ON SOME CURVES IN 3-DIMENSIONAL HYPERBOLIC GEOMETRY AND SOLVGEOMETRY

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Dedicated to professor Koji Matsumoto on the occasion of his 80th birthday

ABSTRACT. We study curve geometry in para-Sasakian 3-manifolds, especially in the hyperbolic 3-space and the space Sol_3 of solvgeometry. Parametric expression for φ-trajectories in the hyperbolic 3-space is given.

INTRODUCTION

As it is well known odd-dimensional hyperbolic space $\mathbb{H}^{2n+1}$ admits a special normal almost contact structure compatible to the metric. The resulting space is a homogeneous Kenmotsu manifold. Based on this fundamental fact, submanifold geometry, especially curve theory in Kenmotsu manifolds have been developed extensively.

On the other hand, Sato [25] introduced the notion of almost paracontact structure. Let $M = (M, \varphi, \xi, \eta)$ be an almost paracontact manifold in the sense of Sato. Then there are two ways to introduce "compatible metric" to this structure.

(1) positive definite Riemannian metric g compatible to the structure. In this case φ is *self-adjoint* with respect to g.
(2) indefinite Riemannian metric g compatible to the structure. In this case φ is *skew-adjoint* with respect to g.

There is a large number of publications in curve geometry in almost paracontact manifolds equipped with *indefinite compatible metric* (see, *e.g.*, [7] and references therein). On the contrary curve geometry in almost paracontact manifolds equipped with *positive definite compatible metric* are not well developed, yet.

The hyperbolic space $\mathbb{H}^n$ (for both even n and odd n) admits a particular almost paracontact structures compatible to the metric. The resulting space is a special para-Sasakian manifold. On the other hand, the model space Sol_3 of solvgeometry in the sense of Thurston also admits para-Sasakian structure.

In this paper we study curve geometry in para-Sasakian manifolds, especially in $\mathbb{H}^3$.

1. PARA-SASAKIAN STRUCTURES

1.1. **Almost paracontact structures.** According to Sato [25, 26, 27], an m-manifold M is said to have an *almost paracontact structure* if it admits a triplet (φ, ξ, η) consisting of an

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endomorphism field φ, vector field ξ and a 1-form η satisfying

$$\varphi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1.$$

A manifold M equipped with an almost paracontact structure is called an *almost paracontact manifold*. One can see that

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0 \quad \text{and} \quad \operatorname{rank} \varphi = m - 1.$$

As we have mentioned in Introduction, there are two options to introduce compatibility of metric. In this paper we consider *positive definite Riemannian metrics*.

A Riemannian metric g of an almost paracontact manifold M is said to be *compatible* if

$$\eta(X) = g(\xi, X), \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all smooth vector fields X and Y on M. The resulting manifold $(M, \varphi, \xi, \eta, g)$ is called an *almost paracontact Riemannian manifold* [25, 26] or *almost paracontact metric manifold*. To distinguish our compatible metric with indefinite compatible metrics (and also to avoid confusions), we use the terminology almost paracontact Riemannian manifold throughout this article. One can see that φ is self-adjoint with respect to any compatible metric, *i.e.*,

$$g(\varphi X, Y) = g(X, \varphi Y)$$

for all smooth vector fields X and Y on M. The *fundamental symmetric form* Φ of M is defined by

$$\Phi(X, Y) = g(X, \varphi Y) = g(\varphi X, Y).$$

Kaneyuki and Williams [13] investigated almost paracontact structures on circle bundles over para-Hodge manifolds.

On an almost paracontact Riemannian manifold M, the endomorphism field φ has constant eigenvalues ± 1 and 0. The multiplicity of 0 is 1. The *type* (p, q) of an almost paracontact Riemannian manifold is the signature of φ, that is, p is the multiplicity of 1 and q is the multiplicity of -1 [24]. One can see that $\mathrm{tr}_g\varphi = p - q$, where tr_g means the *metric trace operator* with respect to g, that is,

$$\mathrm{tr}_g F = \sum_{i=1}^{m} g(Fe_i, e_i)$$

for any endomorphism field F on M. Here $\{e_1, e_2, \cdots, e_m\}$ is a local orthonormal frame field.

1.2. **Paracontact structures.** Sasaki [24] introduced the notion of paracontact Riemannian structure. An almost paracontact Riemannian manifold M is said to be a *paracontact Riemannian manifold* in the sense of Sasaki if it satisfies

$$\nabla\xi = \varphi.$$

On the other hand, M is said to be a *paracontact Riemannian manifold* in the sense of Sato [25] if

$$g(\varphi X, Y) = \frac{1}{2}\{(\nabla_X \eta)Y + (\nabla_Y \eta)X\}$$

for all smooth vector fields X and Y on M. Obviously, paracontact Riemannian in the sense of Sasaki is stronger than the one of Sato. Note that almost paracontact Riemannian manifolds satisfying $\nabla\xi = \varphi$ are called *special paracontact Riemannian manifolds* by Sato. In this article we use Sasaki's definition for paracontact Riemannian.

If M is a paracontact Riemannian manifold, then

$$2\mathrm{d}\eta(X,Y) = (\nabla_X\eta)Y - (\nabla_Y\eta)X = g(\nabla_X\xi, Y) - g(\nabla_Y\xi, X)$$
$$= g(\varphi X, Y) - g(X, \varphi Y) = 0,$$

since φ is self-adjoint.

Definition 1.1 ([28])**.** An paracontact Riemannian manifold is said to be a *para-Sasakian manifold* if it satisfies

$$(\nabla_X\varphi)Y = -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi. \tag{1.1}$$

It should be remarked that para-Sasakian manifolds satisfy $\mathrm{d}\eta = 0$.

Proposition 1.1. *Let M be an almost paracontact Riemannian manifold. Assume that M is a paracontact Riemannian manifold in the sense of Sato. Then M is para-Sasakian if and only if $\mathrm{d}\eta = 0$ and*

$$(\nabla_X\varphi)Y = -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi.$$

Sato obtained the following characterization of para-Sasakian structure.

Theorem 1.1. *Let (M, g) be a Riemannian manifold. Assume that there exists a 1-form η satisfying*

$$(\nabla_X\eta)Y = -\varepsilon\{g(X,Y) - \eta(X)\eta(Y)\}, \quad \varepsilon = \pm 1. \tag{1.2}$$

Then the structure (φ, ξ, η, g) is a para-Sasakian structure on M. Here ξ is the metrical dual vector field of η and $\varphi = \nabla\xi$.

Proof. From (1.2) we deduce that

$$\varphi X = \nabla_X\xi = -\varepsilon(X - \eta(X)\xi).$$

One can see that (φ, ξ, η) is almost paracontact from this formula. Direct computation of $\nabla\varphi$ yields (1.1). □

It should be remarked that general para-Sasakian manifolds do not satisfy (1.2). For this reason, para-Sasakian manifolds satisfying (1.2) are called *special para-Sasakian manifolds* [28]. After introduction of para-Sasakian structure [28], Adati and his collaborators published many articles concerning para-Sasakian manifolds (large part of those papers were published in TRU Math. and Tensor N. S.). Here we only refer [1] and [2]. Matsumoto published some papers [16, 17, 20] in Bull. Yamagata Univ. (see also Ogata's paper [23]). See also [10, 11, 17, 18, 19].

1.3. **Kenmotsu manifolds.** To give an important example, here we recall the notion of Kenmotsu manifold.

Definition 1.2. A $(2n+1)$-manifold M is said to have an *almost contact structure* if it admits a triplet (ψ, ζ, ω) consisting of an endomorphism field ψ, vector field ζ and a 1-form ω satisfying

$$\psi^2 = -I + \omega \otimes \zeta, \quad \omega(\zeta) = 1.$$

A Riemannian metric g of an almost contact manifold M is said to be *compatible* if

$$\omega(X) = g(\zeta, X), \quad g(\psi X, \psi Y) = g(X,Y) - \omega(X)\omega(Y)$$

for all smooth vector fields X and Y on M. The resulting manifold (M,ψ,ζ,ω,g) is called an *almost contact Riemannian manifold.*

Example 1.1 (Sasakian manifold). An almost contact Riemannian manifold (M,ψ,ζ,ω,g) is said to be a *Sasakian manifold* if it satisfies

$$(\nabla_X\varphi)Y = g(X,Y)\xi - \eta(Y)X.$$

Sasakian manifolds satisfy $\nabla\zeta = -\psi$. In particular ζ is a unit Killing vector field.

Remark 1. A Sasakian analogue of Theorem 1.1 is well known. If a Riemannian manifold (M,g) admits a unit Killing vector field ζ satisfying

$$\nabla_X\nabla_Y\zeta - \nabla_{\nabla_XY}\zeta = g(Y,\zeta)X - g(X,Y)\zeta.$$

Then $(\psi = -\nabla\zeta,\zeta,\omega,g)$ is a Sasakian structure on M. Here ω is the metrical dual of ζ.

Example 1.2 (Kenmotsu manifold). An almost contact Riemannian manifold (M,ψ,ζ,ω,g) is said to be a *Kenmotsu manifold* if it satisfies

$$(\nabla_X\psi)Y = h(\psi X,Y)\zeta - \omega(Y)X. \tag{1.3}$$

Kenmotsu manifolds satisfy

$$\nabla_X\zeta = X - \omega(X)\zeta. \tag{1.4}$$

Theorem 1.2. *Let $M = (M,\psi,\xi,\eta,g)$ be a Kenmotsu $(2n+1)$-manifold. Then the structure (φ,ξ,η,g) defined by $\varphi = \nabla\xi$ is a special para-Sasakian structure of type $(2n,0)$.*

Proof. First of all

$$\varphi^2X = \nabla_{\varphi X}\xi = \nabla_{\nabla_X\xi}\xi = \nabla_{X-\eta(X)\xi}\xi = \nabla_X\xi - \eta(X)\nabla_\xi\xi = X - \eta(X)\xi.$$

Hence (φ,ξ,η) is almost paracontact. Next,

$$g(\varphi X,\varphi Y) = g(X-\eta(X)\xi, Y-\eta(Y)\xi) = g(X,Y) - \eta(X)\eta(Y).$$

Thus g is compatible to (φ,ξ,η). Since $\nabla\xi = \varphi$, (φ,ξ,η,g) is paracontact Riemannian. Since $\varphi X = X - \eta(X)\xi$, the type of this almost paracontact structure is $(2n,0)$. We compute the covariant derivative $\nabla\varphi$. From

$$\begin{aligned}\nabla_X(\varphi Y) =& \nabla_X\nabla_Y\xi = \nabla_X(Y-\eta(Y)\xi)\\ =& \nabla_XY - X\eta(Y)\xi - \eta(Y)(X-\eta(X)\xi),\\ \varphi(\nabla_XY) =& \nabla_{\nabla_XY}\xi = \nabla_XY - \eta(\nabla_XY)\xi,\end{aligned}$$

we obtain

$$\begin{aligned}(\nabla_X\varphi)Y =& -(\nabla_X\eta)Y\xi - \eta(Y)X + \eta(X)\eta(Y)\xi\\ =& -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi.\end{aligned}$$

Hence (φ,ξ,η,g) is para-Sasakian. □

1.4. **Model spaces.** We give model spaces of para-Sasakian manifolds.

Example 1.3 (Sasaki-Nemoto's example)**.** Let $F_1 = (F_1^p, g_1)$ and (F_2^q, g_2) be Riemannian manifolds of dimension p and q, respectively. Then we consider double-warped product [8, §3.6], [9]:

$$M(p,q) = \mathbb{R} \times_{e^t} F_1 \times_{e^{-t}} F_2, \quad g = \mathrm{d}t^2 + e^{2t} g_1 + e^{-2t} g_2.$$

We define $\xi = \partial/\partial t$. Then its metrical dual is $\eta = \mathrm{d}t$. One can check that $M(p,q)$ is a non-special para-Sasakian manifold of type (p,q) [21].

Sasaki showed that the maximum dimension of the automorphism group $\mathrm{Aut}(M)$ of a para-Sasakian manifold of type (p,q) is $\{p(p+1)+q(q+1)\}/2+1$. The following example attains the maximum dimension.

Example 1.4 (Solvable Lie groups)**.** Let us choose $F_1 = \mathbb{R}^p$ and $F_2 = \mathbb{R}^q$ (and change the notation from t to z) in the preceding example [24]. Then the resulting space is the Cartesian space $\mathbb{R}^{p+q+1}$ with homogeneous Riemannian metric

$$g = e^{2z}(\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_p^2) + e^{-2z}(\mathrm{d}y_1^2 + \cdots + \mathrm{d}y_q^2) + \mathrm{d}z^2.$$

One can see that this Riemanian manifold is realized as the following solvable Lie group

$$\mathrm{Solv}(p,q) = \left\{ \left(\begin{array}{ccc|ccc|c} e^{-z} & \cdots & 0 & 0 & \cdots & 0 & x_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & e^{-z} & 0 & \cdots & 0 & x_p \\ \hline 0 & \cdots & 0 & e^{z} & \cdots & 0 & y_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & e^{z} & y_q \\ \hline 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \end{array} \right) \middle| \; x_1, \cdots, x_p, y_1, \ldots, y_q, z \in \mathbb{R} \right\}.$$

The para-Sasakian structure is left invariant on $\mathrm{Solv}(p,q)$. The automorphism group $G(p,q) = \mathrm{Aut}(\mathrm{Solv}(p,q))$ acts transitively on $\mathrm{Solv}(p,q)$. The para-Sasakian manifold $\mathrm{Solv}(p,q)$ is represented as $\mathrm{Solv}(p,q) = G(p,q)/\mathrm{O}(p) \times \mathrm{O}(q) \times \mathrm{O}(1)$. Note that $\mathrm{Solv}(1,1)$ is the model space Sol_3 of 3-dimensional *solvgeometry* in the sense of Thurston [29]. In addition the identity component of $G(1,1)$ is $\mathrm{Solv}(1,1)$ itself.

Example 1.5 (Hyperbolic spaces)**.** Let us realize the hyperbolic n-space $\mathbb{H}^n$ as the warped product manifold $\mathbb{R} \times_{e^z} \mathbb{R}^{n-1}$. The warped product $\mathbb{R} \times_{e^z} \mathbb{R}^{n-1}$ is realized as the following solvable Lie group

$$\mathbb{H}^n = \left\{ \left(\begin{array}{ccc|c} e^{-z} & \cdots & 0 & x_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & e^{-z} & x_{n-1} \\ \hline 0 & \cdots & 0 & 1 \end{array} \right) \middle| \; x_1, \cdots, x_{n-1}, z \in \mathbb{R} \right\}.$$

The para-Sasakian structure is special and left invariant.

Remark 2. Manev and Staikova studied para-Sasakian manifolds of type (n,n) [15].

1.5. **Solvable Lie group model.** To describe $\mathbb{H}^3$ and $\mathrm{Solv}(1,1)$ in a unified way, here we give the following solvable Lie group model:

$$G(\delta)=\left\{\left(\begin{array}{ccc} e^{-z} & 0 & x\\ 0 & e^{\delta z} & y\\ 0 & 0 & 1\end{array}\right)\ \middle|\ x,y,z\in\mathbb{R}\right\},\quad \delta=\pm 1$$

equipped with the left invariant metric

$$g=e^{2z}\mathrm{d}x^2+e^{-2\delta z}\mathrm{d}y^2+\mathrm{d}z^2.$$

This family is useful for our study on hyperbolic 3-space and Sol_3. In fact, $G(-1)$ is the hyperbolic 3-space and $G(1)$ is $\mathrm{Solv}(1,1)=\mathrm{Sol}_3$.

The group operation of $G(\delta)$ is given explicitly by

$$(x,y,z)*(\tilde{x},\tilde{y},\tilde{z})=(x+e^{-z}\tilde{x},y+e^{\delta z}\tilde{y},z+\tilde{z}). \tag{1.5}$$

The Lie algebra $\mathfrak{g}(\delta)$ of $G(\delta)$ is

$$\left\{\left(\begin{array}{ccc} -w & 0 & u\\ 0 & \delta w & v\\ 0 & 0 & 0\end{array}\right)\ \middle|\ u,v,w\in\mathbb{R}\right\}.$$

Take an orthonormal basis

$$E_1=\left(\begin{array}{ccc} 0&0&1\\ 0&0&0\\ 0&0&0\end{array}\right),\ E_2=\left(\begin{array}{ccc} 0&0&0\\ 0&0&1\\ 0&0&0\end{array}\right),\ E_3=\left(\begin{array}{ccc} c_1&0&0\\ 0&c_2&0\\ 0&0&0\end{array}\right).$$

We denote by e_i the left invariant vector field on $G(\delta)$ which is obtained by left translation of E_i. Then we have

$$e_1=e^{-z}\frac{\partial}{\partial x},\quad e_2=e^{\delta z}\frac{\partial}{\partial y},\quad e_3=\frac{\partial}{\partial z}, \tag{1.6}$$

$$[e_1,e_2]=0,\quad [e_2,e_3]=\delta\, e_2,\quad [e_3,e_1]=-e_1. \tag{1.7}$$

This commutation relations implies that every $G(\delta)$ is *solvable*.

The Levi-Civita connection ∇ of $G(\delta)$ is described as

$$\begin{gathered}\nabla_{e_1}e_1=-e_3,\quad \nabla_{e_1}e_2=0,\quad \nabla_{e_1}e_3=e_1,\\ \nabla_{e_2}e_1=0,\quad \nabla_{e_2}e_2=\delta\, e_3,\quad \nabla_{e_2}e_3=-\delta\, e_2,\\ \nabla_{e_3}e_1=0,\quad \nabla_{e_3}e_2=0,\quad \nabla_{e_3}e_3=0.\end{gathered} \tag{1.8}$$

Define the endomorphism field φ by $\varphi=\nabla e_3$ and put $\xi=e_3$ and $\eta=\mathrm{d}z$, then we have

$$\varphi e_1=e_1,\quad \varphi e_2=-\delta\, e_2,\quad \varphi e_3=0.$$

One can check that (φ,ξ,η,g) is a left invariant para-Sasakian structure on $G(\delta)$.

The covariant derivative $\nabla\eta$ is computed as

$$(\nabla_X\eta)Y=g(X,e_1)g(Y,e_1)-\delta\, g(X,e_2)g(Y,e_2).$$

On the other hand,

$$g(X,Y)-\eta(X)\eta(Y)=g(X,e_1)g(Y,e_1)+g(X,e_2)g(Y,e_2).$$

Thus $G(c_1,c_2)$ is special para-Sasakian if and only if $\delta=-1$. In particular, on $\mathbb{H}^3$, we have

$$(\nabla_X\eta)Y=g(X,Y)-\eta(X)\eta(Y).$$

2. Slant curves

2.1. **Almost paracontact curves.** We start with the following definition :

Definition 2.1. An arc length parametrized curve $\gamma(s)$ in an almost paracontact Riemannian manifold M is said to be a *slant curve* if its angle function $\theta(s)$ of $\gamma'(s)$ and ξ is constant along it.

In particular, an arc length parametrized curve $\gamma(s)$ is said to be a *almost paracontact curve* (or *para-Legendre curve*) if its unit tangent vector field $\gamma'(s)$ is orthogonal to ξ.

2.2. **Slant curves in $G(\delta)$.** In this subsection we study slant curves in the para-Sasakian group $G(\delta)$. Slant curves in $\mathbb{H}^3$ equipped with Kenmotsu structure are investigated in [6]. The notion of slant curve in $\mathbb{H}^3$ with respecto to para-Sasakian structure coincides with that for Kenmotsu structure. Thus we may use results obtained in [6].

Let $\gamma(s) = (x(s), y(s), z(s))$ be a unit speed curve in $G(\delta)$. Then the unit tangent vector field is given by

$$\begin{aligned} T(s) =&\gamma'(s) = x'(s)\frac{\partial}{\partial x} + y'(s)\frac{\partial}{\partial y} + z'(s)\frac{\partial}{\partial z} \\ =&e^{z(s)}x'(s)e_1 + e^{z(s)}y'(s)e_2 + z'(s)e_3. \end{aligned}$$

We put

$$T_1(s) = e^{z(s)}x'(s), \quad T_2(s) = e^{-\delta\, z(s)}y'(s), \quad T_3(s) = z'(s). \tag{2.1}$$

The last equation implies that $\gamma(s)$ is almost Legendre if and only if z is constant.

By the arc length condition, we have

$$T_1(s)^2 + T_2(s)^2 + T_3(s)^2 = 1.$$

On the other hand the contact angle is given by

$$\cos\theta(s) = z'(s) = T_3(s).$$

Thus we get $T_1(s)^2 + T_2(s)^2 = \sin^2\theta(s)$. Hence $(T_1(s), T_2(s))$ is expressed as

$$(T_1(s), T_2(s)) = (\sin\theta(s)\cos\psi(s), \sin\theta(s)\sin\psi(s))$$

for some function $\psi(s)$.

Since $G(\delta)$ is homogeneous, it suffices to determine slant curves with $\theta \neq 0, \pi$ under the initial condition

$$x(0) = y(0) = z(0) = 0, \quad x'(0) = X, \quad y'(0) = Y, \quad z'(0) = Z. \tag{2.2}$$

First, the z-coordinate is determined as $z(s) = (\cos\theta)s$. Next, since

$$T_1(0) = \sin\theta\cos\psi(0) = X, \quad T_2(0) = \sin\theta\sin\psi(0) = Y,$$

we have

$$x(s) = \int_0^s e^{-z(s)}T_1(s)\,\mathrm{d}s = \sin\theta\int_0^s \exp((\cos\theta)s)\cos\psi(s)\,\mathrm{d}s,$$

$$y(s) = \int_0^s e^{\delta\, z(s)}T_1(s)\,\mathrm{d}s = \sin\theta\int_0^s \exp(\delta(\cos\theta)s)\sin\psi(s)\,\mathrm{d}s.$$

Theorem 2.1 ([6])**.** *A slant curve with $\theta \neq 0$, π in $\mathbb{H}^3$ starting at the origin is represented as*

$$\begin{aligned} x(s) =& \sin\theta \int_0^s \exp((\cos\theta)s)\cos\psi(s)\,\mathrm{d}s,\\ y(s) =& \sin\theta \int_0^s \exp((\cos\theta)s)\sin\psi(s)\,\mathrm{d}s,\\ z(s) =& (\cos\theta)s. \end{aligned}$$

Theorem 2.2. *A slant curve with $\theta \neq 0$, π in Sol_3 starting at the origin is represented as*

$$\begin{aligned} x(s) =& \sin\theta \int_0^s \exp((\cos\theta)s)\cos\psi(s)\,\mathrm{d}s,\\ y(s) =& \sin\theta \int_0^s \exp(-(\cos\theta)s)\sin\psi(s)\,\mathrm{d}s,\\ z(s) =& (\cos\theta)s. \end{aligned}$$

3. φ-TRAJECTORIES

3.1. **Paracontact planar curves.** Let M be an almost paracontact Riemannian manifold. Then a curve $\gamma(t)$ is said to be a *paracontact planar curve* [5] if there exist functions $a(t)$ and $b(t)$ defined along $\gamma(t)$ such that

$$\nabla_{\gamma'}\gamma' = a(t)\gamma'(t) + b(t)\varphi\gamma'(t).$$

Paracontact planar curves are examples of so-called F-geodesics or F-planar curves [3, 4, 22]. In this section we consider curves satisfying

$$\nabla_{\gamma'}\gamma' = c\,\varphi\gamma'. \tag{3.1}$$

Here c is a constant (called the *charge*). Arc length parametrized curves satisfying this ODE are called *φ-trajectories*.

Remark 3. In an almost contact Riemannian manifold $(M, \psi, \zeta, \omega, g)$, one can consider ψ-trajectories:

$$\nabla_{\gamma'}\gamma' = c\,\psi\gamma'.$$

In this case, we have the following *conservation law*:

$$\frac{\mathrm{d}}{\mathrm{d}t}g(\gamma'(t),\gamma'(t)) = 2g(\nabla_{\gamma'}\gamma',\gamma'(t)) = cg(\psi\gamma'(t),\gamma'(t))) = 0,$$

since ψ is skew-adjoint. Thus ψ-trajectories are of constant speed.

On the contrary, when the ambient space is almost paracontact Riemannian, this conservation law does not hold. We can deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}g(\gamma'(t),\gamma'(t)) = cg(\varphi\gamma'(t),\gamma'(t))).$$

For simplicity, let us assume that M is special para-Sasakian then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}g(\gamma'(t),\gamma'(t)) =& -c\varepsilon g(\gamma'(t) - \eta(\gamma'(t))\xi, \gamma'(t)))\\ =& -c\varepsilon\{g(\gamma'(t),\gamma'(t)) - \eta(\gamma'(t))^2\}. \end{aligned}$$

Hence $\gamma(t)$ is of constant speed when and only when $c=0$ or $g(\gamma'(t),\gamma'(t))=\eta(\gamma'(t))^2$. Thus arc length parametrization causes restrictions for φ-trajectories. Thus we do not assume that φ-trajectories are arc length parametrized.

3.2. **φ-trajectories in $G(\delta)$.** Let $\gamma(t)=(x(t),y(t),z(t))$ be a regular curve in $G(\delta)$. As we pointed out before, t is not necessarily arc length parametrized. The tangent vector field $T=\gamma'$ is given by

$$\begin{aligned}T(t)=\gamma'(s)=&x'(t)\frac{\partial}{\partial x}+y'(t)\frac{\partial}{\partial y}+z'(t)\frac{\partial}{\partial z}\\=&e^{z(t)}x'(t)e_1+e^{z(t)}y'(t)e_2+z'(t)e_3.\end{aligned}$$

We put

$$T_1(t)=e^{z(t)}x'(t),\quad T_2(t)=e^{-\delta\, z(t)}y'(t),\quad T_3(t)=z'(t).$$

The acceleration vector field is

$$\nabla_{\gamma'}\gamma'=(T_1'+T_3T_1)e_1+(T_2'-\delta\, T_3T_2)e_2+(T_3'-T_1^2+\delta\, T_2^2)e_3.$$

On the other hand,

$$\varphi\gamma'=T_1e_1-\delta\, T_2e_2.$$

Hence the φ-trajectory equation is the system

$$T_1'+T_3T_1=cT_1,\quad T_2'-\delta\, T_3T_2=-c\delta\, T_2,\quad T_3'-T_1^2+\delta\, T_2^2=0. \tag{3.2}$$

Remark 4. With respect to the Kenmotsu structure of $G(-1)$, ψ-trajectory equation is [12]:

$$T_1'+T_3T_1=-cT_2,\quad T_2'+T_3T_2=cT_1,\quad T_3'-(T_1^2+T_2^2)=0.$$

Let us determine φ-trajectories under the initial condition (2.2).

From the φ-trajectory system we deduce the following 2nd order ODE for T_3:

$$T_3''(t)=2(c-T_3(s))T_3'(s). \tag{3.3}$$

Obviously the constant function $T_3(t)=c$ is a solution to (3.3). First we observe φ-trajectories with $T_3(t)=c$. In this case, we obtain $z(t)=ct$. Next, substituting $T_3=c$ into the first and second equations of the system (3.2), we obtain $T_1'=T_2'=0$. Thus T_1 and T_2 are constant. From the third equation of (3.2), we deduce that $T_1^2=\delta\, T_2$. Under the initial condition (2.2), we get $T_1(t)=X$ and $T_2(t)=Y$. In particular, when $\delta=-1$, $X=Y=0$. In this case the φ-trajectory is a geodesic. On the other hand, when $\delta=1$, $Y=\pm X$. By integrating

$$x'(t)=Xe^{-ct},\quad y'(t)=Ye^{\delta\, ct},$$

we arrive at

$$x(t)=-\frac{X}{c}e^{-ct},\quad y(t)=\frac{\delta\, Y}{c}e^{\delta ct}.$$

Proposition 3.1. *The φ-trajectory $\gamma(t)$ of charge c in the special para-Sasakian $\mathbb{H}^3$ starting at the origin which satisfies the initial condition $z'(0)=c$ is a geodesic parametrized as*

$$\gamma(t)=(0,0,ct).$$

The φ-trajectory $\gamma(t)$ of charge $c \neq 0$ in the para-Sasakian Sol_3 *starting at the origin which satisfies the initial condition $z'(0) = c$ is parametrized as*

$$\gamma(t) = \left(-\frac{X}{c}e^{-ct}, \frac{\mp X}{c}e^{ct}, ct\right).$$

The φ-trajectory is rewritten as

$$\left(-\frac{X}{c}e^{-z}, \frac{\mp X}{c}e^{z}, z\right).$$

Hereafter we look for *non-constant* solutions to (3.3). The ODE (3.3) is rewritten as

$$(T_3(s) - c)'' = -\{(T_3(s) - c)^2\}'.$$

By integration we get

$$(T_3(s) - c)' = -(T_3(s) - c)^2 + k, \quad k \in \mathbb{R}.$$

(1) The Case $k = 0$: Under the initial condition we get

$$z(t) = ct + \log|1 + (Z - c)t|, \quad T_3(t) = c + \frac{Z - c}{1 + (Z - c)t}. \tag{3.4}$$

When $Z = c$, $z(t) = ct$ and $T_3(t) = c$.

(2) The Case $k > 0$: If $k = 1/c_1^2 > 0$, then we have

$$T_3(t) = c + \frac{1}{c_1}\tanh\frac{t + c_2}{c_1}, \quad c_1 \in (\mathbb{R} \setminus \{0\}) \cup \{\pm\infty\}, \quad c_2 \in \mathbb{R}. \tag{3.5}$$

The constant c_2 is determined by

$$c_2 = c_1 \tanh^{-1}\{c_1(Z - c)\}.$$

This formula implies that

$$-1 < c_1(Z - c) < 1. \tag{3.6}$$

The constant solution $T_3(t) = c$ is regarded as a particular solution with $c_1 = \pm\infty$.

(3) The Case $k < 0$: If $k = -1/c_1^2 > 0$, then we have

$$T_3(t) = c - \frac{1}{c_1}\tan\frac{t + c_2}{c_1}, \quad c_1 \in (\mathbb{R} \setminus \{0\}) \cup \{\pm\infty\}, \quad c_2 \in \mathbb{R}. \tag{3.7}$$

The constant c_2 is determined by

$$c_2 = c_1 \tan^{-1}\{c_1(c - Z)\}.$$

The constant solution $T_3(t) = c$ is regarded as a particular solution with $c_1 = \pm\infty$.

Hereafter we assume that $c_1 \neq \pm\infty$ for the cases $k = \pm/c_1^2$.

From the φ-trajectory system, we get

$$\frac{\mathrm{d}T_1}{T_1} = (-\delta)\frac{\mathrm{d}T_2}{T_2} = c - T_3.$$

From the initial condition (2.2), we obtain

$$T_1(t) = X\exp(ct - z(t)), \quad T_2(t) = Y\exp\{(-\delta)(ct - z(t))\}.$$

Hence we get

$$x(t) = X\int_0^t \exp(ct - 2z(t))\,\mathrm{d}t, \tag{3.8}$$

$$y(t) = Y\int_0^t \exp\{(-\delta)(ct - 2z(t))\}\,\mathrm{d}t. \tag{3.9}$$

3.3. Let us consider the case $T_3(t)$ is given as in (3.4). Then we have

$$ct - 2z(t) = -ct - 2\log|1 + (Z - c)t|.$$

Hence we obtain

$$T_1(t) = \frac{X}{1 + (Z - c)t}, \quad T_2(t) = Y\{1 + (Z - c)t\}^{\delta},$$

$$x'(t) = \frac{Xe^{-ct}}{\{1 + (Z - c)t\}^2}, \quad y'(t) = Ye^{\delta\, ct}\{1 + (Z - c)t\}^{2\delta}.$$

The x-coordinate is given by

$$x(t) = \int_0^x \frac{Xe^{-ct}}{\{1 + (Z - c)t\}^2}\,\mathrm{d}t.$$

Here we have

$$\int \frac{e^{-ct}}{\{1 + (Z - c)t\}^2}\,\mathrm{d}t = -\frac{c}{(Z - c)^2}\left\{\frac{e^{-ct}}{ct + \frac{c}{Z-c}} - \exp\left(\frac{c}{Z - c}\right)\mathrm{Ei}\left(1, ct + \frac{c}{Z - c}\right)\right\},$$

where $\mathrm{Ei}(a, z)$ is the *exponential integral function* defined by

$$\mathrm{Ei}(a, z) = \int_1^\infty \exp(-tz)\,t^{-a}\,\mathrm{d}t, \quad z > 0.$$

Next when $\delta = 1$, y-coordinate is integrated as

$$y(t) = \frac{Ye^{-ct}}{c^3}\left\{c^2\{1 + (Z - c)t\}^2 - 2c\{1 + (Z - c)t\} + 2(Z - c)^2\right\} - \frac{Y}{c^3}(c^2 - 2c + 2(Z - c)^2).$$

3.4. Next we consider the case $T_3(t)$ is given as in (3.5). Then z-coordinate is given by

$$z(t) = \int_0^t c + \frac{1}{c_1}\tanh\frac{t + c_2}{c_1}\,\mathrm{d}t = ct + \log\cosh\frac{t + c_2}{c_1} - \log\cosh\frac{c_2}{c_1}.$$

Thus we obtain

$$e^{-z(t)} = \frac{e^{-ct}\cosh\frac{c_2}{c_1}}{\cosh\frac{t+c_2}{c_1}}, \quad \exp(ct - z(t)) = \frac{\cosh\frac{c_2}{c_1}}{\cosh\frac{t+c_2}{c_1}}.$$

Hence T_1 and T_2 are determined as

$$T_1(t) = X\left(\frac{\cosh\frac{c_2}{c_1}}{\cosh\frac{t+c_2}{c_1}}\right), \quad T_2(t) = Y\left(\frac{\cosh\frac{c_2}{c_1}}{\cosh\frac{t+c_2}{c_1}}\right)^{-\delta}.$$

Inserting these into the the third equation of the φ-trajectory system and evaluated at $t = 0$, we deduce that

$$c_1^2\{X^2 - \delta\, Y^2 + (Z - c)^2\} = 1 \tag{3.10}$$

Thus the integral constant c_1 is determined by the initial data X, Y and Z with charge c. Comparing (3.6) and (3.10),

$$0 < \{c_1(Z - c)\}^2 = 1 - c_1^2(X^2 - \delta\, Y^2) < 1.$$

This implies that $X^2 - \delta Y^2 > 0$. When $\delta = -1$, this condition is automatically satisfied. In case $\delta = 1$, we obtain $X^2 - Y^2 > 0$.

Since

$$x'(t) = Xe^{-ct}\left(\frac{\cosh \frac{c_2}{c_1}}{\cosh \frac{t+c_2}{c_1}}\right)^2, \quad y'(t) = Ye^{\delta\, ct}\left(\frac{\cosh \frac{c_2}{c_1}}{\cosh \frac{t+c_2}{c_1}}\right)^{-2\delta},$$

the x-coordinate is integrated as

$$x(t) = X \cosh^2 \frac{c_2}{c_1} \int_0^t \frac{e^{-ct}}{\cosh^2 \frac{t+c_2}{c_1}}\, \mathrm{d}t.$$

When $\delta = 1$, the y-coordinate is given by

$$y(t) = \frac{Y}{\cosh^2 \frac{c_2}{c_1}} \int_0^t e^{ct} \cosh^2 \frac{t + c_2}{c_1}\, \mathrm{d}t.$$

Note that

$$\exp \frac{c_2}{c_1} = \frac{|1 + c_1(Z - c)|}{\sqrt{1 - c_1^2(Z - c)^2}}.$$

3.5. Finally we consider the case $T_3(t)$ is given as in (3.7). Then z-coordinate is given by

$$z(t) = \int_0^t c - \frac{1}{c_1} \tan \frac{t + c_2}{c_1}\, \mathrm{d}t = ct + \log \cos \left|\frac{t + c_2}{c_1}\right| - \log \cos \left|\frac{c_2}{c_1}\right|.$$

Thus we obtain

$$e^{-z(t)} = e^{-ct}\frac{|\cos \frac{c_2}{c_1}|}{|\cos \frac{t+c_2}{c_1}|}, \quad \exp(ct - z(t)) = \frac{|\cos \frac{c_2}{c_1}|}{|\cos \frac{t+c_2}{c_1}|}.$$

Hence T_1 and T_2 are determined as

$$T_1(t) = X\frac{|\cos \frac{c_2}{c_1}|}{|\cos \frac{t+c_2}{c_1}|}, \quad T_2(t) = Y\left(\frac{|\cos \frac{c_2}{c_1}|}{|\cos \frac{t+c_2}{c_1}|}\right)^{-\delta}.$$

Inserting these into the the third equation of the φ-trajectory system and evaluated at $t = 0$, we deduce that

$$c_1^2\{X^2 - \delta\, Y^2 - (c - Z)^2\} = 1 \tag{3.11}$$

Thus the integral constant c_1 is determined by the initial data X, Y and Z with charge c.

The x-coordinate is given by

$$x(t) = X \cos^2 \frac{c_2}{c_1} \int_0^t \frac{e^{-ct}}{\cos^2 \frac{c_1 t+c_2}{c_1}}\, \mathrm{d}t.$$

When $\delta = 1$, the y-coordinate is given by

$$y(t) = \frac{Y}{\cos^2 \frac{c_2}{c_1}} \int_0^t e^{ct} \cos^2 \frac{t+c_2}{c_1}\, \mathrm{d}t.$$

We carry out these integrations. For x-coordinate, we have

$$\int e^{ct} \cos^2 \frac{t+c_2}{c_1}\, \mathrm{d}t = \frac{ce^{ct}}{2} \frac{\cos \frac{2(t+c_2)}{c_1}}{(c^2 + \frac{4}{c_1^2})} + \frac{e^{ct} \sin \frac{2(t+c_2)}{c_1}}{c_1(c^2 + \frac{4}{c_1^2})} + \frac{e^{ct}}{2c}.$$

Next, for y-coordinate, we get

$$\int \cos^2 \frac{t+c_2}{c_1} e^{ct}\, \mathrm{d}t = \int \frac{1}{2}\left(1 + \cos \frac{2(t+c_2)}{c_1}\right) e^{ct}\, \mathrm{d}t$$
$$= \frac{e^{ct}}{2c(c^2c_1^2+4)} \left(c^2c_1^2 \cos \frac{2(t+c_2)}{c_1} + 2cc_1 \sin \frac{2(t+c_2)}{c_1} + c^2c_1^2 + 4\right).$$

By using these, the x-coordinate is integrated as

$$\begin{aligned} x(t) =& \frac{e^{-ct}|\cos \frac{c_2}{c_1}|}{|\cos \frac{t+c_2}{c_1}|} \int_0^t X \frac{|\cos \frac{c_2}{c_1}|}{|\cos \frac{t+c_2}{c_1}|}\, \mathrm{d}t = \frac{2c_1 X\, e^{-ct} \cos^2 \frac{c_2}{c_1}}{|\cos \frac{t+c_2}{c_1}|} \log \frac{1+\sin \frac{t+c_2}{c_1}}{\cos \frac{t+c_2}{c_1}} \\ =& \frac{2c_1 X\, e^{-ct} \cos^2 \frac{c_2}{c_1}}{|\cos \frac{t+c_2}{c_1}|} \left(\log \frac{1+\sin \frac{t+c_2}{c_1}}{\cos \frac{t+c_2}{c_1}} - \log \frac{1+\sin \frac{c_2}{c_1}}{\cos \frac{c_2}{c_1}}\right). \end{aligned}$$

Note that

$$\exp \frac{c_2}{c_1} = \frac{|1 + c_1(Z-c)|}{\sqrt{1 - c_1^2(Z-c)^2}}.$$

When $\delta = 1$, the y-coordinate is given by

$$y(t) = \frac{e^{-ct}|\cos \frac{c_2}{c_1}|}{|\cos \frac{t+c_2}{c_1}|} \int_0^t Y \frac{|\cos \frac{t+c_2}{c_1}|}{|\cos \frac{c_2}{c_1}|}\, \mathrm{d}t = \frac{c_1 Y\, e^{-ct}}{|\cos \frac{t+c_2}{c_1}|} \left(\sin \frac{t+c_2}{c_1} - \sin \frac{c_2}{c_1}\right).$$

4. Main theorem

Now we state our main results.

Theorem 4.1. *The non-geodesic φ-trajectories in the special para-Sasakian $\mathbb{H}^3$ are congruent to one of the following curves*:

- *The curve parametrized as*

$$\begin{aligned} x(t) =& \frac{Xe^{-ct} \log|1 + (Z-c)t|}{(Z-c)\{1 + (Z-c)t\}}, \\ y(t) =& \frac{Ye^{-ct} \log|1 + (Z-c)t|}{(Z-c)\{1 + (Z-c)t\}}, \\ z(t) =& ct + \log|1 + (Z-c)t|, \end{aligned}$$

  where $Z \neq c$.

- *The curve parametrized as*

$$\begin{aligned}
x(t) =& \frac{2c_1 X\, e^{-ct} \cosh^2 \frac{c_2}{c_1}}{\cosh \frac{t+c_2}{c_1}} (\tan^{-1} \exp \frac{t+c_2}{c_1} - \tan^{-1} \exp \frac{c_2}{c_1}).\\
y(t) =& \frac{2c_1 Y\, e^{-ct} \cosh^2 \frac{c_2}{c_1}}{\cosh \frac{t+c_2}{c_1}} (\tan^{-1} \exp \frac{t+c_2}{c_1} - \tan^{-1} \exp \frac{c_2}{c_1}).\\
z(t) =& ct + \log\cosh \frac{t+c_2}{c_1} - \log\cosh \frac{c_2}{c_1},
\end{aligned}$$

where the constants c_1 and c_2 satisfies

$$-1 < c_1(Z-c) < 1, \quad c_1^2\{X^2+Y^2+(Z-c)^2\} = 1, \quad \tanh \frac{c_2}{c_1} = c_1(Z-c).$$

- *The curve parametrized as*

$$\begin{aligned}
x(t) =& \frac{2c_1 X\, e^{-ct} \cos^2 \frac{c_2}{c_1}}{|\cos \frac{t+c_2}{c_1}|} \left(\log \frac{1+\sin \frac{t+c_2}{c_1}}{\cos \frac{t+c_2}{c_1}} - \log \frac{1+\sin \frac{c_2}{c_1}}{\cos \frac{c_2}{c_1}} \right),\\
y(t) =& \frac{2c_1 Y\, e^{-ct} \cos^2 \frac{c_2}{c_1}}{|\cos \frac{t+c_2}{c_1}|} \left(\log \frac{1+\sin \frac{t+c_2}{c_1}}{\cos \frac{t+c_2}{c_1}} - \log \frac{1+\sin \frac{c_2}{c_1}}{\cos \frac{c_2}{c_1}} \right),\\
z(t) =& ct + \log\cos \left| \frac{t+c_2}{c_1} \right| - \log\cos \left| \frac{c_2}{c_1} \right|,
\end{aligned}$$

where the constants c_1 and c_2 satisfies

$$c_1^2\{X^2+Y^2-(c-Z)^2\} = 1, \quad \tan \frac{c_2}{c_1} = c_1(c-Z).$$

Theorem 4.2. *The φ-trajectories in the para-Sasakian* Sol_3 *are congruent to one of the following curves*:

- *The curve parametrized as*

$$\left(-\frac{X}{c} e^{-ct}, \frac{\mp X}{c} e^{ct}, ct \right)$$

  with initial condition $Z = c \neq 0$.
- *The curve parametrized as*

$$\begin{aligned}
x(t) =& \frac{X e^{-ct} \log|1+(Z-c)t|}{(Z-c)\{1+(Z-c)t\}},\\
y(t) =& \frac{Y e^{-ct}}{1+(Z-c)t} \left(t + \frac{Z-c}{2} t^2 \right),\\
z(t) =& ct + \log|1+(Z-c)t|,
\end{aligned}$$

  where $Z \neq c$.

- *The curve parametrized as*

$$\begin{aligned}
x(t) =& \frac{2c_1 X\, e^{-ct} \cosh^2 \frac{c_2}{c_1}}{\cosh \frac{t+c_2}{c_1}} (\tan^{-1} \exp \frac{t+c_2}{c_1} - \tan^{-1} \exp \frac{c_2}{c_1}). \\
y(t) =& \frac{c_1 Y e^{-ct}}{\cosh \frac{t+c_2}{c_1}} \left(\sinh \frac{t+c_2}{c_1} - \sinh \frac{c_2}{c_1} \right), \\
z(t) =& ct + \log \cosh \frac{t+c_2}{c_1} - \log \cosh \frac{c_2}{c_1},
\end{aligned}$$

where the constants c_1 *and* c_2 *satisfies*

$$-1 < c_1(Z-c) < 1, \quad c_1^2\{X^2 - Y^2 + (Z-c)^2\} = 1, \quad \tanh \frac{c_2}{c_1} = c_1(Z-c).$$

- *The curve parametrized as*

$$\begin{aligned}
x(t) =& \frac{2c_1 X\, e^{-ct} \cos^2 \frac{c_2}{c_1}}{|\cos \frac{t+c_2}{c_1}|} \left(\log \frac{1+\sin \frac{t+c_2}{c_1}}{\cos \frac{t+c_2}{c_1}} - \log \frac{1+\sin \frac{c_2}{c_1}}{\cos \frac{c_2}{c_1}} \right), \\
y(t) =& \frac{c_1 Y\, e^{-ct}}{|\cos \frac{t+c_2}{c_1}|} \left(\sin \frac{t+c_2}{c_1} - \sin \frac{c_2}{c_1} \right), \\
z(t) =& ct + \log \cos \left| \frac{t+c_2}{c_1} \right| - \log \cos \left| \frac{c_2}{c_1} \right|,
\end{aligned}$$

where the constants c_1 *and* c_2 *satisfies*

$$c_1^2\{X^2 - Y^2 - (c-Z)^2\} = 1, \quad \tan \frac{c_2}{c_1} = c_1(c-Z).$$

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