{g}$. For a Legendre magnetic curve $\gamma(s)$, we have

$$\widetilde{\nabla}_{\gamma'}\gamma' = q\varphi\gamma' + \frac{\sigma'}{\sigma}\gamma' - \frac{1}{2\sigma}\ \mathrm{grad}\,\sigma\big|_{\gamma}.$$

Thus, under the assumption "$\sigma' = \alpha' = 0$", $\gamma$ is also a Legendre magnetic curve with the same strength with respect to $\tilde{g}$.

For example, let us consider the pseudo-conformal deformation of Wełyczko's space with $\sigma = 1/x^2$ and $\varepsilon = -1$. The resulting Sasakian manifold is the Sasakian space form $\mathbb{R}^3(-3)$ with the metric

$$dx^2 + dy^2 + (dz + 2xdy)^2.$$

Thus the Legendre magnetic helix in Wełyczko's space corresponds to Legendre magnetic helix in the Sasakian space form $\mathbb{R}^3(-3)$ under this pseudo-conformal deformation. For Legendre magnetic curves in $\mathbb{R}^3(-3)$, we refer to [4].

In fact, we have obtained a more general result:

**Proposition 4.1.** *Let $M$ be an almost contact metric manifold satisfying $\nabla\xi = -\alpha\varphi$ for some function $\alpha$ with $\xi(\alpha) = 0$. Take a positive smooth function $a$ and smooth function $b$ such that $\tilde{g} = ag + b\eta\otimes\eta$ is a Riemannian metric. Assume that $da(\xi) = db(\xi) = 0$ and $a+b$ is a non-zero constant. Then the Levi-Civita connection $\widetilde{\nabla}$ of $\tilde{g}$ is*

$$\begin{aligned}\widetilde{\nabla}_X Y = \nabla_X Y &+ \frac{1}{2a}\left\{da(X)Y + db(Y)X) - g(X,Y)\mathrm{grad}\, a\right\} \\ &+ \frac{1}{2a}\left\{(db(X)\eta(Y) + db(Y)\eta(X))\xi - \eta(X)\eta(Y)\mathrm{grad}\, b\right\} \\ &- \frac{\alpha b}{a}\left\{\eta(X)\varphi Y + \eta(Y)\varphi X\right\}.\end{aligned}$$

In particular, if we assume that $a$ is a constant and choosing $b = a(a-1)$, then the $\tilde{g}$ is a $D$-homothetic deformation of $g$. The Levi-Civita connection is

$$\widetilde{\nabla}_X Y = \nabla_X Y + (1-a)\alpha\left\{\eta(X)\varphi Y + \eta(Y)\varphi X\right\}.$$

Next, if we assume that $a = \sigma > 0$ is a positive smooth function and setting $b = 1 - \sigma$, then we obtain (4.1).

## 5. Magnetic curves and Okumura type connection

**5.1.** In his paper [16], Okumura defined a class of linear connections on a Sasakian manifold $(M, \varphi, \xi, \eta, g)$ in such a way that $g$, $\eta$, $\xi$ and $\varphi$ are covariant constant. Such a connection was called a $(\varphi, \eta, g)$ *connection* in [16]. Using the same idea, we can define a $(\varphi, \eta, g)$ connection on a quasi-Sasakian 3-manifold $(M, \varphi, \xi, \eta, g)$.

Let us consider the following $(1, 2)$-type tensor field $P$ on $M$:

$$P(X, Y) = a\,\eta(X)\varphi Y + b\,\eta(Y)\varphi X + c\,\Phi(X, Y)\xi,$$

where $a$, $b$ and $c$ are smooth functions on $M$ which shall be determined such that the connection defined by

$$\overline{\nabla}_X Y = \nabla_X Y + P(X, Y), \tag{5.1}$$

is a $(\varphi, \eta, g)$ connection.

Equation (5.1) yields

$$\begin{cases} (\overline{\nabla}_X g)(Y, Z) = (c - b)\big[\eta(Y)\Phi(X, Z) + \eta(Z)\Phi(X, Y)\big], \\ \overline{\nabla}_X \xi = \nabla_X \xi + b\varphi X, \end{cases}$$

for every $X$, $Y$ tangent to $M$. From the assumption that $\xi$ and $g$ are parallel with respect to the connection $\overline{\nabla}$, we immediately obtain

$$b = c = \alpha.$$

It can be easily proved that $\overline{\nabla}\eta = 0$ and $\overline{\nabla}\varphi = 0$.

We can state the following result:

**Theorem 5.1.** *Let $(M, \varphi, \xi, \eta, g)$ be a quasi-Sasakian 3-manifold. For any function $a$ on $M$, define a linear connection $\overline{\nabla}$ by setting*

$$\overline{\nabla}_X Y = \nabla_X Y + a\,\eta(X)\varphi Y + \alpha\big(\eta(Y)\varphi X + \Phi(X, Y)\xi\big), \tag{5.2}$$

*for all $X$, $Y$ tangent to $M$. Then the tensor fields $g$, $\eta$, $\xi$ and $\varphi$ are parallel with respect to $\overline{\nabla}$.*

We will call this connection the *Okumura type connection*.

**Remark 5.1.** In fact, this connection exists when $M$ belongs to the wider class of almost contact metric manifolds. See [6, Theorem 8.2].

**5.2.** Inspired by the Ikawa's paper [7] we would like to study the properties of magnetic curves on $M$ with respect to the Okumura type connection $\overline{\nabla}$.

Let $\gamma$ be a normal magnetic curve on $M$ with strength $q$, where $q \in \mathbb{R}$, that is the curve $\gamma$ is parametrized by the arclength and its velocity fulfills the Lorentz equation

$$\nabla_{\gamma'}\gamma' = q\varphi\gamma'.$$

As we have already proved, every magnetic curve in a quasi-Sasakian 3-manifold is slant, that is the contact angle $\theta$ is constant.

Equation (5.2) leads to

$$\overline{\nabla}_X X = \nabla_X X + (a + \alpha)\eta(X)\varphi X,$$

for any $X$ tangent to $M$. If we consider $X = \gamma'$ and the covariant derivative along $\gamma$ induced by $\overline{\nabla}$ we obtain

$$\overline{\nabla}_{\gamma'}\gamma' = \big[q + (a + \alpha)\cos\theta\big]\varphi\gamma'.$$

Hence we obtain the following.

**Proposition 5.1.** **(i)** *Any Legendre magnetic curve satisfies the analogue of the Lorentz equation when Okumura type connection is considered with arbitrary function $\alpha$.*

**(ii)** *Any magnetic curve $\gamma$ satisfies the analogue of the Lorentz equation for the Okumura type connection*

$$\bar{\bar{\bar{\nabla}}}_X Y = \nabla_X Y + \alpha\big(\eta(Y)\varphi X - \eta(X)\varphi Y + \Phi(X,Y)\xi\big).$$

*In particular, any geodesic of $M$ is also geodesic for the Okumura type connection $\bar{\bar{\bar{\nabla}}}$.*

Note that when $M$ is cosymplectic, the connection $\bar{\bar{\bar{\nabla}}}$ is nothing but the Levi-Civita connection.

We obtain the following reinterpretation of non-Legendrian contact magnetic curves in quasi-Sasakian 3-manifolds with constant $\alpha$. Note that under this assumption, $M$ is $\alpha$-Sasakian or cosymplectic.

**Theorem 5.2.** *Let $M$ be a quasi-Sasakian manifold whose structure function $\alpha$ is constant and $\gamma$ a non-Legendre magnetic curve with strength $q$. Then $\gamma$ is a geodesic for the Okumura type connection*

$$\dot{\nabla}_X Y = \bar{\bar{\bar{\nabla}}}_X Y - \frac{q}{\cos\theta}\,\eta(X)\varphi Y.$$

**Remark 5.2.** Conformally flat quasi-Sasakian 3-manifolds with constant $\alpha$ are classified by Olszak [19, Theorem 3.6] as follows:

**Theorem.** *Let $M$ be a quasi-Sasakian 3-manifold. Then the following four statements are equivalent:*

(1) *$M$ is locally symmetric*;
(2) *$M$ is conformally flat and its scalar curvature is constant*;
(3) *$M$ is conformally flat and its structure function $\alpha$ is constant*;
(4) (a) *$M$ is a cosymplectic manifold which is locally a product of the real line $\mathbb{R}$ and a two dimensional Kähler space of constant Gaussian curvature, or*
    (b) *$M$ is of constant positive curvature and its structure can be obtained by a homothetic deformation of a Sasakian structure. In this case $M$ is $\alpha$-Sasakian.*

Let us observe that when $M$ is cosymplectic, non-Legendre magnetic curves are geodesic for the Okumura type connection

$$\dot{\nabla}_X Y = \nabla_X Y - \frac{q}{\cos\theta}\,\eta(X)\varphi Y.$$

In order to prove a converse of Theorem 5.2, let us give the following useful assertion.

**Lemma 5.1.** *Let $\gamma$ be a unit speed curve in a quasi-Sasakian 3-manifold which is a geodesic for the Okumura type connection (5.2). Then $\gamma$ is a slant curve, that is the angle $\theta$ between its velocity vector and the characteristic vector field $\xi$ is constant.*

*Proof.* Since $\gamma$ is unit speed, the contact angle $\theta$ is defined by $\cos\theta = \eta(\gamma')$. We have

$$\frac{d}{ds}\,\eta(\gamma') = \frac{d}{ds}\,g(\gamma',\xi) = g(\overline{\nabla}_{\gamma'}\gamma',\xi) + g(\gamma',\overline{\nabla}_{\gamma'}\xi) = 0.$$

Hence $\theta$ is constant. ∎

Let $\gamma$ be a unit speed curve in a quasi-Sasakian 3-manifold which is a geodesic for the Okumura type connection (5.2). Then, $\gamma$ can be interpreted as a magnetic curve with non-constant strength

$$q(s) = -\big[a(\gamma(s)) + \alpha(\gamma(s))\big]\cos\theta.$$

**Remark 5.3.** The magnetic curves with variable strength generate, in the Euclidean plane $\mathbb{E}^2$, beautiful aesthetic curves. See e.g. [27]. Applications of magnetic curves in $\mathbb{E}^3$ with variable strength to CAD systems are described in [26].

We close this Section with the following result.

**Theorem 5.3.** *Let $\gamma$ be a unit speed curve in a quasi-Sasakian 3-manifold whose structure function $\alpha$ is constant. Suppose that $\gamma$ is a geodesic for the Okumura type connection* (5.2) *with $a \in \mathbb{R}$. Then $\gamma$ is a magnetic curve with strength $q = -(a+\alpha)\cos\theta$.*

**Remark 5.4.** On almost contact metric manifolds of arbitrary dimension, the following one-parameter family $\{\widetilde{\nabla}^t\}_{t\in\mathbb{R}}$ of linear connections was introduced in [8]:

$$\widetilde{\nabla}^t_X Y = \nabla_X Y - \frac{1}{2}\varphi(\nabla_X\varphi)Y - \frac{1}{2}\eta(Y)\nabla_X\xi - t\eta(X)\varphi Y + (\nabla_X\eta)(Y)\xi,$$

for all vector fields $X$ and $Y$. Here $t$ is a real constant. The connection $\widetilde{\nabla}^0$ is the $(\varphi,\xi,\eta)$-connection introduced by Sasaki and Hatakeyama in [20]. More than 30 years later, Cho defined and studied [3] the connection $\widetilde{\nabla}^1$. Note that on a Sasakian manifold of dimension 3, the connection $\widetilde{\nabla}^t$ coincides with the linear connection introduced by Okumura. In particular, $\widetilde{\nabla}^1$ is called the *Okumura connection*. On contact metric 3-manifolds, $\widetilde{\nabla}^{-1}$ coincides with the *Tanaka-Webster connection* [21, 25]. All the structure tensor fields $\varphi$, $\xi$, $\eta$ and $g$ are parallel with respect to $\widetilde{\nabla}^t$.

On a normal almost contact metric 3-manifold $M$, $\widetilde{\nabla}^t_X Y$ is given by

$$\widetilde{\nabla}^t_X Y = \nabla_X Y + \alpha\{g(X,\varphi Y)\xi + \eta(Y)\varphi X\} + \beta\{g(X,Y)\xi - \eta(Y)X\} - t\eta(X)\varphi Y.$$

Thus, for a quasi-Sasakian 3-manifold the connection $\widetilde{\nabla}^t$ coincides with the Okumura type connection $\overline{\nabla}$ defined by (5.2) with $a = -t$.

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