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Generalised discretisation of continuous-time distributions

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Abstract: In this study, the definition of discretisation that was proposed recently for continuous-time distributions is made applicable not only to ordinary functions but to a variety of distributions including weak derivatives such that they could be viewed from a unified perspective under useful theorems. While it is not absolutely necessary to introduce distributions for discrete-time signals having finite values, it turns out that it is insightful to introduce discrete-time equivalents in appreciating their richness, which culminates into continuous-time distributions as the sampling-interval approaches zero. For instance, a discretisation of a derivative of a distribution can be found as a discrete derivative of a discretisation of a distribution. This is much easier than the traditional approach, where an ordinary function must first be found to approximate the derivative of a distribution. Simulations show that, by changing a single parameter of the proposed model, different types of signals that are similar to traditional ones developed separately by approximating distributions by ordinary functions, such as Dirichlet' kernel, Gaussian distribution and sinc approximation, can be obtained.

1 Introduction

Discretisation of continuous-time functions is an extremely important process for analysing, designing and controlling various phenomena and systems using digital devices. However, a definition of discretisation that was applicable to all existing classes of discrete-time signals did not appear until relatively recently [1], which also pointed out that the impulse-invariant model was not a proper model as known at that time and suggested a simple scaling adjustment to make it valid. To establish a solid ground on discretisation, a few definitions have been proposed and some useful theorems derived, where similarities, rather than differences, among the existing discrete-time models were highlighted and creation of new models were demonstrated [2]. While useful, the definition was applicable only to ordinary functions, and generalised functions, or distributions, were left uncovered. This was not a serious problem for systems expressed in transfer functions, where impulses could be handled in the Laplace transform domain, circumventing the issue of infinite magnitudes to some extent. However, when such signals are needed to be treated in the discrete-time domain, such as in numerical investigations, infinity must be dealt with somehow in a digital processor based only on finite values.

An area where handling of distribution is crucial is the descriptor system, where the initial condition must be chosen precisely [3] to reflect the response of the continuous-time original in the discrete-time analysis. This is because impulsive responses can occur or disappear in the discrete-time calculation irrespectively of the continuous-time behaviour, depending on the initial condition and the input used. To derive a necessary and sufficient condition on the initial state of the descriptor system, a proper definition of distribution was necessary, and this has been achieved in [3]. In the present paper, the discretisation concept is made more general so that it can be applied to weak derivatives of distributions. Related theorems are presented and applied to Dirac's delta and its derivatives. Simulation studies are carried out to show that a variety of signals can be obtained using the proposed method by changing a single parameter. They include traditional methods using Gaussian distribution [4, 5], Dirichlet's kernel [4, 5] and sinc-function [5], which convert a continuous-time distribution into ordinary functions.

The paper is organised as follows. After the introduction section, Section 2 briefs on fundamental definitions. Section 3 introduces the term of discrete-time generalised function, which is not really a distribution, but highly useful as shown in [3] for handling impulsive responses. This is then extended to discretisation of weak derivatives and several useful theorems are derived. In Section 4, the results of Section 3 are applied to the discretisation of Dirac's delta function and its derivatives are presented. Simulation results are presented in Section 5 and conclusions are drawn in Section 6.

2 Preliminaries

The following conventions of symbols are used: $f(k,T) = f_k$ denotes a discrete-time signal, where $k \in \mathbb{Z}$ indicates the step number and $T \in \mathbb{R}^+$ is the sampling period, whereas an upper bar as in $\bar{f}(t) = \bar{f}$ denotes a continuous-time signal, where $t \in \mathbb{R}$ is an independent variable of time. Similarly, an upper bar denotes coefficients and functions related to continuous-time signals and systems. Furthermore, boldface letters denote vectors, such as $\boldsymbol{f}_k$ and $\bar{\boldsymbol{f}}$.

2.1 Discretisation

The conventional definition of signal discretisation is as follows [1].

Definition 1: (Signal discretisation). A discrete-time signal $\boldsymbol{f}(k,T)$ is said to be a discretisation of a continuous-time signal $\bar{\boldsymbol{f}}(t)$ if the following condition is satisfied: for any fixed time τ and $\kappa \in \mathbb{Z}$ that satisfy $\kappa T \le \tau < (\kappa+1)T$, the following holds:

$$\lim_{T \to 0} \| \bar{\boldsymbol{f}}(\tau) - \boldsymbol{f}(\kappa, T) \| = \boldsymbol{0}, \tag{1}$$

where the sampling instant is synchronised at $t = kT = 0$.□

Discretisation of a continuous-time signal $\bar{\boldsymbol{f}}(t)$ is usually considered as a process that involves loss of information between two successive sampling instants, discarding everything between kT and $(k+1)T$. The above definition offers a slightly different view in that a continuous-time instant τ and $\bar{\boldsymbol{f}}(\tau)$ are fixed. If the discrete-time signal $\boldsymbol{f}(\kappa, T)$ at time-instant κT that is closest to τ from left (in the present definition) approaches $\bar{\boldsymbol{f}}(\tau)$ as the sampling period T is reduced, $\boldsymbol{f}(k, T)$ is said to be a discretisation of $\bar{\boldsymbol{f}}(t)$. In this process, the domain of $\boldsymbol{f}(\kappa, T)$ changes, which is unusual. However, by repeating this for all τ in the domain of interests, point-wise convergence can be considered. In this sense, all the time-instants are taken into account, although the convergence is not of the uniform type but only point-wise.

As in the differential operator $\bar{\mathscr{D}} := \mathrm{d}/\mathrm{d}t$ for the continuous-time domain, the following delta operator, $\underline{\delta}$, used for the discrete-time domain in the paper [6, 7].

Definition 2: (Delta operator, $\underline{\delta}$*)*: The discrete-time delta operator, *δ, is defined as*

$$\underline{\delta} := \frac{q-1}{T}, \tag{2}$$

where q is the conventional shift-operator that satisfies $q\boldsymbol{f}(k+1) := \boldsymbol{f}(k+1, T)$.□

The shift operator q is commonly used in discrete-time models, as it makes the mechanism and implementation of algorithms simple to interpret. In turn, it makes the relationship between the continuous-time and discrete-time domains unclear; discrete-time models do not approach continuous-time models even when the sampling period goes to zero. The delta operator is better in this aspect, as well as in numerical properties, compared with the shift operator [6]. Since the relationship between the delta operator $\underline{\delta}$ and the shift operator q is algebraic, their modelling flexibilities are the same.

The following definition of a discrete-time convolution contains the scaling factor T [8].

Definition 3: (Discrete-time convolution): *A discrete-time convoluted* $\boldsymbol{v}(k, T)$ *of* $\boldsymbol{f}(k, T)$ *and* $\boldsymbol{g}(k, T)$ *is defined as*

$$\boldsymbol{v}(k, T) := \sum_{i=-\infty}^{\infty} T\boldsymbol{f}(k-i, T)\boldsymbol{g}(i, T) = \boldsymbol{f}_k * \boldsymbol{g}_k. \tag{3}$$

2.2 Distribution

In the present study, distributions of single-variable and real-valued functions are considered [9, 10].

Definition 4: (Distribution): Let $\mathscr{D}(\mathbb{R})$ *be a space of test-functions defined as*

$$\mathscr{D}(\mathbb{R}) := \bigcup_{a<b} \mathscr{D}[a,\, b], \tag{4}$$

where $\mathscr{D}[a,\, b]$ is defined as

$$\mathscr{D}[a,\, b] := \{\bar{\phi} : \bar{\phi} \in \mathrm{C}^{\infty} \text{ and if } t \notin [a, b], \text{then } \bar{\phi} = 0\}. \tag{5}$$

(The test function $\bar{\phi} \in \mathscr{D}(\mathbb{R})$ is a real-valued function of a real variable which may be continuously differentiable for an infinite number of times and have compact support.) Then a distribution is defined as continuous and linear functional $\bar{T}$ on the space $\mathscr{D}(\mathbb{R})$, such that

$$\bar{T} : \mathscr{D}(\mathbb{R}) \ni \bar{\phi} \mapsto \bar{T}(\bar{\phi}) \in \mathbb{R}. \tag{6}$$

Since distribution $\bar{T}$ is a linear form on $\mathscr{D}(\mathbb{R})$, let the value of $\bar{T}$ on $\bar{\phi} \in \mathscr{D}(\mathbb{R})$, that is $\bar{T}(\bar{\phi})$, be denoted $\langle \bar{T}, \bar{\phi} \rangle$. Introducing a locally integrable function in $(-\infty, \infty)$ as $\bar{f}(t)$, the distribution $\bar{f}$ is defined through the following convergent integral and called regular distribution:

$$\langle \bar{f},\, \bar{\phi} \rangle := \int_{-\infty}^{\infty} \bar{f}(t)\bar{\phi}(t)\mathrm{d}t. \tag{7}$$

A distribution is called singular if no such function exists, in which case, the right-hand side of (7) is only symbolic [11].□

A vector-valued distribution can be considered by applying the above definition to each component of the vector with a common scalar $\bar{\phi}$, as

$$\bar{\boldsymbol{f}} : \mathscr{D}(\mathbb{R}) \ni \bar{\phi} \mapsto \bar{\boldsymbol{f}}(\bar{\phi}) \in \mathbb{R}^p, \tag{8}$$

$$\bar{\boldsymbol{f}}(\bar{\phi}) = \langle \bar{\boldsymbol{f}}, \bar{\phi} \rangle \int_{-\infty}^{\infty} \bar{\boldsymbol{f}}(t)\bar{\phi}(t)\mathrm{d}t, \tag{9}$$

where integration of vector-valued function calculates each elements integration that is right-hand side of (9) is rearranged the following:

$$\int_{-\infty}^{\infty} \bar{\boldsymbol{f}}(t)\bar{\phi}(t)\mathrm{d}t = \begin{bmatrix} \int_{-\infty}^{\infty} \bar{f}_1(t)\bar{\phi}(t)\mathrm{d}t \\ \int_{-\infty}^{\infty} \bar{f}_2(t)\bar{\phi}(t)\mathrm{d}t \\ \vdots \\ \int_{-\infty}^{\infty} \bar{f}_p(t)\bar{\phi}(t)\mathrm{d}t \end{bmatrix}, \tag{10}$$

where $\bar{\boldsymbol{f}}(t) = [\bar{f}_1(t)\ \bar{f}_2(t)\ \ldots \bar{f}_p(t)]^{\mathrm{T}}$.

Since a distribution is a functional, it should be distinguished from a function of time and not be expressed with (t). However, in the paper, such a notation may be used for simplicity when no ambiguity is suspected.

A definition of a distributional differentiation, which is also called a weak derivative, is the following [9].

Definition 5: (Weak derivative: distributional differentiation): For an arbitrary distribution $\bar{T}$, $\bar{\mathscr{D}}\bar{T}$ defines a distributional differentiation by

$$\langle \bar{\mathscr{D}}\bar{T}, \bar{\phi} \rangle := -\langle \bar{T}, \bar{\mathscr{D}}\bar{\phi} \rangle, \tag{11}$$

where $\bar{\mathscr{D}}$ is a weak differentiator.□

Since the weak derivative $\bar{\mathscr{D}}\bar{T}$ is also a distribution, distributions are differentiable for an arbitrary number of times. A definition of a distribution convolution is shown below [9].

Definition 6: (Distribution convolution): The convolution of distribution $\bar{T}$ and $\bar{S}$ *is defined as*

$$\langle \bar{T} * \bar{S}, \bar{\phi} \rangle := \langle \bar{T}_{(\xi)}\bar{S}_{(\eta)}, \bar{\phi}(\xi + \eta) \rangle \tag{12}$$

$$= \langle \bar{T}_{(\xi)}, \langle \bar{S}_{(\eta)}, \bar{\phi}(\xi + \eta) \rangle \rangle, \tag{13}$$

where $\xi := t - \tau$, $\eta := \tau$ and $\bar{T}_{(\xi)}, \bar{S}_{(\eta)}$ are functions of ξ and η.□

The following describes Dirac's delta function and its derivatives used to verify the validity of the proposed definition in the paper at Section 4 and its characteristics.

The definition of Dirac's delta function, $\bar{\delta}$, that is one of the test signals are widely used in control theory for analysis and design systems called impulse signal, is the following [10].

Definition 7: (Dirac's delta function $\bar{\delta}$*)*: *Dirac's delta function,* $\bar{\delta}$*, is defined as*

$$\bar{\delta} : \mathscr{D}(\mathbb{R}) \ni \bar{\phi} \mapsto \langle \bar{\delta}, \bar{\phi} \rangle = \bar{\phi}(0) \in \mathbb{R}. \tag{14}$$

Since Dirac's delta function is a singular distribution, it should not really be denoted as $\bar{\delta}(t)$. However, it is a common practice to express it as $\bar{\delta}(t)$ and may be used in the paper.

Differentiation of Dirac's delta function is derived in the following with distribution differentiation Definition 5 [10].

Theorem 1: (Differentiation of Dirac's delta function $\bar{\delta}^{(i)}$): An i - th derivative of Dirac's delta function, $\bar{\delta}^{(i)}$, is given by

$$\bar{\delta}^{(i)}: \mathscr{D}(\mathbb{R}) \ni \bar{\phi} \mapsto \langle \bar{\delta}^{(i)}, \bar{\phi} \rangle = (-1)^i \bar{\phi}^{(i)}(0) \in \mathbb{R}, \tag{15}$$

where $i \geq 1$ and $\cdot^{(i)}$ is expressed ith derivative.□

Characteristics of Dirac's delta function and its derivatives convolutions are reviewed as follows [10].

Theorem 2: (Identity element of convolution): A convolution of an arbitrary distribution $\bar{T}$ and the Dirac'*s delta function $\bar{\delta}$ is given by*

$$\bar{\delta} * \bar{T} = \bar{T}. \tag{16}$$

Dirac's delta function derivatives $\bar{\delta}(i)$ behave differentiator with convolution, as [10].

Theorem 3: (Differential operator): A convolution of an arbitrary distribution $\bar{T}$ and derivatives of the Dirac'*s delta function $\bar{\delta}(i)$ are given by*

$$\bar{\delta}^{(i)} * \bar{T} = \bar{T}^{(i)}. \tag{17}$$

3 Extended definition and theorem

This section defines discrete-time generalised function and extension of signal discretisation and derives several theorems with the definitions.

Although it may not be necessary to use distributions in the discrete-time domain, as they take finite values only, it was found quite useful to define discrete-time functionals that approach distributions as the sampling period approaches zero. Therefore, the following definition is proposed in the present study as a discrete-time version of the continuous-time distribution defined by (7). This is achieved by considering multiple sampling points, rather than a single point as used previously, in comparing discrete-time and continuous-time signals, as follows [3].

Definition 8: (Discrete-time generalised function): A discrete-time generalised function $f(\varphi)$ is defined as a functional that assigns a value according to

$$f_k: \phi(k, T) \mapsto f(\phi), \tag{18}$$

$$f(\phi) = \langle f_k, \phi_k \rangle := \sum_{k=-\infty}^{\infty} T f(k, T) \phi(k, T), \tag{19}$$

where $\phi(k, T)$ is an arbitrary discrete-time function, which satisfies $\phi(k, T) = 0$ when $k \notin [\alpha, \beta]$, $\alpha, \beta \in \mathbb{Z}$. The inner product defined in (19) is convergent for any finite sequence, since $\phi(k, T)$ has a finite support.□

Equation (19) is forward or backward difference approximation of continuous-time regular distribution (7).

A definition of signal discretisation based on Definition 8 is proposed as follows [3]: it can be considered in the definition that a multi-step discretisation signal such as Dirac's delta function derivatives described in Section 4.

Definition 9: (Extension of signal discretisation [3]): A discrete-time signal $f(k, T)$ is said to be a discretisation of a continuous-time signal $\bar{f}(t)$ if the following condition is satisfied:

$$\lim_{T \to 0} \| \langle \bar{f}, \bar{\phi} \rangle - \langle f_k, \phi_k \rangle \| = 0, \tag{20}$$

where $\phi_k = \bar{\phi}(kT)$.

Since $\phi_k = \bar{\phi}(kT)$, ϕ_k has support such that $\phi_k = 0, k \notin [\alpha, \beta]$ and α, β are satisfied $(\alpha - 1)T < a \leq \alpha T$, $\beta T \leq b < (\beta + 1)T$, $\lim_{T \to 0} \alpha = a$, $\lim_{T \to 0} \beta = b$.

Definition 9 is indeed an extension of the conventional Definition 1.

Theorem 4: (Definition 9 is extension of Definition 1): Let a discrete-time signal $f(k, T)$ be a discretisation of $\bar{f}(t)$ in the sense of conventional *Definition 1*, then $f(k, T)$ also satisfies *Definition 9*.

Proof: It is shown below that when the discrete-time $f(k, T)$ satisfies Definition 1 for $\bar{f}(t)$, the discrete-time signal $f(k, T)$ meets Definition 9; i.e. $\| \langle \bar{f}, \bar{\phi} \rangle - \langle f_k, \phi_k \rangle \|$ approaches zero as $T \to 0$. Let $f(k, T)$ satisfy conventional signal discretisation Definition 1 for $\bar{f}(t)$ that is $\bar{f}(t)$ is locally integrable and distribution $\bar{f}$ is regular.

The continuous-time signals $\bar{f}^*(t), \bar{\phi}^*(t)$ are the outputs of the zero-order-hold when its inputs $f(t), \bar{\phi}(t)$ are introduced and satisfy the following conditions:

$$\bar{f}^*(t) = \bar{f}(kT), \tag{21}$$

$$\bar{f}^*(t) \to \bar{f}(t), \ (T \to 0), \tag{22}$$

$$\bar{\phi}^*(t) = \bar{\phi}(kT), \tag{23}$$

$$\bar{\phi}^*(t) \to \bar{\phi}(t), \ (T \to 0), \tag{24}$$

It should be noted that while $\bar{f}^*(t), \bar{\phi}^*(t)$ is locally integrable, $\bar{\phi}^*(t)$ is not continuously differentiable. The use of these conditions yields

$$\lim_{T \to 0} \| \langle \bar{f}, \bar{\phi} \rangle - \langle f_k, \phi_k \rangle \| \tag{25}$$

$$\begin{aligned} = \lim_{T \to 0} \| &\langle \bar{f}, \bar{\phi} \rangle - \langle \bar{f}^*, \bar{\phi} \rangle \\ &+ \langle \bar{f}^*, \bar{\phi} \rangle - \int_{-\infty}^{\infty} \bar{f}^*(t) \bar{\phi}^*(t) \mathrm{d}t \\ &+ \int_{-\infty}^{\infty} \bar{f}^*(t) \bar{\phi}^*(t) \mathrm{d}t - \langle f_k, \phi_k \rangle \| . \end{aligned} \tag{26}$$

Since norm is subadditive, triangle inequality holds and it yields

$$\begin{aligned} \leq &\lim_{T \to 0} \| \langle \bar{f}, \bar{\phi} \rangle - \langle \bar{f}^*, \bar{\phi} \rangle \| \\ &+ \lim_{T \to 0} \| \langle \bar{f}^*, \bar{\phi} \rangle - \int_{-\infty}^{\infty} \bar{f}^*(t) \bar{\phi}^*(t) \mathrm{d}t \| \\ &+ \lim_{T \to 0} \| \int_{-\infty}^{\infty} \bar{f}^*(t) \bar{\phi}^* \mathrm{d}t - \langle f_k, \phi_k \rangle \| . \end{aligned} \tag{27}$$

$\bar{f}$ and $\bar{f}^*$ are regular distributions, expanding the above equation leads to

$$\begin{aligned} = &\lim_{T \to 0} \| \int_{-\infty}^{\infty} \bar{f}(t) \bar{\phi} \mathrm{d}t - \int_{-\infty}^{\infty} \bar{f}^*(t) \bar{\phi}(t) \mathrm{d}t \| \\ &+ \lim_{T \to 0} \| \int_{-\infty}^{\infty} \bar{f}^*(t) \bar{\phi}(t) \mathrm{d}t - \int_{-\infty}^{\infty} \bar{f}^*(t) \bar{\phi}^*(t) \mathrm{d}t \| \\ &+ \lim_{T \to 0} \| \int_{-\infty}^{\infty} \bar{f}^*(t) \bar{\phi}^*(t) \mathrm{d}t - \sum_{k=-\infty}^{\infty} T f(k, T) \phi(k, T) \| \end{aligned} \tag{28}$$

$$= \lim_{T \to 0} \left\| \int_{-\infty}^{\infty} (\bar{f}(t) - \bar{f}^*(t))\bar{\phi}(t)\mathrm{d}t \right\| + \lim_{T \to 0} \left\| \int_{-\infty}^{\infty} \bar{f}^*(t)(\bar{\phi}(t) - \bar{\phi}^*(t))\mathrm{d}t \right\| + \lim_{T \to 0} \left\| \sum_{k=-\infty}^{\infty} T\bar{f}(kT)\bar{\phi}(kT) - \sum_{k=-\infty}^{\infty} Tf(k,T)\phi(k,T) \right\| . \quad (29)$$

$\phi(k,T) = \bar{\phi}(kT)$ yields

$$= \lim_{T \to 0} \left\| \int_{-\infty}^{\infty} (\bar{f}(t) - \bar{f}^*(t))\bar{\phi}\mathrm{d}t \right\| + \lim_{T \to 0} \left\| \int_{-\infty}^{\infty} \bar{f}^*(t)(\bar{\phi}(t) - \bar{\phi}^*(t))\mathrm{d}t \right\| + \lim_{T \to 0} \left\| \sum_{k=-\infty}^{\infty} T(\bar{f}(kT) - f(k,T))\phi(k,T) \right\| . \quad (30)$$

The first and second terms are shown in (22) and (24) to converge to 0 as $T \to 0$. Since $f(k,T)$ is a discretisation of $\bar{f}(t)$ in the sense of Definition 1, the third term disappears as $T \to 0$. Therefore, $\| \langle \bar{f}, \bar{\phi}, \rangle - \langle f_k, \phi_k \rangle \|$ approaches 0 as $T \to 0$ and $\bar{f}(t)$, $f(k,T)$ that satisfied Definition 1 also satisfied Definition 9. Thus, it can be said that Definition 9 is an extension of conventional Definition 1.□

Discretisation of weak derivatives is defined as follows.

Definition 10: (Weak derivative discretisation): Let a discrete-time signal $f(k,T)$ be a discretisation of the continuous-time signal $\bar{f}(t)$. Then the discrete-time signal $\underline{\mathscr{D}} f_k$ is said to be a discretisation of the continuous-time weak derivative signal $\bar{\mathscr{D}}\bar{f}$, if the following condition is satisfied:

$$\lim_{T \to 0} \| \langle \bar{\mathscr{D}}\bar{f}, \bar{\phi} \rangle - \langle \underline{\mathscr{D}} f_k, \phi_k \rangle \| = \mathbf{0} . \quad (31)$$

$\underline{\mathscr{D}}$ is an operator on $f(k,T)$, such as the delta operator $\underline{\delta}$ and w-prime operator w' [8]. While there is an infinite number of operations that can be used in the above definition, a general difference operation $\underline{\delta}/(\mu T\underline{\delta} + 1)$ called the mapping discrete-time model [3, 12] is used in the present paper.

To prove the general difference satisfies Definition 10, at first, its characteristic of discrete-time generalised function in Definition 8 corresponded continuous-time weak derivative (Definition 5) is shown below.

Lemma 1: (Characteristic of general difference): $(\underline{\delta}/(\mu T\underline{\delta} + 1))^i f(k,T)$*, which is difference of* $f(k,T)$*, satisfies*

$$\left\langle \left(\frac{\underline{\delta}}{\mu T\underline{\delta} + 1} \right)^i f_k, \phi_k \right\rangle = (-1)^i \left\langle \left(\frac{1}{\mu T\underline{\delta} + 1} \right)^i f_k, \underline{\delta}^i \phi_{k-i} \right\rangle . \quad (32)$$

Proof: Induction on *i* prove this theorem.

For $i = 1$, the left-hand-side of (32) can be rewritten as

$$\left\langle \frac{\underline{\delta}}{\mu T\underline{\delta} + 1} f_k, \phi_k \right\rangle \quad (33)$$

$$= \sum_{k=-\infty}^{\infty} T \frac{\underline{\delta}}{\mu T\underline{\delta} + 1} f(k,T)\phi(k,T) . \quad (34)$$

Since $(\underline{\delta}/(\mu T\underline{\delta} + 1))f_k = (\underline{\delta} f_k/(\mu T\underline{\delta} + 1))$, the above equation can be rewritten as

$$= \sum_{k=-\infty}^{\infty} T \frac{1}{\mu T\underline{\delta} + 1} \frac{f(k+1,T)\phi(k,T)}{T} - \sum_{k=-\infty}^{\infty} T \frac{1}{\mu T\underline{\delta} + 1} \frac{f(k,T)\phi(k,T)}{T} . \quad (35)$$

By shifting the first term infinite sum for $f(k+1,T)$ using a variable transformation

$$= \sum_{k=-\infty}^{\infty} T \frac{1}{\mu T\underline{\delta} + 1} \frac{f(k,T)\phi(k-1,T)}{T} - \sum_{k=-\infty}^{\infty} T \frac{1}{\mu T\underline{\delta} + 1} \frac{f(k,T)\phi(k,T)}{T}, \quad (36)$$

rearranging the above equation taking $f(k,T)$ as the common term leads to

$$= \sum_{k=-\infty}^{\infty} T \frac{1}{\mu T\underline{\delta} + 1} f(k,T) \frac{\phi(k-1,T) - \phi(k,T)}{T} \quad (37)$$

$$= -\sum_{k=-\infty}^{\infty} T \frac{1}{\mu T\underline{\delta} + 1} f(k,T)\underline{\delta}\phi_{k-1} \quad (38)$$

$$= -\left\langle \frac{1}{\mu T\underline{\delta} + 1} f_k, \underline{\delta}\phi_{k-1} \right\rangle, \quad (39)$$

the right-hand-side of (32) is obtained.

Assume that (32) holds for $i = j$ and let us get the following condition:

$$\left\langle \left(\frac{\underline{\delta}}{\mu T\underline{\delta} + 1} \right)^j f_k, \phi_k \right\rangle = \sum_{k=-\infty}^{\infty} T \left(\frac{\underline{\delta}}{\mu T\underline{\delta} + 1} \right)^j f(k,T)\phi(k,T) \quad (40)$$

$$= (-1)^j \left\langle \left(\frac{1}{\mu T\underline{\delta} + 1} \right)^j f_k, \underline{\delta}^j \phi_{k-j} \right\rangle \quad (41)$$

$$= (-1)^j \sum_{k=-\infty}^{\infty} T \left(\frac{1}{\mu T\underline{\delta} + 1} \right)^j f_k \underline{\delta}^j \phi_{k-j} . \quad (42)$$

We then have to show for $i = j + 1$ that

$$\left\langle \left(\frac{\underline{\delta}}{\mu T\underline{\delta} + 1} \right)^{j+1} f_k, \phi_k \right\rangle \quad (43)$$

$$= \sum_{k=-\infty}^{\infty} T \frac{\underline{\delta}}{\mu T\underline{\delta} + 1} \left(\frac{\underline{\delta}}{\mu T\underline{\delta} + 1} \right)^j f(k,T)\phi(k,T), \quad (44)$$

by using the induction hypothesis

$$= (-1)^j \sum_{k=-\infty}^{\infty} T \frac{\underline{\delta}}{\mu T\underline{\delta} + 1} \left(\frac{1}{\mu T\underline{\delta} + 1} \right)^j f(k,T)\underline{\delta}^j \phi(k-j,T), \quad (45)$$

rearranging the above equation in the same manner as the case of $i = 1$ leads to

$$= (-1)^j \sum_{k=-\infty}^{\infty} T \left(\frac{1}{\mu T\underline{\delta} + 1} \right)^{j+1} \frac{f(k+1,T)\underline{\delta}^j \phi(k-j,T)}{T} - (-1)^j \sum_{k=-\infty}^{\infty} T \left(\frac{1}{\mu T\underline{\delta} + 1} \right)^{j+1} \frac{f(k,T)\underline{\delta}^j \phi(k-j,T)}{T} \quad (46)$$

$$= (-1)^j \sum_{k=-\infty}^{\infty} T\left(\frac{1}{\mu T\underline{\delta}+1}\right)^{j+1} \frac{f(k,T)\underline{\delta}^j \phi(k-j-1,T)}{T}$$
$$-(-1)^j \sum_{k=-\infty}^{\infty} T\left(\frac{1}{\mu T\underline{\delta}+1}\right)^{j+1} \frac{f(k,T)\underline{\delta}^j \phi(k-j,T)}{T} \quad (47)$$

$$= (-1)^{j+1} \sum_{k=-\infty}^{\infty} T\left(\frac{1}{\mu T\underline{\delta}+1}\right)^{j+1} f(k,T) \frac{\underline{\delta}^j \phi(k-j,T) - \underline{\delta}^j \phi(k-j-1,T)}{T} \quad (48)$$

$$= (-1)^{j+1} \sum_{k=-\infty}^{\infty} T\left(\frac{1}{\mu T\underline{\delta}+1}\right)^{j+1} f(k,T) \underline{\delta}^{j+1}\phi(k-(j+1),T) \quad (49)$$

$$= (-1)^{j+1}\left\langle \left(\frac{1}{\mu T\underline{\delta}+1}\right)^{j+1} f_k, \underline{\delta}^{j+1}\phi_{k-(j+1)}\right\rangle, \quad (50)$$

which complete the proof.□

The above result is similar to a continuous-time weak derivative (Definition 5).

It can be shown below that difference of $f(k,T)$ is a discretisation of $\bar{f}(t)$ derivatives. It is important that this can discretise weak derivatives with a general difference. As a conventional definition, since the general difference is an approximation of differentiation, it is assumed that the differentiable function should be differentiable for a sufficient number of times.

Theorem 5: (Discretisation of weak derivative): If a discrete-time signal $f(k,T)$ is a discretisation of a continuous-time signal $\bar{f}(t)$, then a discrete-time signal $f^{(i)}(k,T)$ is also a discretisation of a continuous-time signal $\bar{f}^{(i)}(t)$ in the sense of *Definition 9*, where $f^{(i)}(k,T)$ is defined as

$$\bar{f}^{(i)}(k,T) := \left(\frac{\underline{\delta}}{\mu T\underline{\delta}+1}\right)^i f(k,T). \quad (51)$$

Proof: It is shown below that the discrete-time signal $f^{(i)}(k,T)$ satisfies Definition 9, where $\phi_k^{(i)}$ satisfies $\phi_k^{(i)} = \bar{\phi}^{(i)}(kT)$.

Weak derivative definition (Definition 5) and characteristic of general difference (Lemma 1) yields

$$\lim_{T\to 0} \left\| \left\langle \bar{f}^{(i)}, \bar{\phi}\right\rangle - \left\langle \left(\frac{\underline{\delta}}{\mu T\underline{\delta}+1}\right)^i f_k, \phi_k \right\rangle \right\| \quad (52)$$

$$= \lim_{T\to 0} \left\| (-1)^i \left\langle \bar{f}, \bar{\phi}^{(i)}\right\rangle - (-1)^i \left\langle \left(\frac{1}{\mu T\underline{\delta}+1}\right)^i f_k, \underline{\delta}^i \phi_{k-i}\right\rangle \right\| \quad (53)$$

$$= \lim_{T\to 0} \left\| \left\langle \bar{f}, \bar{\phi}^{(i)}\right\rangle - \left\langle \left(\frac{1}{\mu T\underline{\delta}+1}\right)^i f_k, \underline{\delta}^i \phi_{k-i}\right\rangle \right\| \quad (54)$$

$$= \lim_{T\to 0} \left\| \left\langle \bar{f}, \bar{\phi}^{(i)}\right\rangle - \left\langle f_k, \phi_k^{(i)}\right\rangle + \left\langle f_k, \phi_k^{(i)}\right\rangle - \left\langle f_k, \underline{\delta}^i \phi_{k-i}\right\rangle \left\langle f_k, \underline{\delta}^i \phi_{k-i}\right\rangle - \left\langle \left(\frac{1}{\mu T\underline{\delta}+1}\right)^i f_k, \underline{\delta}^i \phi_{k-i}\right\rangle \right\|. \quad (55)$$

Since norm is subadditive, triangle inequality holds and it yields

$$\le \lim_{T\to 0} \left\| \left\langle \bar{f}, \bar{\phi}^{(i)}\right\rangle - \left\langle f_k, \phi_k^{(i)}\right\rangle \right\| + \lim_{T\to 0} \left\| \left\langle f_k, \phi_k^{(i)}\right\rangle - \left\langle f_k, \underline{\delta}^i \phi_{k-1}\right\rangle \right\| + \lim_{T\to 0} \left\| \left\langle f_k, \underline{\delta}^i \phi_{k-i}\right\rangle - \left\langle \left(\frac{1}{\mu T\underline{\delta}+1}\right)^i f_k, \underline{\delta}^i \phi_{k-i}\right\rangle \right\|, \quad (56)$$

rearranging the above equation leads to

$$= \lim_{T\to 0} \left\| \left\langle \bar{f}, \bar{\phi}^{(i)}\right\rangle - \left\langle f_k, \phi_k^{(i)}\right\rangle \right\| + \lim_{T\to 0} \left\| \sum_{k=-\infty}^{\infty} T f_k\left(\phi_k^{(i)} - \underline{\delta}^i \phi_{k-i}\right) \right\| + \lim_{T\to 0} \left\| \sum_{k=-\infty}^{\infty} T\left(f_k - \left(\frac{1}{\mu T\underline{\delta}+1}\right)^i f_k\right)\underline{\delta}^i \phi_{k-i} \right\|. \quad (57)$$

Since $f(k,T)$ is a discretisation of $\bar{f}(t)$ in the sense of Definition 9, the first term converges to 0 as T approaches 0. The second term is shown in the definition of differentiation to converge to 0 as T goes to 0. The third term approaches 0 as T goes to 0 with $(1/(\mu T\underline{\delta}+1))^i$ satisfies the following condition: $\lim_{T\to 0}(1/(\mu T\underline{\delta}+1))^i = 1$. Therefore, $f^{(i)}(k,T)$ is a discretisation of $\bar{f}^{(i)}(t)$.□

The above theorem becomes to be able to discretise weak derivatives. Since conventional interpretation of general difference $\underline{\delta}/(\mu T\underline{\delta}+1)$ is a finite difference approximation of differentiation, differentiated signals should be sufficiently smooth.

The validity of the proposed Definition 9 is explained below:

- The proposed definition of the discrete-time generalised function (Definition 8) approaches (7) of the continuous-time distribution definition (Definition 4) as $T \to 0$.
- It is shown in Theorem 4 that the proposed Definition 9 is an extension of conventional Definition 1.
- Section 4 describes to verify the validity of the proposed definition with Dirac's delta function that is one of the typical singular distribution.

4 Dirac's delta function and its derivatives discretisation

This section derives a discretisation and some characteristics of Dirac's delta function and its derivatives to present adaption examples and verify the validity of the proposed definition.

Deriving a discretisation of Dirac's delta function $\bar{\delta}$ in the sense of Definition 9 is shown below.

Theorem 6: (Discretisation of Dirac's delta function δ_k): A discretisation of a continuous-time Dirac's delta function $\bar{\delta}$ (14) in the sense of Definition 9 is given by

$$\delta_k := \begin{cases} \frac{1}{T}, & k = 0, \\ 0, & k \neq 0. \end{cases} \quad (58)$$

Proof: Substituting (58) in Definition 9 ((20)) yields

$$\lim_{T\to 0} \left\| \left\langle \bar{\delta}, \bar{\phi}\right\rangle - \left\langle \delta_k, \phi_k\right\rangle \right\| = \lim_{T\to 0} \left\| \bar{\phi}(0) - \sum_{k=-\infty}^{\infty} T\delta_k \phi_k \right\| \quad (59)$$

$$= \lim_{T\to 0} \| \bar{\phi}(0) - \phi_0 \| \quad (60)$$

$$= 0. \quad (61)$$

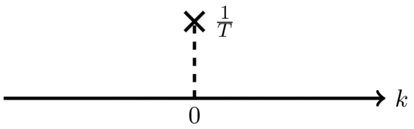

Fig. 1 *Discrete-time Dirac's delta function δ_k*

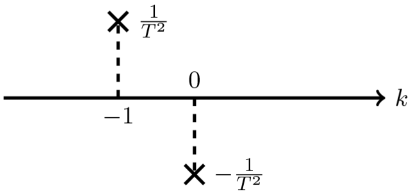

Fig. 2 *Discrete-time Dirac's delta function derivative $\delta^{(1)}_{0,\,k}$*

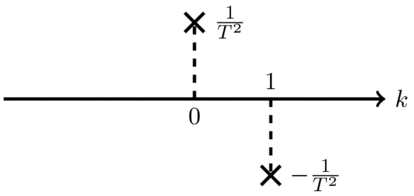

Fig. 3 *Discrete-time Dirac's delta function derivative $\delta^{(1)}_{1,\,k}$*

Since the discrete-time signal, δ_k (58) satisfied Definition 9, it is a discretisation of continuous-time Dirac's delta function, $\bar{\delta}$ (14), in the sense of Definition 9.□

Fig. 1 shows δ_k (58).

While (58) is not identical to Kronecker's delta, or unit pulse [13, 14], which is often used in digital signal processing as a discrete-time version of impulse function, it is identical to a popular discrete-time impulse function in delta form digital control [6, 7].

Deriving a discretisation of Dirac's delta function derivatives $\bar{\delta}$ in the sense of Definition 9 with Theorem 6 and 5 is shown below.

Theorem 7: (Discretisation of Dirac's delta function derivatives $\delta^{(i)}_{\mu,k}$): A discrete-time signal $\delta^{(i)}_{\mu}(k,T)$ *defined as the following is a discretisation in the sense of Definition 9 of Dirac's delta function derivatives*

$$\delta^{(i)}_{\mu,k} := \left(\frac{\delta}{\mu T\underline{\delta}+1}\right)^{i}\delta_k\,. \tag{62}$$

Proof: The proof is easily verified with Theorem 6 and Theorem 5.

Figs. 2–5 show $\delta^{(i)}_{\mu,k}$ whose parameters are chosen as $i = 1,\,2,\,\mu = 0,\,1$.

When $\mu = 0$, the above signal is not proper.

It is shown below that convolution characteristics of derived discrete-time signals, δ_k, (58) and $\delta^{(i)}_{\mu,k}$, (62) correspond to continuous-time characteristics.

Theorem 8: (Convolution of δ_k, $\delta^{(i)}_{\mu,k}$): Discrete-time convolutions of δ_k (58), $\delta^{(i)}_{\mu,k}$ (62) *are given by*

$$\delta_k * f_k = f_k, \tag{63}$$

$$\delta^{(i)}_{\mu,k} * f_k = \left(\frac{\delta}{\mu T\underline{\delta}+1}\right)^{i} f_k, \tag{64}$$

Proof: . Induction on i prove this theorem.

For $i = 0$, equation is given by

$$\delta_k * f_k = \sum_{n=-\infty}^{\infty} T\delta_n f_{k-n} \tag{65}$$

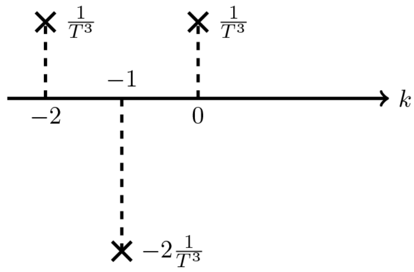

Fig. 4 *Discrete-time Dirac's delta function derivative $\delta^{(2)}_{0,\,k}$*

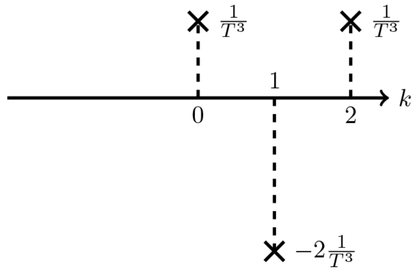

Fig. 5 *Discrete-time Dirac's delta function derivative $\delta^{(2)}_{1,\,k}$*

$$= (f_k + f_{k-1} + \ldots) - (f_{k-1} + f_{k-2} + \ldots) \tag{66}$$

$$= f_k\,. \tag{67}$$

Assume that the theorem is true for $i = j$, the equation is given by

$$\delta^{(j)}_{\mu,k} * f_k = \sum_{n=-\infty}^{\infty} T\delta^{(j)}_{\mu,n} f_{k-n} = \left(\frac{\delta}{\mu T\underline{\delta}+1}\right)^{j} f_k\,. \tag{68}$$

We then have to show for $i = j + 1$ that

$$\delta^{(j+1)}_{\mu,k} * f_k = \sum_{n=-\infty}^{\infty} T\delta^{(j+1)}_{\mu,n} f_{k-n} \tag{69}$$

$$= \sum_{n=-\infty}^{\infty} T\frac{\delta}{\mu T\underline{\delta}+1}\delta^{(j)}_{\mu,n} f_{k-n} \tag{70}$$

$$= \frac{\delta}{\mu T\underline{\delta}+1}\sum_{n=-\infty}^{\infty} T\delta^{(j)}_{\mu,n} f_{k-n} \tag{71}$$

$$= \frac{\delta}{\mu T\underline{\delta}+1}\left(\frac{\delta}{\mu T\underline{\delta}+1}\right)^{j} f_k \tag{72}$$

$$= \left(\frac{\delta}{\mu T\underline{\delta}+1}\right)^{j+1} f_k, \tag{73}$$

which is exactly the right-hand-side of the equation.□

The above theorem leads to the following. It is shown below that convolutions of discrete-time Dirac's delta function and its derivatives are discretisation of continuous-time.

Theorem 9: (Convolutions of Dirac's delta function and its derivatives): If a discrete-time signal $\boldsymbol{f}(k,T)$ is a discretisation of a continuous-time signal $\bar{\boldsymbol{f}}(t)$, then discrete-time convolutions $\delta_k * \boldsymbol{f}_k$, $\delta^{(i)}_{\mu,k} * \boldsymbol{f}_k$ are discretisation of continuous-time convolutions $\bar{\delta} * \bar{\boldsymbol{f}}$, $\bar{\delta}^{(i)} * \bar{\boldsymbol{f}}$.

Proof: As in the case of $\delta_k * \boldsymbol{f}_k$, since $\boldsymbol{f}(k,T)$ is a discretisation of $\bar{\boldsymbol{f}}(t)$, it is clear with Theorem 2 and Theorem 8. Similarly, as in the case of $\delta^{(i)}_{\mu,k} * \boldsymbol{f}_k$, it is clear with Theorem 3, Theorem 5 and Theorem 8.□

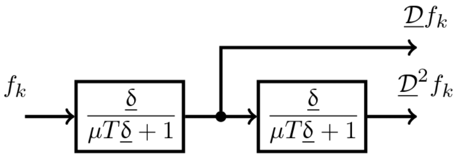

Fig. 6 *Block diagram of simulation*

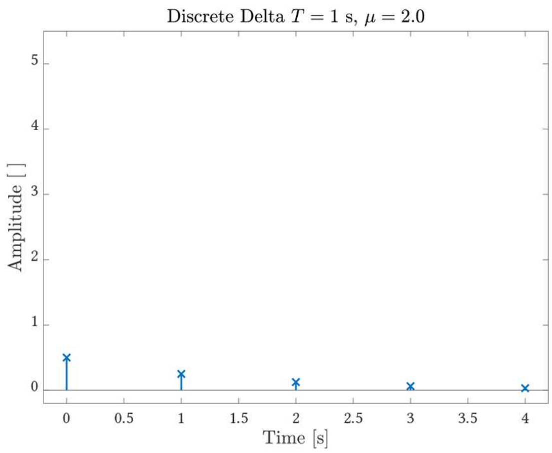

Fig. 7 *Discrete-time delta – general difference of step function,* $\mu = 2.0, T = 1$ s

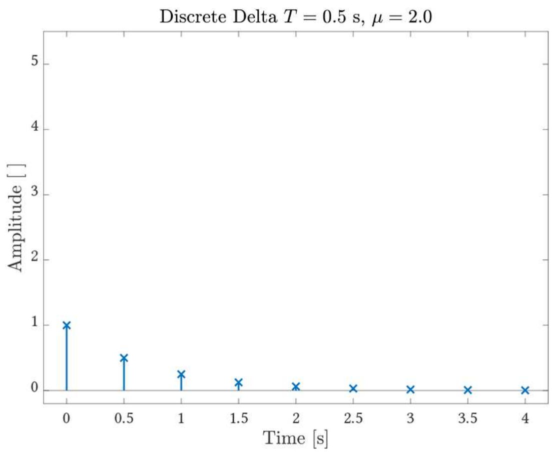

Fig. 8 *Discrete-time delta – general difference of step function,* $\mu = 2.0, T = 0.5$ s

5 Simulations

In this section, simulations are carried out to show that different types of signals can be obtained by changing parameter μ of the proposed model. Fig. 6 shows the series discrete-time differentiators for this purpose. When the step input h_k defined as

$$h_k := \begin{cases} 0, & k < 0, \\ 1, & k \geq 0, \end{cases} \tag{74}$$

is used, the outputs of the first and the second blocks are discrete-time models of the impulse and doublet signals, respectively. Shown in Figs. 7–9 are the results obtained for $\mu = 2.0$ with $T = 1$, 0.5and0.1 s, respectively. Their magnitudes at $t = 0$ increase as T decreases, and these responses approach zero monotonically, which look similar to the impulse obtained by Gaussian's approximation (It might be easier to visualise the continuous-time stair-case waveform obtained through the zero-order-hold.). Figs. 10–12 are the results obtained under the same condition except for $\mu = 1.0$. This is a special case of the above with the responses converging to zero in a single step. Figs. 13–15 are for $\mu = 0.7$, where the responses converge to zero with both positive and negative swings and look similar to those obtained using Dirichlet's kernel. Figs. 16–18 show the outputs of the second block for three values of μ and with $T = 0.1$ s, which are valid discrete-time models of the continuous-time doublet signals under the definition given in the present paper, without which such conclusions are difficult to draw.

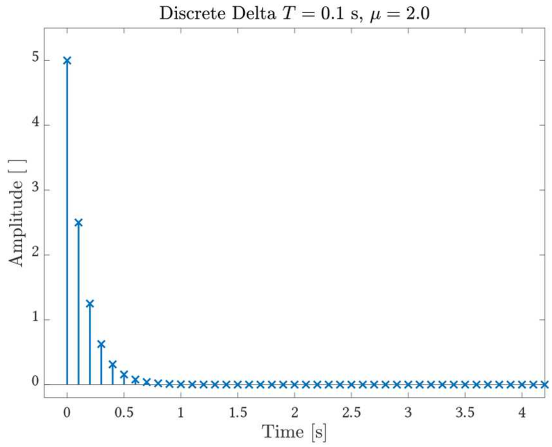

Fig. 9 *Discrete-time delta – general difference of step function,* $\mu = 2.0, T = 0.1$ s

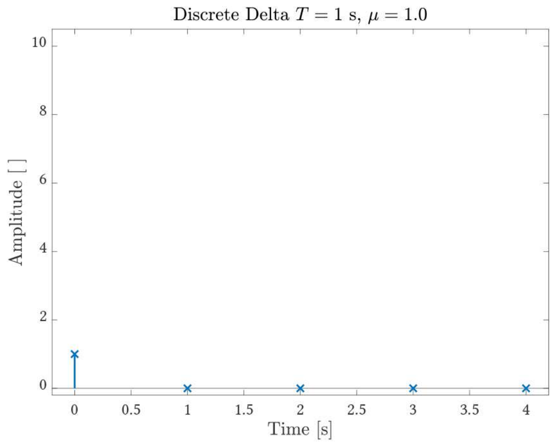

Fig. 10 *Discrete-time delta – general difference of step function,* $\mu = 1.0, T = 1$ s

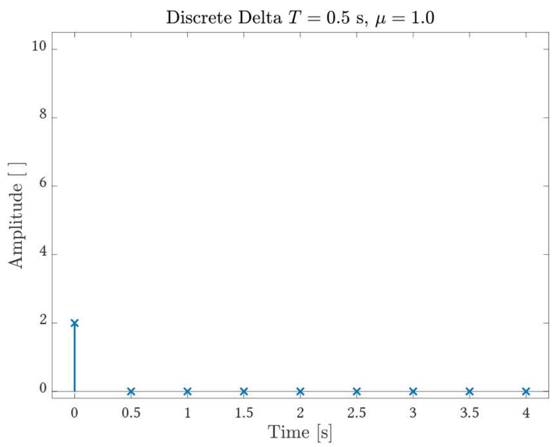

Fig. 11 *Discrete-time delta – general difference of step function,* $\mu = 1.0, T = 0.5$ s

6 Conclusions

The concept of distribution is abstract but useful idealisation in simplifying expressions and handling real phenomena, even though no such physical signals may exist exactly. Discrete-time signals have finite values and it is not absolutely necessary to introduce generalised functions as in continuous-time signals. However, it turns out to be useful to introduce a discrete-time functional that

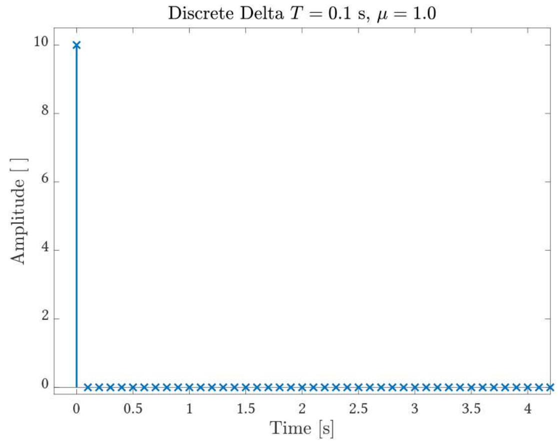

Fig. 12 *Discrete-time delta – general difference of step function,* $\mu = 1.0, T = 0.1\,\mathrm{s}$

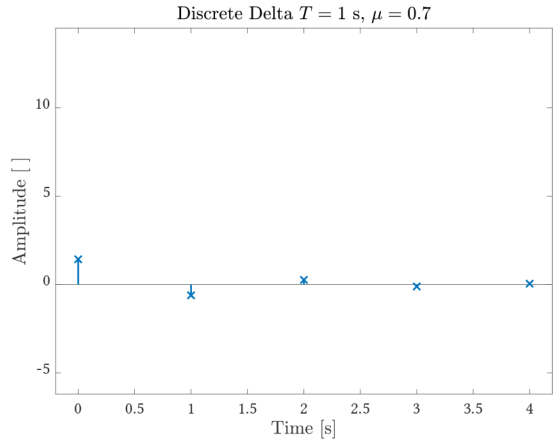

Fig. 13 *Discrete-time delta – general difference of step function,* $\mu = 0.7, T = 1\,\mathrm{s}$

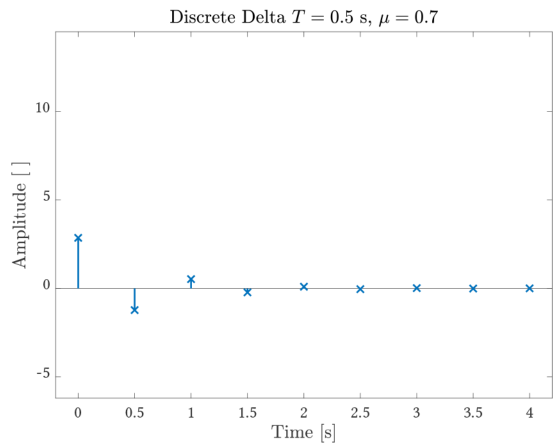

Fig. 14 *Discrete-time delta – general difference of step function,* $\mu = 0.7, T = 0.5\,\mathrm{s}$

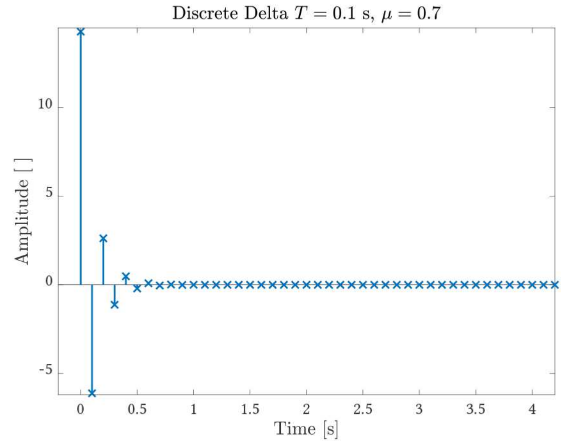

Fig. 15 *Discrete-time delta – general difference of step function,* $\mu = 0.7, T = 0.1\,\mathrm{s}$

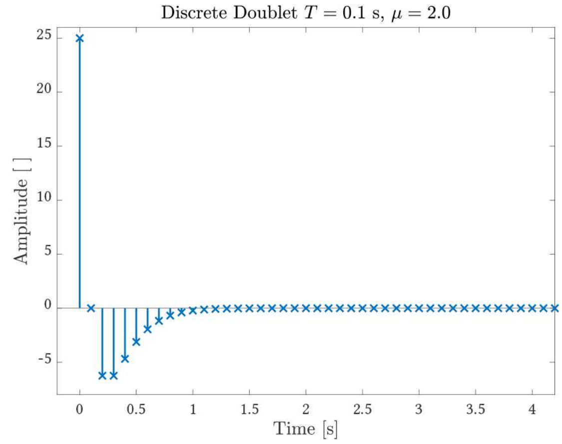

Fig. 16 *Discrete-time doublet – general difference of delta function,* $\mu = 2.0, T = 0.1\,\mathrm{s}$

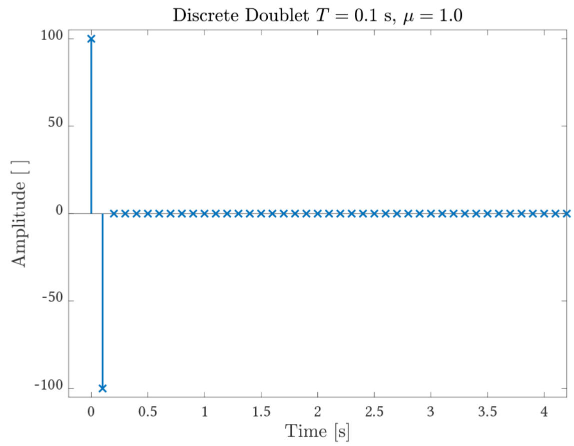

Fig. 17 *Discrete-time doublet – general difference of delta function,* $\mu = 1.0, T = 0.1\,\mathrm{s}$

approaches continuous-time distribution as the discrete-time interval approaches zero. A key in this extension is to take multiple sampling points into account, rather than a single point as has been considered previously. Using the results obtained in the present paper, discrete-time signals that approach a continuous-time distribution in the sense defined in the paper can now be created easily. For instance, the discretisation of derivatives of a distribution can be found as the discrete derivatives of a discretisation of a distribution. This is much easier than the traditional approach, where ordinary functions must first be found as appropriate approximations of the derivatives of a distribution and assessments of their validity must be conducted for each such approximation. Without such endeavours, one is not sure if the discrete-time model can be expected to behave as in the continuous-time case at the limit of discrete-time interval approaching zero. These proposed models form a class of discretised systems and an infinite number of other and new models can be obtained. As an example, Dirac's delta function and its derivatives, which are typical distributions and often appear as input signals and system responses, are discretised so that the results obtained using on-line discrete-time computations approach those of continuous-time originals as the discretisation period approaches zero. This is what has been expected but not achieved for systems expressed in the descriptor form with arbitrary initial conditions and inputs. It is indeed crucial to choose a proper initial condition on and calculate the responses of, descriptor systems,

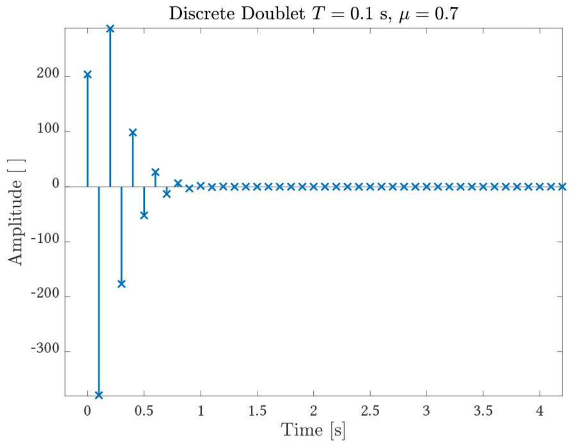

Fig. 18 *Discrete-time doublet – general difference of delta function,* $\mu = 0.7, T = 0.1$ s

where impulsive responses can be preserved or avoided corresponding to the continuous-time case [3]. Simulations have also been carried out and shown that by changing a single parameter, different types of signals can be obtained, including those known with the traditional methods that uses Gaussian distributions and Dirichlet kernels.

7 Acknowledgments

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