ALMOST COSYMPLECTIC 3-MANIFOLDS WITH PSEUDO-PARALLEL CHARACTERISTIC JACOBI OPERATOR

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ABSTRACT. In this paper, we classify almost cosymplectic 3-manifolds with pseudo-parallel characteristic Jacobi operator. The only simply connected and complete non-cosymplectic almost cosymplectic 3-manifolds with pseudo parallel characteristic Jacobi operator is the Minkowski motion group.

INTRODUCTION

In almost contact metric geometry, the Jacobi operator along the characteristic vector field (called the *characteristic Jacobi operator*) plays an important role. In particular, the characteristic Jacobi operator of contact metric 3-manifolds has been paid much attention by researchers of contact manifolds. See [6, 7] and references therein.

On the other hand, in almost contact metric geometry, the parallelism of tensor fields often causes very strong restrictions for almost contact metric manifolds. For instance, Sasakian manifolds with parallel Riemannian curvature (*i.e.* locally symmetric Sasakian manifolds) are of constant curvature 1. Analogously locally symmetric Kenmotsu manifolds are of constant curvature -1.

The parallelism of the characteristic Jacobi operator is also a strong restriction for almost contact metric manifolds. In fact, if the characteristic Jacobi operator ℓ of an almost cosymplectic 3-manifold is parallel, then $\ell = 0$ (see [14] or Proposition 9 of the present paper).

Thus we need to find appropriate relaxations of parallelism for the characteristic Jacobi operator to characterize nice classes of almost contact metric manifolds.

From the viewpoint of almost contact structure, η-parallelism was proposed and studied intensively for some tensor fields on almost contact metric manifolds. In our previous paper [6], Cho and the first named author studied contact metric 3-manifolds with η-parallel characteristic Jacobi operator. In addition almost cosymplectic 3-manifolds with η-parallel characteristic Jacobi operator are investigated in [14]

On the other hand, from Riemannian geometric viewpoint, semi-parallelism, and pseudo-parallelism are introduced. On a Riemannian manifold (M, g) with Levi- Civita connection ∇, every curvature-like tensor field acts on tensor fields of type $(1, 1)$ as a derivation. For

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instance the derivative $R \cdot P$ of P by the Riemannian curvature R is given by

$$(R \cdot P)(Z; X, Y) = R(X, Y)(PZ) - PR(X, Y)Z.$$

A tensor field P of type $(1, 1)$ is said to be *semi-paralle* if $R \cdot P = 0$. More generally, P is said to be *pseudo-parallel* (*cf.* [11]) if $R \cdot P = LR_1 \cdot P$ for some function L, where R_1 is the curvature-like tensor field

$$R_1(X, Y)Z = (X \wedge Y)Z := g(Y, Z)X - g(Z, X)Y.$$

We proposed the following problems in [7]:

(1) *Classify unit tangent sphere bundles with semi-parallel characteristic Jacobi operator.*
(2) *Classify contact metric* 3*-manifolds with pseudo-parallel characteristic Jacobi operator.*

Concerning this problem, Cho and Chun [5] investigated unit tangent sphere bundles with pseudo-parallel characteristic Jacobi operator (for 2-dimensional base manifold case, see [15]).

On the second problem, Wang and Dai [24] studied pseudo-parallelism and semi-parallelism of the characteristic Jacobi operator on contact metric 3-manifolds.

Motivated by [7, 24], in the present paper we study pseudo-parallelism of characteristic Jacobi operator of almost cosymplectic 3-manifolds.

The purpose of this paper is to classify almost cosymplectic 3-manifolds with pseudo-parallel characteristic Jacobi operator. Our main result is stated as follows:

Theorem. *Almost cosymplectic* 3*-manifolds with pseudo-parallel characteristic Jacobi operator are cosymplectic or non-cosymplectic and locally isomorphic to the Minkowski motion group* $\mathrm{E}_{1,1}$.

Our results will be given in Section 5 and 6. More precisely, in Section 5, we show that almost cosymplectic 3-manifolds with pseudo-parallel characteristic Jacobi operator are cosymplectic or locally isomorphic to the Minkowski motion group $\mathrm{E}_{1,1}$ (Theorem 9). It should be mentioned that the model space Sol_3 of the solvgeometry in the sense of Thurston has pseudo-parallel characteristic Jacobi operator.

In Section 6, we study characteristic Jacobi operator of homogeneous almost cosymplectic 3-manifolds. In particular, we give a complete classification of all homogeneous almost cosymplectic 3-manifolds with pseudo-parallel Jacobi operator (Corollary 6).

1. Preliminaries

1.1. **Pseudo-parallelism.** Let (M, g) be a Riemannian manifold with its Levi-Civita connection ∇. Then the *Riemannian curvature* R of M is defined by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

The *Ricci tensor field* S of (M, g) is a symmetric tensor field defined by

$$S(X, Y) = \mathrm{tr}\,(Z \longmapsto R(Z, Y)X).$$

The Ricci operator Q is a self-adjoint endomorphism field metrically equivalent to S, that is

$$S(X, Y) = g(QX, Y) = g(X, QY).$$

The smooth function $r = \mathrm{tr}\, S = \mathrm{tr}\, Q$ is called the *scalar curvature* of (M, g).

On a Riemannian manifold (M, g), We define a curvature-like tensor field R_1 by

$$R_1(X, Y)Z = (X \wedge Y)Z = g(Y, Z)X - g(Z, X)Y.$$

As is well known, every curvature-like tensor field acts on tensor fields as a derivation.

Definition 1. A tensor field P of type $(1, 1)$ is said to be *pseudo-parallel* if there exits a smooth function L such that

$$R \cdot P = LR_1 \cdot P.$$

In particular, P is said to be *semi-parallel* if

$$R \cdot P = 0.$$

Obviously, the following relations hold:

$$P \text{ is parallel} \Rightarrow P \text{ is semi-parallel} \Rightarrow P \text{ is pseudo-parallel}.$$

1.2. **Harmonic vector fields.** Let (M, g) be a Riemannian manifold with unit tangent sphere bundle UM. We equip the Sasaki-lift metric g^{s} on UM. Denote by $\mathfrak{X}_1(M)$ the space of all smooth unit vector fields on M. Every unit vector field $V \in \mathfrak{X}_1(M)$ is regarded as an immersion of M into UM.

A unit vector field $V \in \mathfrak{X}_1(M)$ is said to be *minimal* if it is a critical point of the volume functional on $\mathfrak{X}_1(M)$. It is known that V is a minimal unit vector field if and only if it is a minimal immersion with respect to the pull-backed metric V^*g^{s}.

On the other hand, a unit vector field V is said to be a *harmonic unit vector field* if it is a critical point of the energy functional restricted $\mathfrak{X}_1(M)$. It should be remarked that for a unit vector field, to be a harmonic map into UM is stronger than to be a harmonic vector field. In fact, the following result is known.

Proposition 1 ([13])**.** *A unit vector field $V : M \to UM$ is a harmonic map if and only if it is a harmonic vector field and in addition, satisfies*

$$\mathrm{tr}_g R(\nabla V, V) = 0.$$

2. Almost contact metric manifolds

In this section, we recall the fundamental ingredients of almost contact metric geometry. For general information on almost contact metric geometry, we refer to [1].

2.1. **Almost contact metric structures.** An *almost contact metric structure* of a $(2n+1)$-manifold M is a quartet (φ, ξ, η, g) of structure tensor fields which satisfies:

$$\eta(\xi) = 1, \quad d\eta(\xi, \cdot) = 0, \tag{1}$$

$$\varphi^2 = -\mathrm{I} + \eta \otimes \xi, \ \varphi\xi = 0, \tag{2}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{3}$$

A $(2n + 1)$-manifold $M = (M, \varphi, \xi, \eta, g)$ equipped with an almost contact metric structure is called an *almost contact metric manifold*. The vector field ξ is called the *characteristic vector field* of M. The 2-form

$$\Phi(X, Y) = g(X, \varphi Y)$$

is called the *fundamental* 2*-form* of M.

An almost contact metric manifold M is said to be *normal* if

$$[\varphi, \varphi](X, Y) + 2d\eta(X, Y)\xi = 0,$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ.

Definition 2. Let $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold. A tangent plane Π_x at $x \in M$ is said to be *holomorphic* if it is invariant under φ_x.

It is easy to see that a tangent plane Π_x is holomorphic if and only if ξ_x is orthogonal to Π_x. The sectional curvature $K(\Pi_x)$ of a holomorphic plane Π_x is called the *holomorphic sectional curvature*. In case $\dim M = 3$, holomorphic sectional curvature is a smooth function on M and we denote it by H.

Here we recall an auxiliary tensor field h which is very useful for the study of almost contact metric manifolds. The endomorphism field h is defined by $h = (\pounds_\xi \varphi)/2$. Here $\pounds_\xi$ denotes the Lie differentiation by ξ.

2.2. **The characteristic Jacobi operator.** In addition we introduce a self-adjoint endomorphism field ℓ on an almost contact metric manifold M of dimension $2n+1 \geq 3$ by

$$\ell(X) = R(X, \xi)\xi, \quad X \in \mathfrak{X}(M).$$

The self-adjoint operator ℓ is called the *characteristic Jacobi operator* of M. Note that our ℓ has the opposite sign to the one in [20].

2.3. **Almost cosymplectic structures.** Now we turn our attention to almost cosymplectic manifolds (see [1, 12]).

Definition 3. An almost contact metric manifold M is said to be *almost cosymplectic* (or *almost coKähler*) if $d\eta = 0$ and $d\Phi = 0$. An almost cosymplectic manifold is said to be *cosymplectic* (or *coKähler*) if it is normal.

The distribution

$$\mathcal{D} = \{X \in TM \mid \eta(X) = 0\}$$

on an almost cosymplectic manifold M is integrable and hence it defines a foliation $\mathcal{F}$ on M. The foliation $\mathcal{F}$ is called the *canonical foliation* of M. The almost cosymplectic structure induces an almost Kähler structure on leaves. An almost cosymplectic manifold M is said to be an *almost cosymplectic manifold with Kähler leaves* if leaves of the canonical foliation are Kähler manifolds. Clearly if $\dim M = 3$ or M is cosymplectic, then all the leaves are Kähler.

Theorem 1 ([18])**.** *An almost cosymplectic manifold has Kähler leaves if and only if*

$$(\nabla_X \varphi)Y = g(X, hY)\xi - \eta(Y)hX. \tag{4}$$

Let us introduce an endomorphism field A by $A = -\nabla\xi$. One can see that on every leaf, A is nothing but the shape operator of the leaf derived from the unit normal ξ. Dacko and Olszak obtained the following formulas:

$$h = A\varphi, \quad A = \varphi h, \quad A\xi = 0, \quad \eta \circ A = 0.$$

Moreover A is self-adjoint operator, *i.e.*,

$$g(AX, Y) = g(X, AY)$$

for all vector fields X and Y on M.

The cosymplectic property is characterized as follows:

Proposition 2. *An almost contact metric manifold M is cosymplectic if and only if φ is parallel.*

In particular, ξ is parallel on every cosymplectic manifold. Here we recall the following fundamental fact (see *e.g.* [2, Theorem 3.11], [12]).

Theorem 2. *Let M be an almost cosymplectic manifold. Then the following properties are mutually equivalent*:

- $h = 0$.
- $\nabla \xi = 0$.
- *ξ is a Killing vector field.*
- *M is locally isomorphic to a direct product of an almost Kähler manifold and the real line.*

2.4. **Cosymplectic manifolds.** A complete cosymplectic manifold M of constant holomorphic sectional curvature c is called a *cosymplectic space form*.

Example 1. Let $\overline{M} = (\overline{M}, \bar{g}, J)$ be an almost Kähler manifold. Consider the Riemannian product $M = (\overline{M} \times \mathbb{R}, g)$ with $g = \bar{g} + dt^2$. Then we can equip an almost cosymplectic structure of M by

$$\xi = \frac{d}{dt}, \quad \eta = dt, \quad \varphi\left(X, f\frac{d}{dt}\right) = (JX, 0), \quad X \in \mathfrak{X}(\overline{M}).$$

The almost cosymplectic manifold M is cosymplectic if and only if $\overline{M}$ is Kähler. In particular when $\overline{M}$ is a complex space form, that is, a Kähler manifold of constant holomorphic sectional curvature, then M is a cosymplectic manifold of constant holomorphic sectional curvature. Now let $\mathbb{C}P_n(\bar{c})$, $\mathbb{C}^n$ and $\mathbb{C}H_n(\bar{c})$ be complex projective n-space of constant holomorphic sectional curvature $\bar{c} > 0$, complex Euclidean n-space and complex hyperbolic n-space of constant holomorphic sectional curvature $\bar{c} < 0$, respectively. Then the cosymplectic manifolds

$$\mathbb{C}P_n(\bar{c}) \times \mathbb{R}, \quad \mathbb{E}^{2n+1} = \mathbb{C}^n \times \mathbb{R}, \quad \mathbb{C}H_n(\bar{c}) \times \mathbb{R}$$

are cosymplectic space forms. In particular

$$\mathbb{S}^2(\bar{c}) \times \mathbb{R}, \quad \mathbb{E}^3 = \mathbb{E}^2 \times \mathbb{R}, \quad \mathbb{H}^2(\bar{c}) \times \mathbb{R}$$

are 3-dimensional cosymplectic space forms.

2.5. **$\mathcal{D}$-homothetic deformations.** On an almost cosymplectic manifold M, we define a subring $\mathcal{R}_\eta(M)$ of the commutative ring $C^\infty(M)$ of all smooth functions on M by

$$\mathcal{R}_\eta(M) = \{f \in C^\infty(M) \mid df \wedge \eta = 0\}.$$

For any positive constant t and $\beta \in \mathcal{R}_\eta(M)$ satisfying $\beta \neq 0$, we deform the structure tensors (φ, ξ, η, g) as

$$\tilde{\varphi} = \varphi, \quad \tilde{\xi} = \frac{1}{\beta}\xi, \quad \tilde{\eta} = \beta\eta, \quad \tilde{g} = tg + (\beta^2 - t)\eta \otimes \eta. \tag{5}$$

Then the resulting structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is also almost cosymplectic with fundamental 2-form $\tilde{\Phi} = t\Phi$. This new structure is called a *$\mathcal{D}$-homothetic deform* of the original structure. The procedure $(\varphi, \xi, \eta, g) \longmapsto (\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is called a *$\mathcal{D}$-homothetic deformation*. If two almost cosymplectic structures are related by a *$\mathcal{D}$-homothetic deformation*, then those structures are said to be *$\mathcal{D}$-homothetically equivalent*.

Dacko and Olszak obtained the following result.

Proposition 3 ([9])**.** *The Levi-Civita connection* $\tilde{\nabla}$ *of the* $\mathcal{D}$*-homothetic deform is given by*

$$\tilde{\nabla}_X Y = \nabla_X Y - \frac{\beta^2 - t}{\beta^2} g(AX, Y)\xi + \frac{\xi(\beta)}{\beta}\eta(X)\eta(Y)\xi.$$

The endomorphisms $\tilde{h} = \pounds_{\tilde{\xi}}\tilde{\varphi}/2$ *and* $\tilde{A} = -\tilde{\nabla}\tilde{\xi}$ *are given by*

$$\tilde{h} = \frac{1}{\beta}h, \quad \tilde{A} = \frac{1}{\beta}A.$$

The Riemannian curvature $\tilde{R}$ *of the* $\mathcal{D}$*-homothetic deform satisfies*

$$\tilde{R}(X,Y)\tilde{\xi} = \frac{1}{\beta}R(X,Y)\xi + \frac{\xi(\beta)}{\beta^2}(\eta(X)AY - \eta(Y)AX). \tag{6}$$

From this proposition we can deduce that the characteristic Jacobi operator $\tilde{\ell}$ of $(M, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is

$$\tilde{\ell}(X) = \frac{1}{\beta}\left(\ell(X) - \frac{\xi(\beta)}{\beta}AX\right).$$

3. Generalized almost cosymplectic (κ, μ, ν)-spaces

In this section, we collect fundamental facts on generalized almost cosymplectic (κ, μ, ν)-spaces.

3.1. H-almost cosymplectic manifolds.

Here we introduce the following notion.

Definition 4. An almost cosymplectic manifold M is said to be an *H-almost cosymplectic manifold* if its characteristic vector field ξ is a harmonic vector field.

Perrone clarified relations between harmonicity and minimality of unit vector fields ([20, Theorem 4.3], [21, Theorem 4.2], *cf.* [12]):

Theorem 3. *Let M be an almost cosymplectic* 3*-manifold. Then ξ is a minimal unit vector field if and only if it is a harmonic unit vector field.*

On the other hand, on an almost cosymplectic 3-manifold $(M, \varphi, \xi, \eta, g)$, the minimality of ξ is characterized in terms of Ricci operator as follows.

Theorem 4 ([21, 12])**.** *On an almost cosymplectic* 3*-manifold M, ξ is a minimal unit vector field if and only if ξ is an eigenvector field of Q.*

Perrone proved the following fact ([22, Remark 3.1]):

Proposition 4. *Let $(M, \varphi, \xi, \eta, g)$ be an almost cosymplectic* 3*-manifold with minimal ξ. Then for any positive constant t and $\beta \in \mathcal{R}_\eta(M)$ with $\beta \neq 0$, the $\mathcal{D}$-homothetic deform $(M, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ defined by* (5) *has minimal characteristic vector field $\tilde{\xi}$.*

3.2. Generalized almost cosymplectic (κ, μ, ν)-spaces.

Next, we recall the notion of almost cosymplectic (κ, μ, ν)-space.

Definition 5. An almost cosymplectic manifold M is said to be a *generalized almost cosymplectic (κ, μ, ν)-space* if

$$\begin{aligned} R(X,Y)\xi = \ & \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \\ & + \nu(\eta(Y)\varphi hX - \eta(X)\varphi hY). \end{aligned} \tag{7}$$

for some smooth functions κ, μ and ν. Generalized almost cosymplectic $(\kappa, \mu, 0)$-spaces are called *generalized almost cosymplectic* (κ, μ)-*spaces.*

Definition 6. Let M be a generalized almost cosymplectic (κ, μ, ν)-space. If all the functions κ, μ and ν are constants, then M is called an *almost cosymplectic* (κ, μ, ν)-*space.* A generalized almost cosymplectic (κ, μ, ν)-space is said to be *proper* if $|d\kappa|^2 + |d\mu|^2 + |d\nu|^2 \neq 0$.

Remark 1. Generalized almost cosymplectic (κ, μ, ν)-space in this paper are called *almost cosymplectic* (κ, μ, ν)-space in [22]. On the other hand, an almost cosymplectic (κ, μ, ν)-space in the sense of Dacko and Olszak [9] is a generalized almost cosymplectic (κ, μ, ν)-space in the sense of the present paper satisfying the additional condition:

$$d\kappa \wedge \eta = 0, \quad d\mu \wedge \eta = 0, \quad \text{and} \quad d\nu \wedge \eta = 0. \tag{8}$$

Dacko and Olszak [9] showed that if the dimension of a generalized almost cosymplectic (κ, μ, ν)-space is greater than 3, then κ, μ and ν satisfy this additional condition. In this paper, we do not require this condition for 3-dimensional generalized almost cosymplectic (κ, μ, ν)-spaces.

The endomorphism field A of a generalized almost cosymplectic (κ, μ, ν)-space M satisfies (see [9]):

$$A^2 X = -\kappa(X - \eta(X)\xi). \tag{9}$$

This equation implies the following fact.

Proposition 5. *A* 3*-dimensional generalized almost cosymplectic* $(0, \mu, \nu)$*-space is cosymplectic.*

Proof. If $\kappa = 0$, then $A^2 = 0$ from (9). Since A is self-adjoint, for any non-zero vector field X, we have

$$0 = g(A^2 X, X) = g(AX, AX) = \|AX\|^2.$$

This implies that $A = 0$, *i.e.*, $\nabla \xi = 0$. By Theorem 2, M is cosymplectic. ☐

We notice that $\kappa \leq 0$. Dacko and Olszak showed that if $\kappa = 0$ at some point, then $\kappa = 0$ on whole M [9, Lemma 3].

The generalized (κ, μ, ν)-property is invariant under $\mathcal{D}$-homothetic deformation. Indeed, from (6) and (7), one can check the following theorem.

Theorem 5 ([9])**.** *Assume that* $(M, \varphi, \xi, \eta, g)$ *be a generalized almost cosymplectic* (κ, μ, ν)*-space. Then its* $\mathcal{D}$*-homothetic deform* $(M, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ *defined by* (5) *is a generalized almost cosymplectic* $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})$*-space with*

$$\tilde{\kappa} = \frac{\kappa}{\beta^2}, \quad \tilde{\mu} = \frac{\mu}{\beta}, \quad \tilde{\nu} = \frac{\nu\beta - \xi(\beta)}{\beta^2}.$$

4. Almost cosymplectic 3-manifolds

Hereafter we concentrate on almost cosymplectic 3-manifolds.

4.1. **Cosymplectic 3-manifolds.** Cosymplectic 3-manifolds have some particular geometric properties.

Proposition 6 ([17])**.** *An almost contact metric* 3*-manifold* M *is cosymplectic if and only if* ξ *is parallel.*

Proposition 7. *Let* M *be a cosymplectic* 3*-manifold. Then the Ricci operator* Q *of* M *has the form*

$$Q = \frac{r}{2}(\mathrm{I} - \eta \otimes \xi).$$

The principal Ricci curvatures are $r/2$*,* $r/2$ *and* 0*. The Ricci operator* Q *commutes with* φ*. The Riemannian curvature* R *satisfies*

$$R(X, Y)\xi = 0$$

for any vector fields X *and* Y*. Hence, the characteristic Jacobi operator vanishes. The holomorphic sectional curvature is* $H = r/2$*. For a unit vector* X *in* T_xM *such that* $\eta(X) = 0$*, then the sectional curvatures* $K(X \wedge \xi)$ *is always* 0*.*

This proposition tells us differences between Sasakian manifolds, Kenmotsu manifolds and cosymplectic manifolds.

Structure	Sasakian	Cosymplectic	Kenmotsu
Characteristic Jacobi operator	$\ell = -\varphi^2$	$\ell = 0$	$\ell = \varphi^2$

Corollary 1. *The following properties of a cosymplectic* 3*-manifold* M *are mutually equivalent.*

- M *is locally symmetric,*
- *the scalar curvature* r *is constant,*
- *the holomorphic sectional curvature* H *is constant,*
- M *is locally isomorphic to a Riemannian product* $\overline{M}(\bar{c}) \times \mathbb{R}$*, where* $\overline{M}(\bar{c})$ *is a* 2*-dimensional Riemannian manifold of constant curvature* $\bar{c}$*.*

Corollary 2. *A cosymplectic* 3*-manifold* M *is of constant curvature if and only if it is locally isomorphic to Euclidean* 3*-space* $\mathbb{E}^3 = \mathbb{E}^2 \times \mathbb{R}$*.*

4.2. **The Levi-Civita connection.** Let M be an almost cosymplectic 3-manifold. Denote by $\mathcal{U}_1$ the open subset of M consisting of points x such that $h \neq 0$ around x. Next, let $\mathcal{U}_0$ be the open subset of M consisting of points $x \in M$ such that $h = 0$ around x. Since h is smooth, $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_0$ is an open dense subset of M. Thus any property satisfied in $\mathcal{U}$ is also satisfied in the whole M. For any point $x \in \mathcal{U}$, there exists a local orthonormal frame field $\mathcal{E} = \{e_1, e_2 = \varphi e_2, e_3 = \xi\}$ around x, where e_1 is an eigenvector field of h.

Lemma 1 (*cf.* [19, 21])**.** *Let* M *be an almost cosymplectic* 3*-manifold. Then there exists a local orthonormal frame field* $\mathcal{E} = \{e_1, e_2, e_3\}$ *on* $\mathcal{U}$ *such that*

$$he_1 = \lambda e_1, \quad e_2 = \varphi e_1, \quad e_3 = \xi$$

for some locally defined smooth function λ*. The Levi-Civita connection* ∇ *is described as*

$$\nabla_{e_1} e_1 = \frac{1}{2\lambda}(e_2(\lambda) + \sigma(e_1))e_2, \quad \nabla_{e_1} e_2 = -\frac{1}{2\lambda}(e_2(\lambda) + \sigma(e_1))e_1 + \lambda\xi, \quad \nabla_{e_1} e_3 = -\lambda e_2,$$

$$\nabla_{e_2} e_1 = -\frac{1}{2\lambda}(e_1(\lambda) + \sigma(e_2))e_2 + \lambda\xi, \quad \nabla_{e_2} e_2 = \frac{1}{2\lambda}(e_1(\lambda) + \sigma(e_2))e_1, \quad \nabla_{e_2} e_3 = -\lambda e_1, \tag{10}$$

$$\nabla_{e_3} e_1 = \alpha e_2, \quad \nabla_{e_3} e_2 = -\alpha e_1, \quad \nabla_{e_3} e_3 = 0,$$

where σ *is the* 1*-form metrically equivalent to* $Q\xi$*, that is,*

$$\sigma = g(Q\xi, \cdot) = S(\xi, \cdot).$$

The commutation relations are

$$(11) \qquad \begin{aligned} [e_1, e_2] =& -\frac{1}{2\lambda}\left\{(e_2(\lambda) + \sigma(e_1))e_1 - (e_1(\lambda) + \sigma(e_2))e_2\right\}, \\ [e_2, e_3] =& (\alpha - \lambda)e_1, \quad [e_3, e_1] = (\alpha + \lambda)e_2. \end{aligned}$$

The Jacobi identity is described as

$$(12) \qquad -e_1(\alpha - \lambda) + \xi(p) + q(\alpha - \lambda) = 0, \quad -e_2(\alpha + \lambda) - \xi(q) + p(\alpha + \lambda) = 0,$$

where

$$p = \frac{1}{2\lambda}(e_2(\lambda) + \sigma(e_1)), \quad q = \frac{1}{2\lambda}(e_1(\lambda) + \sigma(e_2)).$$

4.3. **The Riemannian curvature.** The Riemannian curvature R is computed by the table of Levi-Civita connection in Lemma 1:

$$\begin{aligned} &R(e_1, e_2)e_1 = -\left(\frac{r}{2} + 2\lambda^2\right)e_2 - \sigma(e_2)\xi, \quad R(e_1, e_2)e_2 = \left(\frac{r}{2} + 2\lambda^2\right)e_1 + \sigma(e_1)\xi, \\ &R(e_1, e_2)e_3 = \sigma(e_2)e_1 - \sigma(e_1)e_2, \qquad R(e_2, e_3)e_1 = \sigma(e_1)e_2 - \xi(\lambda)\xi, \\ &R(e_2, e_3)e_2 = -\sigma(e_1)e_1 + (\lambda^2 - 2\alpha\lambda)\xi, \qquad R(e_2, e_3)e_3 = \xi(\lambda)e_1 - (\lambda^2 - 2\alpha\lambda)e_2, \\ &R(e_3, e_1)e_1 = \sigma(e_2)e_2 - (\lambda^2 + 2\alpha\lambda)\xi, \qquad R(e_3, e_1)e_2 = -\sigma(e_2)e_1 + \xi(\lambda)\xi, \\ &R(e_3, e_1)e_3 = (\lambda^2 + 2\alpha\lambda)e_1 - \xi(\lambda)e_2, \end{aligned}$$

where

$$\begin{aligned} \sigma(e_1) =& e_2(\alpha) + \xi(q) - (\alpha - \lambda)p = -e_2(\lambda) + 2p\lambda, \\ \sigma(e_2) =& -e_1(\alpha) + \xi(p) + (\alpha + \lambda)q = -e_1(\lambda) + 2q\lambda. \end{aligned}$$

The sectional curvatures $K_{ij} = K(e_i \wedge e_j)$ are given by

$$\begin{aligned} K_{12} = \lambda^2 + e_1(q) + e_2(p) - p^2 - q^2 = \frac{r}{2} + 2\lambda^2, \\ K_{13} = -\lambda(\lambda + 2\alpha), \quad K_{23} = -\lambda(\lambda - 2\alpha). \end{aligned}$$

4.4. **The Ricci operator.** The Ricci operator Q is described as [21]:

$$\begin{aligned} Qe_1 =& \left(\frac{r}{2} + \lambda^2 - 2\alpha\lambda\right)e_1 + \xi(\lambda)e_2 + \sigma(e_1)\xi, \\ Qe_2 =& \xi(\lambda)e_1 + \left(\frac{r}{2} + \lambda^2 + 2\alpha\lambda\right)e_2 + \sigma(e_2)\xi, \\ Qe_3 =& \sigma(e_1)e_1 + \sigma(e_2)e_2 - 2\lambda^2\xi. \end{aligned}$$

Note that the scalar curvature r is computed as

$$(13) \qquad r = 2\{e_1(q) + e_2(p) - p^2 - q^2 - \lambda^2\}.$$

The characteristic Jacobi operator ℓ is given by

$$\ell e_1 = -(\lambda^2 + 2\alpha\lambda)e_1 + \xi(\lambda)e_2, \quad \ell e_2 = \xi(\lambda)e_1 - (\lambda^2 - 2\alpha\lambda)e_2.$$

From Lemma 1 we deduce that φ commutes with Q if and only if

$$(14) \qquad \xi(\lambda) = 0, \quad \alpha = 0, \quad \sigma(e_1) = \sigma(e_2) = 0.$$

It should be remarked that the commutativity $Q\varphi = \varphi Q$ implies the H-almost cosymplectic property.

Here we recall the following result:

Theorem 6 ([4, 8])**.** *Let M be an almost cosymplectic 3-manifold. If M satisfies $Q\varphi = \varphi Q$ if and only if M is either cosymplectic or locally isomorphic to the Minkowski motion group $\mathrm{E}_{1,1}$ equipped with a left invariant almost cosymplectic structure. In the latter case, the eigenvalue λ of h is constant.*

The left invariant almost cosymplectic structure of $\mathrm{E}_{1,1}$ appeared in Theorem 6 will be described explicitly in Example 3 of Section 6 (with $a = b$, *i.e.*, $c_1 = -c_2$). Note that Dacko [8] showed that 3-dimensional almost cosymplectic $(\kappa, 0)$-spaces with $\kappa < 0$ are locally homogeneous and locally isomorphic to $\mathrm{E}_{1,1}$.

4.5. **Curvatures of H-almost cosymplectic 3-manifolds.** The class of H-almost cosymplectic 3-manifolds is characterized in terms of Riemannian curvature as follows (see [22, Theorem 4.1, Proposition 4.3]):

Theorem 7 ([21])**.** *Let M be an almost cosymplectic 3-manifold. If M is a generalized (κ, μ, ν)-space, then ξ is a minimal unit vector field. Conversely, if ξ is a minimal unit vector field, then M satisfies the generalized (κ, μ, ν)-condition on an open dense subset. In such a case we have*

$$Q\xi = -2\lambda^2\xi, \quad \kappa = -\lambda^2, \quad \mathrm{tr}\,(h^2) = 2\lambda^2.$$

Moreover λ satisfies $d\lambda \wedge \eta = 0$, that is, $X(\lambda) = 0$ for any vector field X orthogonal to ξ. The Ricci operator has the form

$$Q = \left(\frac{r}{2} + \lambda^2\right) \mathrm{I} - \left(\frac{r}{2} + 3\lambda^2\right) \eta \otimes \xi + \mu h + \nu \varphi h.$$

Let us take a local orthonormal field $\{e_1, e_2, e_3\}$ on a 3-dimensional generalized almost cosymplectic (κ, μ, ν)-space as in Lemma 1, we have

$$Q\xi = -2\lambda^2\xi, \quad \kappa = -\lambda^2, \quad \mu = -2\alpha, \quad \lambda\nu = \xi(\lambda).$$

By Theorem 7, the eigenvalue λ satisfies $e_1(\lambda) = e_2(\lambda) = 0$. Hence $p = q = 0$. Thus from (13), the scalar curvature is $r = -\mathrm{tr}(h^2) = -2\lambda^2$. Hence the Ricci operator has the form

$$\begin{aligned} Qe_1 =& -2\alpha\lambda e_1 + \xi(\lambda)e_2, \\ Qe_2 =& \xi(\lambda)e_1 + 2\alpha\lambda e_2, \\ Qe_3 =& -2\lambda^2 e_3. \end{aligned}$$

The Riemannian curvature R of a generalized almost cosymplectic (κ, μ, ν)-space is described as follows ([3, Theorem 3.25]).

$$R = \left(\frac{r}{2} - 2\kappa\right) R_1 + \left(\frac{r}{2} - 3\kappa\right) R_3 + \mu R_4 + \nu R_7,$$

where

$$\begin{aligned} R_3(X, Y)Z =& \eta(Z)\eta(X)Y - \eta(Y)\eta(Z)X + \{g(Z, X)\eta(Y) - g(Y, Z)\eta(X)\}\xi, \\ R_4(X, Y)Z =& g(Y, Z)hX - g(Z, X)hY + g(hY, Z)X - g(Z, hX)Y, \\ R_7(X, Y)Z =& g(Y, Z)\varphi hX - g(Z, X)\varphi hY + g(\varphi hY, Z)X - g(Z, \varphi hX)Y. \end{aligned}$$

Combining this curvature formula and the formula of Q above, we obtain the following curvature formula for 3-dimensional generalized almost cosymplectic (κ,μ,ν)-spaces:

$$R = -\kappa R_1 - 2\kappa R_3 + \mu R_4 + \nu R_7. \tag{15}$$

Remark 2. Carriazo and Martín-Molina showed the curvature formlula (15) under the assumption

$$d\kappa \wedge \eta = 0, \quad d\mu \wedge \eta = 0, \quad d\nu \wedge \eta = 0.$$

See [3, Corollary 3.28].

As we have mentioned before, for ξ, to be a harmonic map is stronger than to be a harmonic vector field.

Proposition 8. *Let M be almost cosymplectic 3-manifold. Then its characteristic vector field is a harmonic map into the unit tangent sphere bundle UM of M if and only if $\sigma(X) = 0$ for all vector fields X orthogonal to ξ and M satisfies the generalized $(\kappa,\mu,0)$-condition on an open dense subset of M.*

From this we notice that

$$Q\varphi = \varphi Q \implies \xi \text{ is a harmonic map } \implies \xi \text{ is minimal.}$$

Now let M be an almost cosymplectic 3-manifold whose characteristic vector field ξ is a unit minimal vector field. Then M satisfies the generalized (κ,μ,ν)-condition on an open dense subset of M. Take a local orthonormal frame field $\{e_1, e_2, e_3\}$ as in Lemma 1 on an open subset on which $h \neq 0$. Then from Theorem 7, $\kappa = -\lambda^2 < 0$ and $d\lambda \wedge \eta = 0$. We perform a local $\mathcal{D}$-homothetic deformation (5) with $t = 1$ and $\beta = \lambda$. The resulting structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ satisfies the generalized almost cosymplectic $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})$-structure such that

$$\tilde{\kappa} = -1, \quad \tilde{\mu} = \frac{\mu}{\lambda} = \pm\frac{\mu}{\sqrt{-\kappa}}, \quad \tilde{\nu} = 0.$$

One can see that $\{\tilde{e}_1 = e_1, \tilde{e}_2 = e_2, \tilde{e}_3 = \tilde{\xi}\}$ is a local orthonormal frame field of $(M, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ and satisfies

$$\tilde{h}\tilde{e}_1 = \tilde{e}_1, \quad \tilde{h}\tilde{e}_2 = -\tilde{e}_2, \quad \tilde{h}\tilde{e}_3 = 0.$$

This procedure is valid for any 3-dimensional generalized almost cosymplectic (κ,μ,ν)-space with $\kappa < 0$.

Theorem 8. *Let M be a 3-dimensional generalized almost cosymplectic (κ,μ,ν)-space with $\kappa < 0$. Then there exists a $\mathcal{D}$-homothetic deformation of M to a generalized almost cosymplectic $(-1, \mu/\sqrt{-\kappa}, 0)$-space.*

In [9, 10], Dacko and Olszak constructed a local model $N^3(\mu) = (-\varepsilon, \varepsilon) \times U \subset \mathbb{R}^3$ for generalized almost cosymplectic $(-1,\mu,0)$-spaces satisfying $d\mu \wedge \eta = 0$. In particular, $N^3(\mu)$ for constant μ is extended to the whole $\mathbb{R}^3$. Those model spaces are realized as universal covering $\widetilde{\mathrm{E}}_2$ of the Euclidean motion group E_2 for $|\mu| > 2$, Heisenberg group for $|\mu| = 2$ and, Minkowski motion group $\mathrm{E}_{1,1}$ for $|\mu| < 2$, respectively. We discuss again these Lie groups in Section 6.

5. Pseudo-parallelism of the characteristic Jacobi operator

In this section, we study almost cosymplectic 3-manifolds with pseudo-parallel characteristic Jacobi operator.

5.1. **Almost cosymplectic 3-manifolds with vanishing ℓ.** From Lemma 1 we obtain

$$(\nabla_{e_1}\ell)e_3 = \lambda\ell e_2, \quad (\nabla_{e_2}\ell)e_3 = \lambda\ell e_1.$$

These equations imply the following discouraging fact.

Proposition 9. *Let M be an almost cosymplectic 3-manifold. Then its characteristic Jacobi operator is parallel if and only if $\ell = 0$ and the structure is cosymplectic.*

Motived from this fact, we study semi-parallelism and pseudo-parallelism of the characteristic Jacobi operator.

5.2. **Semi-parallelism.** First, the derivative of ℓ by $R(X,Y)$ is defined by

$$(R(X,Y)\cdot\ell)Z = R(X,Y)\ell Z - \ell R(X,Y)Z,$$

and we calculate $R(X,Y)\cdot\ell$ as follows:

$$\begin{aligned}
(R(e_1,e_2)\cdot\ell)e_1 &= \left(\frac{r}{2}+2\lambda^2\right)\{2\xi(\lambda)e_1 + 4\alpha\lambda e_2\} + \{(\lambda^2+2\alpha\lambda)\sigma(e_2)+\xi(\lambda)\sigma(e_1)\}\xi,\\
(R(e_1,e_2)\cdot\ell)e_2 &= (\frac{r}{2}+2\lambda^2)\{4\alpha\lambda e_1 - 2\xi(\lambda)e_2\} - \{\xi(\lambda)\sigma(e_2)+(\lambda^2-2\alpha\lambda)\sigma(e_1)\}\xi,\\
(R(e_1,e_2)\cdot\ell)e_3 &= \{(\lambda^2+2\alpha\lambda)\sigma(e_2)+\xi(\lambda)\sigma(e_1)\}e_1 - \{\xi(\lambda)\sigma(e_2)+(\lambda^2-2\alpha\lambda)\sigma(e_1)\}e_2,\\
(R(e_2,e_3)\cdot\ell)e_1 &= -2\sigma(e_1)\xi(\lambda)e_1 - 4\alpha\lambda\sigma(e_1)e_2 + 2\lambda^2\xi(\lambda)\xi,\\
(R(e_2,e_3)\cdot\ell)e_2 &= -4\alpha\lambda\sigma(e_1)e_1 + 2\sigma(e_1)\xi(\lambda)e_2 - \{\xi(\lambda)^2+(\lambda^2-2\alpha\lambda)^2\}\xi,\\
(R(e_2,e_3)\cdot\ell)e_3 &= 2\lambda^2\xi(\lambda)e_1 - \{\xi(\lambda)^2+(\lambda^2-2\alpha\lambda)^2\}e_2,\\
(R(e_3,e_1)\cdot\ell)e_1 &= -2\sigma(e_2)\xi(\lambda)e_1 - 4\alpha\lambda\sigma(e_2)\lambda e_2 + \{\xi(\lambda)^2+(\lambda^2+2\alpha\lambda)^2\}\xi,\\
(R(e_3,e_1)\cdot\ell)e_2 &= -4\alpha\lambda\sigma(e_2)e_1 + 2\sigma(e_2)\xi(\lambda)e_2 - 2\lambda^2\xi(\lambda)\xi,\\
(R(e_3,e_1)\cdot\ell)e_3 &= \{(\lambda^2+2\alpha\lambda)^2+\xi(\lambda)^2\}e_1 - 2\lambda^2\xi(\lambda)e_2.
\end{aligned}$$

From $(R(e_2,e_3)\cdot\ell)e_3 = 0$ and $(R(e_3,e_1)\cdot\ell)e_1 = 0$, we get $\lambda = 0$ and hence we, have the following classification of almost cosympectic 3-manifolds with semi-parallel ℓ.

Proposition 10. *An almost cosymplectic 3-manifolds M with semi-parallel characteristic Jacobi operator is a cosympletic manifold and $\ell = 0$.*

5.3. **Pseudo-parallelism.** It turned out that semi-parallelism is still a strong restriction for ℓ. Next, we consider pseudo-parallelism. The derivative $R_1\cdot\ell$ is given by

$$\begin{aligned}
(R_1(X,Y)\cdot\ell)Z =&((X\wedge Y)\cdot\ell)Z = (X\wedge Y)\ell Z - \ell((X\wedge Y)Z)\\
=&g(Y,\ell Z)X - g(\ell Z,X)Y - g(Y,Z)\ell X + g(Z,X)\ell Y.
\end{aligned}$$

and using this we calculate $(X \wedge Y) \cdot \ell$ as follows:

$$\begin{aligned}
((e_1 \wedge e_2) \cdot \ell)e_1 &= 2\xi(\lambda)e_1 + 4\alpha\lambda e_2,\\
((e_1 \wedge e_2) \cdot \ell)e_2 &= 4\alpha\lambda e_1 - 2\xi(\lambda)e_2,\\
((e_1 \wedge e_2) \cdot \ell)e_3 &= 0,\\
((e_2 \wedge e_3) \cdot \ell)e_1 &= -\xi(\lambda)\xi,\\
((e_2 \wedge e_3) \cdot \ell)e_2 &= (\lambda^2 - 2\alpha\lambda)\xi,\\
((e_2 \wedge e_3) \cdot \ell)e_3 &= -\xi(\lambda)e_1 + (\lambda^2 - 2\alpha\lambda)e_2,\\
((e_3 \wedge e_1) \cdot \ell)e_1 &= -(\lambda^2 + 2\alpha\lambda)\xi,\\
((e_3 \wedge e_1) \cdot \ell)e_2 &= \xi(\lambda)\xi,\\
((e_3 \wedge e_1) \cdot \ell)e_3 &= -(\lambda^2 + 2\alpha\lambda)e_1 + \xi(\lambda)e_2.
\end{aligned}$$

Now we suppose that $\lambda \neq 0$ and M satisfies the pseudo-parallel condition:

$$R(X,Y) \cdot \ell = L(X \wedge Y) \cdot \ell, \quad X, Y \in \mathfrak{X}(M). \tag{16}$$

From the list above, $R(e_3, e_1) \cdot \ell = L(e_3 \wedge e_1) \cdot \ell$ holds if and only if

$$\alpha\lambda\sigma(e_2) = 0, \quad \sigma(e_2)\xi(\lambda) = 0, \quad (L + 2\lambda^2)\xi(\lambda) = 0, \tag{17}$$

and

$$(L + \lambda^2 + 2\alpha\lambda)(\lambda^2 + 2\alpha\lambda) + \xi(\lambda)^2 = 0. \tag{18}$$

Next, $R(e_2, e_3) \cdot \ell = L(e_2 \wedge e_3) \cdot \ell$ holds if and only if

$$\alpha\lambda\sigma(e_1) = 0, \quad \sigma(e_1)\xi(\lambda) = 0, \quad (L + 2\lambda^2)\xi(\lambda) = 0, \tag{19}$$

and

$$(L + \lambda^2 - 2\alpha\lambda)(\lambda^2 - 2\alpha\lambda) + \xi(\lambda)^2 = 0. \tag{20}$$

Finally, $R(e_1, e_2) \cdot \ell = L(e_1 \wedge e_2) \cdot \ell$ holds if and only if

$$\xi(\lambda)\left\{L - \left(\frac{r}{2} + 2\lambda^2\right)\right\} = 0, \quad \alpha\lambda\left\{L - \left(\frac{r}{2} + 2\lambda^2\right)\right\} = 0 \tag{21}$$

and

$$(\lambda^2 + 2\alpha\lambda)\sigma(e_2) + \xi(\lambda)\sigma(e_1) = 0, \quad (\lambda^2 - 2\alpha\lambda)\sigma(e_1) + \xi(\lambda)\sigma(e_2) = 0. \tag{22}$$

From the above equations we have our main theorem.

Theorem 9. *Let M be almost cosymplectic 3-manifold with pseudo-parallel characteristic Jacobi operator. Then M is cosymplectic with $L = 0$ or locally isomorphic to the Minkowski motion group $\mathrm{E}_{1,1}$ equipped with a left invariant non-normal almost cosymplectic $(\kappa, 0)$-structure and $L = \kappa = -\mathrm{tr}(h^2)/2$ is a negative constant.*

Proof. Let us consider the open subsets

$$\begin{aligned}
\mathcal{U}_0 =& \{x \in M \mid h = 0 \text{ in a neighborhood of } x\},\\
\mathcal{U}_1 =& \{x \in M \mid h \neq 0 \text{ in a neighborhood of } x\}.
\end{aligned}$$

Suppose that $M = \mathcal{U}_0$ then M is cosymplectic. Obviously, when M is cosymplectic, ℓ is pseudo-parallel with $L = 0$ (The case (1)). Hereafter we assume that $\mathcal{U}_1$ is non-empty. Then we can take a local orthonormal frame field $\{e_1, e_2, e_3\}$ as in Lemma 1. To analyze the system of pseudo-parallelism for ℓ, we set

$$\mathcal{U}_2 = \{x \in \mathcal{U}_1 \mid \xi(\lambda) = 0 \text{ in a neighborhood of } x\},$$
$$\mathcal{U}_3 = \{x \in \mathcal{U}_1 \mid \xi(\lambda) \neq 0 \text{ in a neighborhood of } x\},$$

where $\mathcal{U}_2 \cup \mathcal{U}_3$ is open and dense in the closure of $\mathcal{U}_1$.

(1) In $\mathcal{U}_2$, the equation $R(e_3, e_1) \cdot \ell = L(e_3 \wedge e_1) \cdot R$ is reduced to

$$\alpha\sigma(e_2) = 0, \quad (L + \lambda^2 + 2\alpha\lambda)(\lambda^2 + 2\alpha\lambda) = 0.$$

the equation $R(e_2, e_3) \cdot \ell = L(e_2 \wedge e_3) \cdot R$ is reduced to

$$\alpha\sigma(e_1) = 0, \quad (L + \lambda^2 - 2\alpha\lambda)(\lambda^2 - 2\alpha\lambda) = 0.$$

Finally, the equation $R(e_1, e_2) \cdot \ell = L(e_1 \wedge e_2) \cdot R$ is reduced to

$$\alpha \left\{ L - \left(\frac{r}{2} + 2\lambda^2 \right) \right\} = 0, \quad (\lambda - 2\alpha)\sigma(e_1) = (\lambda + 2\alpha)\sigma(e_2) = 0.$$

We consider, in addition, open subsets of $\mathcal{U}_2$;

$$\mathcal{U}_4 = \{x \in \mathcal{U}_2 \mid \alpha = 0 \text{ in a neighborhood of } x\},$$
$$\mathcal{U}_5 = \{x \in \mathcal{U}_2 \mid \alpha \neq 0 \text{ in a neighborhood of } x\}.$$

Here $\mathcal{U}_4 \cup \mathcal{U}_5$ is open and dense in the closure of $\mathcal{U}_2$.

- In $\mathcal{U}_4$, the pseudo-parallel condition is the system:

  $$L = -\lambda^2, \quad \sigma(e_1) = \sigma(e_2) = 0.$$

  In this case, $\mathcal{U}_4$ is H-almost cosymplectic. Since we assumed that $\xi(\lambda) = 0$ and $\alpha = 0$, $\mathcal{U}_4$ is locally a generalized almost cosymplectic $(\kappa, 0)$-space. Moreover, from (14), φ commutes with Q. Hence Theorem 6 implies that $\mathcal{U}_4$ is locally isomorphic to $\mathrm{E}_{1,1}$ (The case (2)).
- In $\mathcal{U}_5$, we have the system

$$\sigma(e_1) = \sigma(e_2) = 0, \quad (L + \lambda^2 + 2\alpha\lambda)(\lambda + 2\alpha) = 0, \quad (L + \lambda^2 - 2\alpha\lambda)(\lambda - 2\alpha) = 0, \quad L = \frac{r}{2} + 2\lambda^2.$$

  From these we deduce that $\mathcal{U}_5$ is H-almost cosymplectic and

  $$L = -2\lambda^2, \quad r = -8\lambda^2, \quad \lambda = \pm 2\alpha.$$

  On the other hand, from (13) we have $r = -2\lambda^2$. Hence, we have $\lambda = 0$ and this is a contradiction. Hence $\mathcal{U}_5$ is empty.

Thus $\mathcal{U}_4$ is open and dense in the closure of $\mathcal{U}_2$.

(2) In $\mathcal{U}_3$, the pseudo-parallel condition is the system:

$$\sigma(e_1) = \sigma(e_2) = 0, \quad L = -2\lambda^2 = \frac{r}{2} + 2\lambda^2$$

and

$$(L + \lambda^2 + 2\alpha\lambda)(\lambda^2 + 2\alpha\lambda) + \xi(\lambda)^2 = 0, \quad (L + \lambda^2 - 2\alpha\lambda)(\lambda^2 - 2\alpha\lambda) + \xi(\lambda)^2 = 0.$$

Hence $\mathcal{U}_3$ is H-almost cosymplectic and $L = -2\lambda^2$, $r = -8\lambda^2$. From (13) we have $r = -2\lambda^2$, hence, we have $\lambda = 0$ and it is contradiction. Thus $\mathcal{U}_3$ is empty. Henceforth $\mathcal{U}_2$ is open and dense in the closure of $\mathcal{U}_1$. Thus $\mathcal{U}_1$ is locally isomorphic to $\mathrm{E}_{1,1}$. □

Concerning on the harmonicity of ξ, we have the following corollary.

Corollary 3. *Let M be an almost cosymplectic 3-manifold with pseudo-parallel characteristic Jacobi operator, then its characteristic vector field ξ is a harmonic map into UM.*

The Minkowski motion group $\mathrm{E}_{1,1}$ mentioned in Theorem 6 is a standard example of homogeneous almost cosymplectic 3-manifolds. In the next section, we give explicit models of homogeneous almost cosymplectic 3-manifolds, especially $\mathrm{E}_{1,1}$.

6. Homogeneous almost cosymplectic 3-manifolds

6.1. **Simply connected homogeneous almost cosymplectic 3-manifolds.** In this section, we study the characteristic Jacobi operator of homogeneous almost cosymplectic 3-manifolds.

Definition 7 (*cf.* [20])**.** An almost contact metric manifold $M = (M, \varphi, \xi, \eta, g)$ is said to be a *homogeneous almost contact metric manifold* if there exists a Lie group G of isometries which acts transitively on M such that every element f of G preserves η, that is $f^*\eta = \eta$.

Perrone obtained the following classification.

Theorem 10 ([20])**.** *Let M be a simply connected homogeneous cosymplectic 3-manifold, then M is either*

- *M is one of the product Riemannian symmetric spaces*
$$\mathbb{S}^2(\bar{c}) \times \mathbb{R}, \quad \mathbb{H}^2(\bar{c}) \times \mathbb{R},$$
where $\mathbb{S}^2(\bar{c})$ and $\mathbb{H}^2(\bar{c})$ are sphere of curvature $\bar{c} > 0$ and hyperbolic plane of curvature $\bar{c} < 0$ or
- *M itself is a Lie group G equipped with left invariant almost cosymplectic structure.*

6.2. **Unimodular Lie groups.** Let G be a 3-dimensional unimodular Lie group with a left invariant metric $\langle \cdot, \cdot \rangle$. Then there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of the Lie algebra $\mathfrak{g}$ such that

$$[e_1, e_2] = c_3 e_3, \quad [e_2, e_3] = c_1 e_1, \quad [e_3, e_1] = c_2 e_2, \qquad c_i \in \mathbb{R}. \tag{23}$$

Three-dimensional unimodular Lie groups are classified by Milnor as [16] follows:

Signature of (c_1, c_2, c_3)	Simply connected Lie group	Property
$(+,+,+)$	SU_2	compact and simple
$(-,-,+)$ or $(-,+,+)$	$\widetilde{\mathrm{SL}}_2\mathbb{R}$	non-compact and simple
$(+,+,0)$	$\widetilde{\mathrm{E}}_2$	solvable
$(-,+,0)$	$\mathrm{E}_{1,1}$	solvable
$(0,+,0)$	Heisenberg group	nilpotent
$(0,0,0)$	$(\mathbb{R}^3,+)$	Abelian

To describe the Levi-Civita connection ∇ of G, we introduce the following constants:

$$\mu_i = \frac{1}{2}(c_1 + c_2 + c_3) - c_i.$$

Proposition 11. *The Levi-Civita connection is given by*

$$\begin{aligned}
\nabla_{e_1} e_1 &= 0, & \nabla_{e_1} e_2 &= \mu_1 e_3, & \nabla_{e_1} e_3 &= -\mu_1 e_2, \\
\nabla_{e_2} e_1 &= -\mu_2 e_3, & \nabla_{e_2} e_2 &= 0, & \nabla_{e_2} e_3 &= \mu_2 e_1, \\
\nabla_{e_3} e_1 &= \mu_3 e_2, & \nabla_{e_3} e_2 &= -\mu_3 e_1 & \nabla_{e_3} e_3 &= 0.
\end{aligned}$$

The Riemannian curvature R is given by

$$R(e_1,e_2)e_1 = (\mu_1\mu_2 - c_3\mu_3)e_2, \quad R(e_1,e_2)e_2 = -(\mu_1\mu_2 - c_3\mu_3)e_1,$$
$$R(e_2,e_3)e_2 = (\mu_2\mu_3 - c_1\mu_1)e_3, \quad R(e_2,e_3)e_3 = -(\mu_2\mu_3 - c_1\mu_1)e_2,$$
$$R(e_1,e_3)e_1 = (\mu_3\mu_1 - c_2\mu_2)e_3, \quad R(e_1,e_3)e_3 = -(\mu_3\mu_1 - c_2\mu_2)e_1.$$

The basis $\{e_1,e_2,e_3\}$ diagonalizes the Ricci operator Q. The principal Ricci curvatures are given by

$$S_1 = 2\mu_2\mu_3, \quad S_2 = 2\mu_1\mu_3, \quad S_3 = 2\mu_1\mu_2.$$

6.3. **Perrone invariants.** According to a result due to Perrone, simply connected 3-dimensional unimodular Lie groups equipped with left invariant almost cosymplectic structure are classified by *Perrone invariant* $\mathcal{P} = \|\pounds_\xi h\| - 2\|h\|^2$ as follows:

Theorem 11 ([20])**.** *Let (G,φ,ξ,η,g) be a simply connected 3-dimensional Lie group equipped with left invariant almost cosymplectic structure. If G is unimodular, then G is one of the following Lie groups*:

1. *If G is cosympletic then $\mathcal{P} = 0$ and $G = \widetilde{\mathrm{E}}_2$ with flat metric or abelian group $\mathbb{R}^3$ equipped with Euclidean metric.*
2. *If G is non-cosympletic, then*
   - (a) *$G = \widetilde{\mathrm{E}}_2$ if $\mathcal{P} > 0$.*
   - (b) *$G =$ Heisenberg group if $\mathcal{P} = 0$.*
   - (c) *$G = \mathrm{E}_{1,1}$ if $\mathcal{P} < 0$.*

The Lie algebra $\mathfrak{g}$ of G is generated by an orthonormal basis $\{e_1,e_2,e_3\}$ as in (23) *with $c_3 = 0$. The left invariant cosymplectic structure is determined by*

$$\xi = e_3, \quad \varphi e_1 = e_2, \quad \varphi e_2 = -e_1, \ \varphi\xi = 0.$$

Hereafter we denote by $G(c_1,c_2)$ the 3-dimensional unimodular Lie group (whose Lie algebra is determined by (23) with $c_3 = 0$) equipped with a left invariant almost cosymplectic structure. The global orthonormal frame field $\{e_1,e_2,e_3\}$ on $G(c_1,c_2)$ is an example of the frame field given in Lemma 1 with $\alpha = \mu_3 = (c_1+c_2)/2$.

Proposition 12. *The endomorphism field h of a unimodular Lie group $G(c_1,c_2)$ equipped with a left invariant homogeneous almost cosymplectic structure is given by*

$$he_1 = -\frac{1}{2}(c_1-c_2)e_1, \quad he_2 = \frac{1}{2}(c_1-c_2)e_2.$$

The sectional curvatures of G are given by

$$H = K_{12} = \frac{1}{4}(c_1-c_2)^2, \quad K_{13} = \frac{1}{4}(c_1-c_2)(c_1+3c_2), \quad K_{23} = -\frac{1}{4}(c_1-c_2)(3c_1+c_2).$$

The principal Ricci curvatures are

$$S_1 = \frac{1}{2}(c_1^2-c_2^2), \quad S_2 = -\frac{1}{2}(c_1^2-c_2^2), \quad S_3 = -\frac{1}{2}(c_1-c_2)^2.$$

The scalar curvature is

$$r = -2\lambda^2 = -\frac{1}{2}(c_1-c_2)^2.$$

In particular, $G(c_1,c_2)$ is scalar flat if and only if $c_1 = c_2$.

The Perrone invariant is computed as

$$\mathcal{P} = |c_1 - c_2| \left(\sqrt{c_1^2 + c_2^2} - |c_1 - c_2| \right).$$

Corollary 4. *The non-cosymplectic unimodular Lie group $G(c_1, c_2)$ is an almost cosymplectic (κ, μ)-space with*

$$\kappa = -\frac{1}{4}(c_1 - c_2)^2, \quad \mu = -(c_1 + c_2).$$

Comparing the model spaces $N^3(\mu)$ of almost cosymplectic $(-1, \mu)$-space constructed in [10] with $G(c_1, c_2)$ we obtain the following corollary.

Corollary 5. *Let $N^3(\mu)$ be a 3-dimensional simply connected almost cosymplectic $(-1, \mu)$-space. Then $N^3(\mu)$ is isomorphic to one of the following almost cosymplectic Lie groups*:

- $\widetilde{\mathrm{E}}_2$ *if* $|\mu| > 2$,
- *Heisenberg group if* $|\mu| = 2$,
- $\mathrm{E}_{1,1}$ *if* $|\mu| < 2$.

6.4. **Explicit models.** Here we give explicit expressions of these unimodular Lie groups.

Example 2 (Euclidean motion group)**.** Let us denote by $\widetilde{\mathrm{E}}_2$ the universal covering of the *Euclidean motion group* E_2. Then $\widetilde{\mathrm{E}}_2$ is realized as $\mathbb{R}^3(x, y, z)$ with multiplication

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = (x_1 + (\cos z_1)x_2 - (\sin z_1)y_2, y_1 + (\sin z_1)x_2 + (\cos z_1)y_2, z_1 + z_2).$$

For any positive real numbers a, b and c satisfying $a \geq b$, we take a global frame field

$$e_1 = \frac{1}{a}\left(\cos z \frac{\partial}{\partial x} + \sin z \frac{\partial}{\partial y}\right), \quad e_2 = \frac{1}{b}\left(-\sin z \frac{\partial}{\partial x} + \cos z \frac{\partial}{\partial y}\right), \quad e_3 = \frac{1}{c}\frac{\partial}{\partial z}.$$

Then $\{e_1, e_2, e_3\}$ satisfies

$$[e_1, e_2] = 0, \quad [e_2, e_3] = \frac{a}{bc}e_1, \quad [e_3, e_1] = \frac{b}{ca}e_2.$$

The left invariant Riemannian metric g determined by the condition $\{e_1, e_2, e_3\}$ is orthonormal with respect to it and it is

$$g = a^2\omega^1 \otimes \omega^1 + b^2\omega^2 \otimes \omega^2 + c^2\omega^3 \otimes \omega^3,$$

where

$$\omega^1 = \cos z\, dx + \sin z\, dy, \quad \omega^2 = -\sin z\, dx + \cos z\, dy, \quad \omega^3 = dz.$$

Let us introduce a left invariant almost contact structure by

$$\eta = c\omega^3 = c\, dz, \quad \xi = e_3,$$

$$\varphi e_1 = e_2, \quad \varphi e_2 = -e_1, \quad \varphi e_3 = 0.$$

Then the resulting homogeneous almost contact metric 3-manifold $(\widetilde{\mathrm{E}}_2, \varphi, \xi, \eta, g)$ is almost cosymplectic with Perrone invariant

$$\mathcal{P} = \frac{|a^2 - b^2|}{a^2b^2c^2}\left(\sqrt{a^4 + b^4} - |a^2 - b^2|\right) \geq 0.$$

The almost cosymplectic 3-manifold $(\widetilde{\mathrm{E}}_2, \varphi, \xi, \eta, g)$ is cosymplectic when and only when $a = b$. In such a case g is flat and has the form

$$g = a^2(dx^2 + dy^2) + c^2\, dz^2.$$

Example 3 (The Mikowski motion group)**.** The identity component of the isometry group of Minkowski plane $\mathbb{E}^{1,1} = (\mathbb{R}^2(x_1, x_2), dx_1 dx_2)$ is denoted by $\mathrm{E}_{1,1}$ and called the *Minkowski motion group*. The Minkowski motion group $\mathrm{E}_{1,1}$ is realized as $\mathbb{R}^3(x, y, z)$ with multiplication

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = (x_1 + e^{z_1} x_2, y_1 + e^{-z_1} y_2, z_1 + z_2).$$

For any positive numbers a, b, c, we set

$$e_1 = a\left(e^z \frac{\partial}{\partial x} - e^{-z} \frac{\partial}{\partial y}\right), \quad e_2 = b\left(e^z \frac{\partial}{\partial x} + e^{-z} \frac{\partial}{\partial y}\right), \quad e_3 = c\,\frac{\partial}{\partial z}.$$

Then we have

$$[e_1, e_2] = 0, \quad [e_2, e_3] = c_1 e_1, \quad [e_3, e_1] = c_2 e_2$$

with

$$c_1 = -\frac{bc}{a} < 0, \quad c_2 = \frac{ca}{b} > 0.$$

We equip a left invariant metric $g_{a,b,c}$ so that $\{e_1, e_2, e_3\}$ is orthonormal with respect to it. Then

$$g_{a,b,c} = \frac{a^2 + b^2}{4a^2 b^2}(e^{-2z} dx^2 + e^{2z} dy^2) + \frac{dz^2}{c^2}.$$

In particular,

$$g_{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1} = e^{-2z} dx^2 + e^{2z} dy^2 + dz^2$$

is the metric of the model space Sol_3 of solvgeometry in the sense of Thurston [23]. Let us introduce an almost contact structure by $\xi = e_3$, $\eta = g_{a,b,c}(e_3, \cdot)$ and

$$\varphi e_1 = e_2, \quad \varphi e_2 = -e_1, \quad \varphi e_3 = 0,$$

then the Minkowski motion group equipped with this almost cosymplectic structure satisfies:

$$\lambda = \frac{c(a^2 + b^2)}{2ab} \neq 0, \quad \kappa = -\frac{(a^2 + b^2)^2 c^2}{4a^2 b^2}, \quad \mu = \frac{(a^2 - b^2)c}{ab}.$$

$$H = \frac{(a^2 + b^2)c^2}{4a^2 b^2}, \quad K_{13} = \frac{(a^2 + b^2)(3a^2 - b^2)c^2}{4a^2 b^2}, \quad K_{23} = \frac{(a^2 + b^2)(a^2 - 3b^2)c^2}{4a^2 b^2}.$$

The Perrone invariant is computed as

$$\mathcal{P} = \frac{c^2(a^2 + b^2)}{a^2 b^2}\left(\sqrt{a^4 + b^4} - (a^2 + b^2)\right) < 0.$$

The characteristic Jacobi operator is invariant under characteristic flow when and only when $a = b$. In particular, Sol_3 equipped with compatible left invariant almost cosymplectic structure satisfies $\mathcal{L}_\xi \ell = 0$ but not $\ell = 0$ (see [14]).

For the explicit representation of left invariant almost cosymplectic structure on the Heisenberg group, see [14, Example 5.3].

6.5. **The characteristic Jacobi operator.** The characteristic Jacobi operator ℓ of a unimodular Lie group $G(c_1, c_2)$ is computed as

$$\ell(e_1) = \ell_1 e_1, \quad \ell(e_2) = \ell_2 e_2,$$

where

$$\ell_1 = \frac{1}{4}(c_1 - c_2)(c_1 + 3c_2) = K_{13}, \quad \ell_2 = -\frac{1}{4}(c_1 - c_2)(3c_1 + c_2) = K_{23}.$$

In particular, we have

$$\ell_1 + \ell_2 = -\frac{1}{2}(c_1 - c_2)^2, \ \ \ell_1 - \ell_2 = (c_1 - c_2)(c_1 + c_2).$$

Thus $\ell = 0$ if and only if $c_1 = c_2$.

Proposition 13. *A 3-dimensional unimodular Lie group $G(c_1, c_2)$ has vanishing characteristic Jacobi operator if and only if $G(c_1, c_2)$ is locally isometric to $\widetilde{\mathrm{E}}_2$ equipped with flat metric or Euclidean 3-space $\mathbb{E}^3$.*

Proposition 14. *A 3-dimensional unimodular Lie group $G(c_1, c_2)$ equipped with a left invariant almost cosymplectic structure satisfies $\ell = 0$ if and only if $G(c_1, c_2)$ is cosymplectic.*

6.6. **Pseudo-parallelism.** We already know from Theorem 9, the only possible unimodular Lie algebra $\mathfrak{g}(c_1, c_2)$ with pseudo-parallel characteristic Jacobi operator are cosymplectic ones or $\mathfrak{e}_{1,1}$. However as we have exhibited in Example 3, left invariant almost cosymplectic structure on $\mathrm{E}_{1,1}$ are not unique. We need to identify the left invariant almost cosymplectic structures on $\mathrm{E}_{1,1}$ admitting pseudo-parallel characteristic Jacobi operator. In this section, we pursue this task.

Let us investigate the pseudo-parallelism of the characteristic Jacobi operator ℓ of the unimodular Lie group $G(c_1, c_2)$. First of all the covariant derivatives $\nabla \ell$ is described as

$$(\nabla_{e_1}\ell)e_1 = (\nabla_{e_2}\ell)e_2 = (\nabla_{e_3}\ell)e_3 = 0,$$

$$\begin{aligned}
(\nabla_{e_1}\ell)e_2 =& \mu_1 K_{23} e_3 = \frac{1}{8}(c_1 - c_2)^2(3c_1 + c_2)e_3, \\
(\nabla_{e_1}\ell)e_3 =& \mu_1 K_{23} e_2 = \frac{1}{8}(c_1 - c_2)^2(3c_1 + c_2)e_2, \\
(\nabla_{e_2}\ell)e_1 =& -\mu_2 K_{13} e_3 = -\frac{1}{8}(c_1 - c_2)^2(c_1 + 3c_2)e_3, \\
(\nabla_{e_2}\ell)e_3 =& -\mu_2 K_{13} e_1 = -\frac{1}{8}(c_1 - c_2)^2(c_1 + 3c_2)e_1, \\
(\nabla_{e_3}\ell)e_1 =& \mu_3 (K_{13} - K_{23}) e_2 = \frac{1}{2}(c_1 + c_2)^2(c_1 - c_2)e_2, \\
(\nabla_{e_3}\ell)e_2 =& \mu_3 (K_{13} - K_{23}) e_1 = \frac{1}{2}(c_1 + c_2)^2(c_1 - c_2)e_1.
\end{aligned}$$

Thus we obtain

Proposition 15. *Let $G(c_1, c_2)$ be a 3-dimensional unimodular Lie group equipped with a left invariant almost cosymplectic structure. Then $G(c_1, c_2)$ has parallel characteristic Jacobi operator if and only if $G(c_1, c_2)$ is cosymplectic and $\ell = 0$.*

Unimodular almost cosymplectic Lie groups with pseudo-parallel characteristic Jacobi operator are classified as follows.

Theorem 12. *Let $G(c_1, c_2)$ be a 3-dimensional unimodular Lie group equipped with left invariant almost cosymplectic structure. Then $G(c_1, c_2)$ has pseudo-parallel characteristic Jacobi operator if and only if $G(c_1, c_2)$ is one of the following list*:

(1) *cosymplectic Lie groups*: $\mathbb{E}^3$ *or* $\tilde{\mathrm{E}}_2$,
(2) *non-cosymplectic Lie groups*: $\mathrm{E}_{1,1}$ *with* $c_1 + c_2 = 0$. *In this case, the characteristic Jacobi operator is properly pseudo-parallel with* $L = -c_1^2$.

Proof. From Proposition 11, first, we have $R(e_3, e_1) \cdot \ell = L(e_3 \wedge e_1) \cdot \ell$ holds if and only if

$$\mu_1(\mu_1 + 2\mu_3)\{L + \mu_1(\mu_1 + 2\mu_3)\} = 0. \tag{24}$$

Also, we have $R(e_2, e_3) \cdot \ell = L(e_2 \wedge e_3) \cdot \ell$ holds if and only if

$$\mu_1(\mu_1 - 2\mu_3)\{L + \mu_1(\mu_1 - 2\mu_3)\} = 0. \tag{25}$$

Thus, we have $R(e_1, e_2) \cdot \ell = L(e_1 \wedge e_2) \cdot \ell$ holds if and only if

$$\mu_1\mu_3(L - \mu_1^2) = 0. \tag{26}$$

From (26), we have $\mu_1 = 0$, $\mu_3 = 0$ or $L = \mu_1^2$.

First, if $\mu_1 = 0$, then $c_1 = c_2$ and $G(c_1, c_2)$ is cosymplectic.

Next, if $\mu_1 \neq 0$ and $\mu_3 = 0$, then from (24) and (25) we get $L = -\mu_1^2 = -c_1^2$. Lastly, if $\mu_1 \neq 0$ and $\mu_3 \neq 0$ and $L = \mu_1^2$, then it contradicts to (24) and (25). □

Remark 3. The unimodular basis $\{e_1, e_2, e_3\}$ of $G(c_1, c_2)$ is a global orthonormal frame field as in Lemma 1 with $\lambda = \mu_1 = -(c_1 - c_2)/2$, $\alpha = \mu_3 = (c_1 + c_2)/2$, $p = q = 0$ (and hence $\sigma(e_1) = \sigma(e_2) = 0$) and $r = -2\mu_1^2$. Thus Theorem 12 can be verified by using the system (17)–(22) of pseudo-parallelism of ℓ with respect to $\{e_1, e_2, e_3\}$ and applying Theorem 9.

6.7. **Non-unimodular Lie groups.** Now let us consider 3-dimensional non-unimodular Lie groups equipped with left invariant almost cosymplectic structure. Here we recall Perrone's construction [20].

Let G be a (simply connected) 3-dimensional non-unimodular Lie group equipped with a left invariant almost cosymplectic structure. Then one can easily check that $\xi \in \mathfrak{u}$. We take an orthonormal basis $\{e_2, e_3 = \xi\}$ of $\mathfrak{u}$. Then $e_1 = -\varphi e_2 \in \mathfrak{u}^\perp$ and hence $\mathrm{ad}(e_1)$ preserves $\mathfrak{u}$. Express $\mathrm{ad}(e_1)$ as

$$[e_1, e_2] = a_{11}e_2 + a_{21}e_3, \quad [e_1, e_3] = a_{12}e_2 + a_{22}e_3$$

over $\mathfrak{u}$. The closing condition $d\eta = 0$ implies that $a_{21} = 0$. Next, $\nabla_\xi \xi = 0$ implies that $a_{22} = 0$. Moreover one can deduce that $[e_2, e_3] = 0$ from the Jacobi identity. Note that 3-dimensional non-unimodular Lie algebras are classified by *Milnor invariant* $D = \det \mathrm{ad}(e_1)$.

Theorem 13 ([20])**.** *Let G be a 3-dimensional non-unimodular Lie group equipped with a left invariant almost cosymplectic structure. Then the Lie algebra $\mathfrak{g} = \mathfrak{g}(\gamma, \delta)$ satisfies the commutation relations*

$$[e_1, e_2] = \delta e_2, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = -\gamma e_2,$$

with $e_3 = \xi$, $e_1 = -\varphi e_2 \in \mathfrak{u}^\perp$ and $\delta \neq 0$. In particular Milnor invariant of $\mathfrak{g}(\gamma, \delta)$ is 0.

The Lie algebra $\mathfrak{g} = \mathfrak{g}(\gamma, \delta)$ is given explicitly by

$$\mathfrak{g}(\gamma, \delta) = \left\{ \begin{pmatrix} (1+\delta)x & \gamma x & y \\ 0 & x & z \\ 0 & 0 & x \end{pmatrix} \middle| \; x, y, z \in \mathbb{R} \right\}$$

with basis

$$e_1 = \begin{pmatrix} 1+\delta & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The corresponding simply connected Lie group $G(\gamma,\delta) = \exp \mathfrak{g}(\gamma,\delta)$ is given by

$$G(\gamma,\delta) = \left\{ \left(\begin{array}{ccc} e^{(1+\delta)x} & \frac{\gamma}{\delta} e^{x}(e^{\delta x}-1) & \frac{e^{x}\left((\delta y+\gamma z)(e^{\delta x}-1)-\gamma\delta x z\right)}{\delta^2 x} \\ 0 & e^{x} & z e^{x} \\ 0 & 0 & e^{x} \end{array} \right) \middle| \; x,y,z \in \mathbb{R} \right\}.$$

The multiplication law is expressed as

$$\begin{aligned} \left(x_1, \begin{pmatrix} y_1 \\ z_1 \end{pmatrix}\right) \cdot \left(x_2, \begin{pmatrix} y_2 \\ z_2 \end{pmatrix}\right) &= \left(x_1 + x_2, \begin{pmatrix} y_1 \\ z_1 \end{pmatrix} + \exp \begin{pmatrix} \delta x_1 & \gamma x_1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_2 \\ z_2 \end{pmatrix}\right) \\ &= \left(x_1 + x_2, \begin{pmatrix} y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} e^{\delta x_1} & \frac{\gamma}{\delta}(e^{\delta x_1}-1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_2 \\ z_2 \end{pmatrix}\right) \\ &= \left(x_1 + x_2, y_1 + e^{\delta x_1} y_2 + \tfrac{\gamma}{\delta}(e^{\delta x_1}-1) z_2, z_1 + z_2\right). \end{aligned}$$

The left invariant metric is expressed as $\omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \eta \otimes \eta$, where

$$\begin{aligned} \omega^1 &= dx, \\ \omega^2 &= dy + \frac{e^{-\delta x} + \delta x - 1}{\delta^2 x^2} \left\{(\gamma y + \delta z) dx - x(\gamma dz + \delta dy)\right\}, \\ \eta &= dz. \end{aligned}$$

The left invariant vector fields obtained from e_1, e_2 and e_3 by left translation are

$$e_1 = \frac{\partial}{\partial x} - \frac{e^{\delta x}(e^{-\delta x} + \delta x - 1)(\delta y + \gamma z)}{\delta x (e^{\delta x} - 1)} \frac{\partial}{\partial y}, \quad e_2 = \frac{\delta x e^{\delta x}}{e^{\delta x} - 1} \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z} - \frac{\gamma}{\delta}\left(1 + \frac{x}{1 - e^{-\delta x}}\right) \frac{\partial}{\partial y}.$$

The Levi-Civita connection of G is given by the following table:

Proposition 16 ([20])**.**

$$\begin{array}{ccc} \nabla_{e_1} e_1 = 0, & \nabla_{e_1} e_2 = -\frac{\gamma}{2} e_3, & \nabla_{e_1} e_3 = \frac{\gamma}{2} e_2, \\ \nabla_{e_2} e_1 = -\delta e_2 - \frac{\gamma}{2} e_3, & \nabla_{e_2} e_2 = \delta e_1, & \nabla_{e_2} e_3 = \frac{\gamma}{2} e_1, \\ \nabla_{e_3} e_1 = -\frac{\gamma}{2} e_2, & \nabla_{e_3} e_2 = \frac{\gamma}{2} e_1 & \nabla_{e_3} e_3 = 0. \end{array}$$

The global orthonormal frame filed $\{e_1, e_2, e_3\}$ is an example of orthonormal frame field given in Lemma 1 with $\alpha = \lambda = -\gamma/2$.

From this table, we obtain

$$h e_1 = -\frac{1}{2} \gamma e_1, \quad h e_2 = \frac{1}{2} \gamma e_2.$$

Thus $G(\gamma,\delta)$ is cosymplectic if and only if $\gamma = 0$.

The Riemannian curvature R is given by

$$\begin{aligned}
&R(e_1,e_2)e_1=\left(\delta^2-\frac{\gamma^2}{4}\right)e_2+\gamma\delta e_3, \quad R(e_1,e_2)e_2=-\left(\delta^2-\frac{\gamma^2}{4}\right)e_1,\\
&R(e_1,e_3)e_1=\gamma\delta e_2+\frac{3\gamma^2}{4}e_3, \quad R(e_1,e_3)e_3=-\frac{3\gamma^2}{4}e_1,\\
&R(e_2,e_3)e_2=-\frac{\gamma^2}{4}e_3, \quad R(e_2,e_3)e_3=\frac{\gamma^2}{4}e_2,\\
&R(e_1,e_2)e_3=-\gamma\delta e_1.
\end{aligned}$$

Hence

$$H=K_{12}=-\delta^2+\frac{\gamma^2}{4}, \quad K_{13}=-\frac{3\gamma^2}{4}, \quad K_{23}=\frac{\gamma^2}{4}.$$

Thus the characteristic Jacobi operator is given by

$$\ell(e_1)=K_{13}e_1, \quad \ell(e_2)=K_{23}e_2.$$

The Ricci operator Q is described as

$$Qe_1=-\left(\delta^2+\frac{\gamma^2}{2}\right)e_1, \quad Qe_2=-\left(\delta^2-\frac{\gamma^2}{2}\right)e_2-\gamma\delta\xi, \quad Q\xi=-\gamma\delta e_2-\frac{\gamma^2}{2}\xi$$

The principal Ricci curvatures of Q are computed as

$$-\frac{\gamma^2}{2}-\delta^2, \quad \frac{\gamma^2}{2}-\delta^2, \quad -\frac{\gamma^2}{2}.$$

The scalar curvature is

$$r=-\frac{1}{2}\gamma^2-2\delta^2.$$

Proposition 17. *The almost cosymplectic non-unimodular Lie group $G(\gamma,\delta)$ satisfies $\ell=0$ if and only if $\gamma=0$. In this case, the structure is cosymplectic.*

Thus the vanishing of ℓ is a too strong restriction for $G(\gamma,\delta)$. Next we investigate parallelism of ℓ. The covariant derivatives of ℓ are computed as

$$\begin{aligned}
(\nabla_{e_1}\ell)e_1=&0, \quad (\nabla_{e_1}\ell)e_2=-\frac{\gamma}{2}K_{23}e_3=-\frac{\gamma^3}{8}e_3,\\
(\nabla_{e_2}\ell)e_1=&-\delta(K_{13}-K_{23})e_2-\frac{\gamma}{2}K_{13}e_3=\gamma^2\delta e_2+\frac{3\gamma^3}{8}e_3,\\
(\nabla_{e_2}\ell)e_2=&-\delta(K_{13}-K_{23})e_1=\gamma^2\delta e_1.
\end{aligned}$$

Proposition 18. *The characteristic Jacobi operator of the almost cosymplectic non-unimodular Lie group $G(\gamma,\delta)$ is parallel when and only when $\gamma=0$. In such a case $\ell=0$.*

Thus unfortunately parallelism of ℓ is still a strong restriction for $G(\gamma,\delta)$.

From Theorem 9, the only possible $G(\gamma,\delta)$ admitting pseudo-parallel characteristic Jacobi operator is the cosymplectic $G(0,\delta)$. Here we conform this fact by direct approach.

Theorem 14. *An almost cosymplectic non-unimodular Lie group $G(\gamma,\delta)$ has pseudo-parallel characteristic Jacobi operator if and only if ℓ is parallel. In this case, it is $G(0,\delta)$ and cosymplectic. Thus it has left invariant cosymplectic structure which is isometric to $\mathbb{H}^2(-\delta^2)\times\mathbb{R}$.*

Proof. The orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathfrak{g}(\gamma, \delta)$ is regarded as a globally defined orthonormal frame field as in Lemma 1 with $\lambda = \alpha = -\gamma/2$, $p = 0$, $q = \delta$, $\sigma(e_1) = 0$, $\sigma(e_2) = -\gamma\delta$ and $r = -2(\delta^2 + \frac{\gamma^2}{4})$.

From (17) $R(e_3, e_1) \cdot \ell = L(e_3 \wedge e_1) \cdot \ell$ holds if and only if

$$\gamma^3\delta = 0, \quad \left(L + \frac{3}{4}\gamma^2\right)\gamma^2 = 0. \tag{27}$$

Next from (19) $R(e_2, e_3) \cdot \ell = L(e_2 \wedge e_3) \cdot \ell$ holds if and only if

$$\left(L - \frac{1}{4}\gamma^2\right)\gamma^2 = 0. \tag{28}$$

Finally, from (21) and (22), $R(e_1, e_2) \cdot \ell = L(e_1 \wedge e_2) \cdot \ell$ holds if and only if

$$\gamma^2\left(L + \delta^2 - \frac{\gamma^2}{4}\right) = 0, \quad \delta\gamma^3 = 0. \tag{29}$$

The second equation of (29) implies that $\gamma = 0$ since $\delta \neq 0$. Of course, if ℓ is semi-parallel (*i.e.*, $L = 0$), then $\gamma = 0$. □

Remark 4. The simply connected almost cosymplectic Lie group $G(0, \delta)$ is cosymplectic and isometric to the cosymplectic space form $\mathbb{H}^2(-\delta^2) \times \mathbb{R}$. However, as a homogeneous cosymplectic 3-manifold, these two spaces are distinguished. The cosymplectic space form $\mathbb{H}^2(-\delta^2) \times \mathbb{R}$ is represented by $\mathbb{H}^2(-\delta^2) \times \mathbb{R} = (\mathrm{SU}_{1,1} \times \mathbb{R})/\mathrm{U}_1$ as a homogeneous space. On the other hand, $G(0, \delta)$ is represented as $\mathbb{H}^2(-\delta^2) \times \mathbb{R} = G(0, \delta)/\{\mathrm{Id}\}$.

By using Theorem 14 together with Theorem 10, Theorem 11 and Theorem 12, we obtain the following classification of homogeneous almost cosymplectic 3-manifolds.

Corollary 6. *Let M be a homogeneous almost cosymplectic 3-manifold with pseudo-parallel characteristic Jacobi operator. Then M is locally isomorphic to one of the following spaces*:

- *Cosymplectic space forms*: $\mathbb{S}^2(\bar{c}) \times \mathbb{R} = (\mathrm{SU}_2 \times \mathbb{R})/\mathrm{U}_1$, $\mathbb{H}^2(\bar{c}) \times \mathbb{R} = (\mathrm{SU}_{1,1} \times \mathbb{R})/\mathrm{U}_1$, $\mathbb{E}^3 = \mathrm{E}_3/\mathrm{SO}_3$,
- $\tilde{\mathrm{E}}_2$ *equipped with left invariant flat cosymplectic structure,*
- $\mathrm{E}_{1,1}$ *with structure constants (c_1, c_2) satisfying $c_1 + c_2 = 0$ equipped with left invariant non-cosymplectic almost cosymplectic structure.*
- *The non-unimodular Lie group $G(0, \delta)$ equipped with left invariant cosymplectic structure which is isometric to $\mathbb{H}^2(-\delta^2) \times \mathbb{R}$.*

In particular, the model space Sol_3 *of Thurston geometry equipped with compatible almost cosymplectic structure has pseudo-parallel characteristic Jacobi operator.*

References

[1] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Progress in Math. **203**, Birkhäuser Boston, Inc., Boston, second edition, 2010.

[2] B. Cappelletti-Montano, A. De Nicola and I. Yudin, A survey on cosymplectic geometry, Rev. Math. Phys. **25** (2013), 1343002.

[3] A. Carriazo and V. Martín-Molina, Almost cosymplectic and almost Kenmotsu (κ, μ, ν)-spaces, Mediterr. J. Math. **10** (2013), 1551–1571.

[4] J. T. Cho, Reeb flow symmetry on almost cosymplectic three-manifolds, Bull. Korean Math. Soc. **53** (2016), no. 4, 1249–1257.

[5] J. T. Cho and S. H. Chun, The unit tangent sphere bundle whose characteristic Jacobi operator is pseudo-parallel, Bull. Korean Math. Soc. **53** (2016), no. 6, 1715–1723.

[6] J. T. Cho and J. Inoguchi, Characteristic Jacobi operator on contact Riemannian 3-manifolds, Differ. Geom. Dyn. Syst. **17** (2015), 49–71.
[7] J. T. Cho and J. Inoguchi, Contact 3-manifolds with Reeb flow invariant characteristic Jacobi operator, An. Ştiinţ. Univ. Al. I. Cuza Mat. (N. S.) **63** (2017), no. 3, 665–676.
[8] P. Dacko, On almost cosymplectic manifolds with the structure vector field ξ belonging to the k-nullity distributions, Balkan J. Geom. Appl. **5** (2000), 47–60.
[9] P. Dacko and Z. Olszak, On almost cosymplectic (κ, μ, ν)-space, Banach Center Publ. **69** (2005), 211–220.
[10] P. Dacko and Z. Olszak, On almost cosymplectic $(-1, \mu, 0)$-spaces, Central European Science Journals, **3**(2) (2005), 318-330.
[11] R. Deszcz, On pseudosymmetric spaces, Bull. Soc. Math. Belg. Sér. A **44** (1992), no. 1, 1–34.
[12] S. I. Goldberg and K. Yano, Integrability of almost cosymplectic structures, Pacific J. Math. **31** (1969), no. 2, 373–382.
[13] S. D. Han and J. W. Yim, Unit vector fields on spheres which are harmonic maps, Math. Z. **227** (1998), 83–92.
[14] J. Inoguchi, Characteristic Jacobi operator on almost cosymplectic 3-manifolds, Int. Electron. J. Geom. **12** (2019), no. 2, 276–299.
[15] J. Inoguchi and J.-E. Lee, On the characteristic Jacobi operator of the tangent sphere bundles over surfaces, submitted.
[16] J. Milnor, Curvatures of left invariant metrics on Lie groups, Adv. Math. **21**(1976), 293–329.
[17] Z. Olszak, On almost cosymplectic manifolds, Kodai Math. J. **4** (1981), 239–250.
[18] Z. Olszak, Almost cosymplectic manifolds with Kählerian leaves, Tensor N. S. **46** (1987), 117–124.
[19] H. Öztürk, N. Aktan and C. Murathan, Almost α-cosymplectic (κ, μ, ν)-spaces, `arXiv:1007.0527v1 [math.DG]`
[20] D. Perrone, Classification of homogeneous almost cosymplectic three-manifolds, Differential Geom. Appl. **30** (2012), 49–58.
[21] D. Perrone, Almost contact metric manifolds whose Reeb vector field is a harmonic section, Acta Math. Hungar. **138** (2013), no. 1–2, 102–126.
[22] D. Perrone, Minimal Reeb vector fields on almost cosymplectic manifolds, Kodai Math. J. **36** (2013), 258–274.
[23] W. M. Thurston, *Three-dimensional Geometry and Topology* I, Princeton Math. Series., vol. **35** (S. Levy ed.), 1997.
[24] W. Wang, X. Dai, Pseudo-parallel characteristic Jacobi operators on contact metric 3 manifolds, J. Math. **2021** (2021), Article ID 6148940, 6 pages.

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