A two-population game in observable double-ended queuing systems

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Abstract

In this study, the existing game theoretical framework is extended to strategic queuing in search of solutions for a two-population game in observable double-ended queuing systems with zero matching times. We show that multiple Nash equilibria and one unique subgame perfect Nash equilibrium exist in this game.

Keywords: Strategic queuing; game theory; two-population game; Nash equilibrium; subgame perfect Nash equilibrium.

1. Introduction

In this study, the concept of a *two-population game* in a strategic queuing model is introduced. An observable double-ended queuing system is considered with two populations of agents arriving at each side of the queue for one-by-one matching. This queuing model comes with various practical examples, such as taxi-passenger queues or electronic commerce platforms. In an observable double-ended queue with one strategic side considered in the literature [6] (which is identical to an M/M/1/K model), the Nash equilibrium (which is often referred to as "equilibrium threshold strategy") can be instantly derived using Naor's [5] line of research. However, when both types of agents are strategic, such a threshold strategy is no longer guaranteed because waiting times no longer increase with regard to the position of agents upon their arrival, but also depend on the number of agents arriving behind them. A quick check shows that the decision adopted by both sides not to join the system at all also forms an equilibrium, which naturally poses the question: Are there any other outcomes of this game? This problem does not seem to be solvable with the existing literature on strategic queuing (summarized in [2, 4]), which only focuses on the strategic behavior of one population of agents.
This study extends the current game theoretical framework to strategic queuing [1, 3] for solving a more general and realistic problem in which two populations of agents engage in strategic behavior, which affects the strategy of not only individuals in the same population but also those in the other population. The desired output is all possible Nash equilibria, that is, the outcomes of the game, which are helpful for the planning and optimization of the system.

2. Preliminaries

Consider a society $\mathcal{P} = \{1, 2\}$ that consists of two *populations of agents* arriving at a double-ended queuing system based on Poisson processes with rates λ_1 and λ_2. The buffer capacity of population-(i) is denoted by N_i ($i = 1, 2$). The state space is denoted by $\mathcal{S} = \{-N_1, -N_1 + 1, ..., N_2\}$, where a state $s < 0$ prescribes a queue with population-(1) agents, while a state $s > 0$ prescribes a queue with population-(2) agents, and $s = 0$ prescribes an empty system. Matching is performed on a first-come-first-served basis by a pair of a population-(1) agent and a population-(2) agent in zero unit time. The reward upon the completion of a service and the waiting cost per unit time of a population-(i) agent are denoted by R_i and C_i ($i = 1, 2$), respectively.
Agents can choose to join or balk the queue upon arrival. Agents receive a reward of 0 if they choose to balk. Let $\mathcal{A} = \{a_1, a_2\}$ be the set of *pure strategies* of each agent, where a_1 represents "joining" and a_2 represents "balking" upon arrival. The *strategy* of an arbitrary individual in population-(i) is denoted by $\boldsymbol{\sigma}^{(i)} \in [0, 1]^{N_1+N_2+1}$, which gives the probabilities $\sigma_s^{(i)}$ ($s = -N_1, -N_1 + 1, ..., 0, ..., N_2$) with which the agent decides to join the queue when each system state $s \in \mathcal{S}$ is observed upon arrival. Let a vector $\mathbf{x}^{(i)} \in [0, 1]^{N_1+N_2+1}$ ($i = 1, 2$) denote the *population profile* of population-(i), which yields the probabilities $x_s^{(i)}$ with which the strategy *joining* is played at each state s in population-(i). A *social profile*, defined over $[0, 1]^{N_1+N_2+1} \times [0, 1]^{N_1+N_2+1}$ as $\mathbf{X} = \{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\}$, consists of the strategy profiles of the two populations.

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Definition 1 (Limiting Probabilities, Recurrent States, Absorbing States, and Transient States)**.** *(Hassin and Haviv [3])* *Let L_t be the system state at the time of arrival of the t^{th} agent. The limiting probability of $L_t = s$ when $t \to +\infty$, given a social profile $\mathbf{X}$ and $L_0 = s$, is denoted by $\pi(\mathbf{X}, s)$. For a given social profile $\mathbf{X}$, a state s is* recurrent *if $\pi(\mathbf{X}, s) > 0$. If $\pi(\mathbf{X}, s) = 1$, then the recurrent state s is* absorbing*. If $\pi(\mathbf{X}, s) = 0$, the state s is* transient*.*

The *payoff* to a focal individual in population-(i), who adopts a strategy $\boldsymbol{\sigma}^{(i)}$ is given by

$$U^{(i)}(\boldsymbol{\sigma}^{(i)}|\mathbf{X}) = \sum_{s\in\mathcal{S}} \pi(\mathbf{X}, s) U^{(i)}(\boldsymbol{\sigma}^{(i)}|\mathbf{X}, s) = \sum_{s\in\mathcal{S}} \pi(\mathbf{X}, s) \left[\sigma_s^{(i)} U^{(i)}(a_1|\mathbf{X}, s) + (1 - \sigma_s^{(i)}) U^{(i)}(a_2|\mathbf{X}, s)\right]$$
$$= \sum_{s\in\mathcal{S}} \pi(\mathbf{X}, s) \sigma_s^{(i)} U^{(i)}(a_1|\mathbf{X}, s), \tag{1}$$

where $U^{(i)}(a_j|\mathbf{X}, s)$ $(j = 1, 2)$ denotes the payoff to the focal individual in the population-(i), who adopts the pure strategy a_j upon observing state s. By definition, $U^{(i)}(a_2|\mathbf{X}, s) = 0$, while $U^{(i)}(a_1|\mathbf{X}, s)$ is obtained by subtracting the waiting cost from the service value.
A strategy $\boldsymbol{\sigma}^{(i)}$ of a focal individual in population-(i) is called a *best response* against a social profile $\mathbf{X}$ if

$$\boldsymbol{\sigma}^{(i)} \in BR^{(i)}(\mathbf{X}) = \arg\max_{\boldsymbol{\sigma}^{(i)}} U^{(i)}(\boldsymbol{\sigma}^{(i)}|\mathbf{X}), \tag{2}$$

where $BR^{(i)}(\mathbf{X})$ denotes the set of best responses against $\mathbf{X}$ of an arbitrary individual in population-(i).

Definition 2 (Nash Equilibria)**.** *A social profile $\bar{\mathbf{X}} = \left(\bar{\mathbf{x}}^{(1)}, \bar{\mathbf{x}}^{(2)}\right)$ is in a* Nash equilibrium *if all agents in each population respond optimally to $\bar{\mathbf{X}}$, that is,*

$$NE = \{\bar{\mathbf{X}} : \bar{\boldsymbol{\sigma}}^{(i)} \in BR^{(i)}(\bar{\mathbf{X}}) \textit{ for all } i \in \mathcal{P}\}, \tag{3}$$

where $\bar{\boldsymbol{\sigma}}^{(i)}$ is the strategy that generates the population profile $\bar{\mathbf{x}}^{(i)}$ $(\bar{\boldsymbol{\sigma}}^{(i)} = \bar{\mathbf{x}}^{(i)})$.

Definition 3 (Subgame Perfect Nash Equilibria)**.** *A social profile $\bar{\mathbf{X}}^* = \left(\bar{\mathbf{x}}^{*(1)}, \bar{\mathbf{x}}^{*(2)}\right)$ is in a* subgame perfect Nash equilibrium *if all agents in each population respond optimally to $\bar{\mathbf{X}}^*$ at each state in the state space $\mathcal{S}$, that is,*

$$SPNE = \{\bar{\mathbf{X}}^* : \bar{\sigma}_s^{*(i)} \in BR^{(i)}(\bar{\mathbf{X}}^*, s) \textit{ for all } i \in \mathcal{P} \textit{ and } s \in \mathcal{S}\}, \tag{4}$$

where $\bar{\boldsymbol{\sigma}}^{*(i)}$ is the strategy that generates the population profile $\bar{\mathbf{x}}^{*(i)}$ $(\bar{\mathbf{x}}^{*(i)} = \bar{\mathbf{x}}^{*(i)})$, and $BR^{(i)}(\mathbf{X}, s)$ denotes the set of best responses against a population profile $\mathbf{X}$ at state s, calculated as

$$BR^{(i)}(\mathbf{X}, s) = \arg\max_{\sigma_s^{(i)}} U(\sigma^{(i)}|\mathbf{X}, s).$$

The index "$\tilde{i}$" is also used to refer to a population other than i. In other words, $\tilde{i} = 2$ if $i = 1$, and vice versa.
Furthermore, let

$$N_i^e = \max\left\{n : R_i - C_i \frac{n}{\lambda_{\tilde{i}}} \geq 0, n \in \mathbb{N}\right\},$$

and $\mathcal{N}_i = \min\{N_i, N_i^e\}$. In this paper, N_i^e is referred to as 'Naor's threshold'.

Remark 1 (Discussion on Naor's result in the case of finite buffer size)**.** *In addition to considering two strategic populations, the model presented in this paper differs slightly from Naor's in the sense that we assume a finite buffer size for both sides, whereas an infinite buffer size was considered in Naor's model. It should be noted that Naor's result remains the same under a setting with an arbitrarily large finite buffer limit, as long as it is greater than Naor's threshold level. In Naor's model, if there is a limit for the buffer size and it is smaller than Naor's threshold, then the threshold adopted by customers is identical to that buffer limit.*

3. Nash equilibria

The Nash equilibria of this game is of interest, defined as $\bar{\mathbf{X}} = \left(\bar{\mathbf{x}}^{(1)}, \bar{\mathbf{x}}^{(2)}\right)$, where $\bar{\mathbf{x}}^{(i)}$ is generated from strategies $\bar{\boldsymbol{\sigma}}^{(i)}$, defined as

$$\bar{\boldsymbol{\sigma}}^{(i)} = \left(\bar{\sigma}_{-N_1}^{(i)}, \bar{\sigma}_{-N_1+1}^{(i)}, ..., \bar{\sigma}_{-1}^{(i)}, \bar{\sigma}_0^{(i)}, \bar{\sigma}_1^{(i)}, ..., \bar{\sigma}_{N_2-1}^{(i)}, \bar{\sigma}_{N_2}^{(i)}\right), i = 1, 2.$$

It immediately follows that $\bar{\sigma}_{-N_1}^{(1)} = \bar{\sigma}_{N_2}^{(2)} = 0$.

By defining the equilibria in (2) and (3), for $i = 1, 2$, the following condition must be satisfied:

$$U^{(i)}\left(\bar{\boldsymbol{\sigma}}^{(i)}|\bar{\mathbf{X}}\right) = \max_{\boldsymbol{\sigma}^{(i)}} U^{(i)}\left(\boldsymbol{\sigma}^{(i)}|\bar{\mathbf{X}}\right). \tag{5}$$

In equilibrium, the system can be modeled as shown in *Figure 1*. The dashed arrows indicate that the corresponding transition rates may be equal to 0.

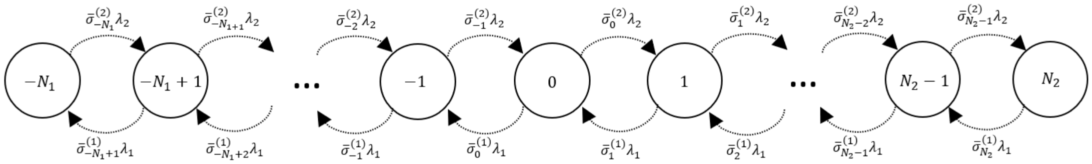

Figure 1: Transition diagram of the system in equilibrium.

The steady state balance equations are given by

$$\bar{\sigma}^{(2)}_s \lambda_2 \pi(\bar{\mathbf{X}}, s) = \bar{\sigma}^{(1)}_{s+1} \lambda_1 \pi(\bar{\mathbf{X}}, s+1) \tag{6}$$

for $s = -N_1, -N_1 + 1, ..., N_2 - 1$.

Lemma 1. *In equilibrium,*
(1) If there exists a recurrent state $s < 0$, then $\bar{\sigma}^{(2)}_s = 1$, $\bar{\sigma}^{(1)}_{s+1} > 0$, and $s + 1$ is also a recurrent state.
(2) If there exists a recurrent state $s > 0$, then $\bar{\sigma}^{(1)}_s = 1$, $\bar{\sigma}^{(2)}_{s-1} > 0$, and $s - 1$ is also a recurrent state.

Proof. (1) Assume that $\bar{\sigma}^{(2)}_s < 1$. Then, an arbitrary population-(2) agent who adopts a strategy $\boldsymbol{\sigma}^{(2)}$, where $\sigma^{(2)}_j = \bar{\sigma}^{(2)}_j$ for all $j \neq s$, and $\sigma^{(2)}_s = 1$, would find the expected payoff of

$$\begin{aligned}
U^{(2)}\left(\boldsymbol{\sigma}^{(2)}|\bar{\mathbf{X}}\right) &= \sum_{j\neq s} \pi(\bar{\mathbf{X}}, j) U^{(2)}\left(\boldsymbol{\sigma}^{(2)}|\bar{\mathbf{X}}, j\right) + \pi(\bar{\mathbf{X}}, s) U^{(2)}\left(\boldsymbol{\sigma}^{(2)}|\bar{\mathbf{X}}, s\right) \\
&= \sum_{j\neq s} \pi(\bar{\mathbf{X}}, j) U^{(2)}\left(\boldsymbol{\sigma}^{(2)}|\bar{\mathbf{X}}, j\right) + \pi(\bar{\mathbf{X}}, s) R_2 \\
&> \sum_{j\neq s} \pi(\bar{\mathbf{X}}, j) U^{(2)}\left(\boldsymbol{\sigma}^{(2)}|\bar{\mathbf{X}}, j\right) + \pi(\bar{\mathbf{X}}, s) \bar{\sigma}^{(2)}_s R_2 \text{ (as } \bar{\sigma}^{(2)}_s < 1 \text{ and } \pi(\bar{\mathbf{X}}, s) > 0), \\
&= U^{(2)}\left(\bar{\boldsymbol{\sigma}}^{(2)}|\bar{\mathbf{X}}\right),
\end{aligned}$$

which contradicts the definition of the best responses and equilibria in (2) and (3). This means that $\bar{\sigma}^{(2)}_s = 1$. Because $\pi(\bar{\mathbf{X}}, s) > 0$ and $\sigma^{(2)}_s = 1$, the left-hand side of (6) is positive, implying that $\bar{\sigma}^{(1)}_{s+1} \lambda_1 \pi(\bar{\mathbf{X}}, s+1) > 0$. Thus, $\bar{\sigma}^{(1)}_{s+1} > 0$ and $\pi(\bar{\mathbf{X}}, s+1) > 0$. This also means that $s + 1$ is recurrent.
(2) This can be similarly proved.

□

From **Lemma 1**, the following important result can be obtained by induction: In equilibrium, if there exists a recurrent state $s < 0$, then all states $s + 1, s + 2, ..., 0$ are also recurrent, and $\sigma^{(2)}_j = 1$ for all $j = s, s + 1, ..., 0$. Similarly, if there exists a recurrent state $s > 0$ in equilibrium, then all states $s - 1, s - 2, ..., 0$ are also recurrent, and $\sigma^{(1)}_j = 1$ for all $j = s, s - 1, ..., 0$. For deriving the joining strategy of a tagged agent, it is necessary to obtain the expected waiting time with respect to the state observed upon arrival, prescribed one-dimensionally by $s \in \mathcal{S}$, which also encodes the number of agents in the same population in front of the tagged agent. This expected waiting time is 0 if there is currently a queue of agents in the opponent population. If a tagged agent arrives and observes a queue of agents in the same population, one more variable that represents the number of agents behind the tagged agent is required to derive the expected waiting time of the tagged agent. This is because the arrival of other agents in the same population behind the tagged agent affects the strategy of agents in the opponent population. According to first step analysis, the expected waiting time $T_i(u, v)$ of a tagged population-(i) agent who is at position $u > 0$ (that is, there are $u - 1$ other population-(i) agents in front of the tagged agent) and observes v other population-(i) agents behind him ($u + v \leq N_i$) is

$$T_i(u, v) = \begin{cases} \frac{1}{\bar{\sigma}^{(i)}_w \lambda_i + \bar{\sigma}^{(\bar{i})}_w \lambda_{\bar{i}}} + \frac{\bar{\sigma}^{(i)}_w \lambda_i}{\bar{\sigma}^{(i)}_w \lambda_i + \bar{\sigma}^{(\bar{i})}_w \lambda_{\bar{i}}} T_i(u, v+1) + \frac{\bar{\sigma}^{(\bar{i})}_w \lambda_{\bar{i}}}{\bar{\sigma}^{(i)}_w \lambda_i + \bar{\sigma}^{(\bar{i})}_w \lambda_{\bar{i}}} T_i(u-1, v) & \text{if } u + v < N_i, \\ \frac{1}{\bar{\sigma}^{(\bar{i})}_w \lambda_{\bar{i}}} + T_i(u-1, v) & \text{if } u + v = N_i, \end{cases} \tag{7}$$

where $T_i(0,v)=0$ is the bound for the recursion, $w=-(u+v)$ if $i=1$, and $w=u+v$ if $i=2$. The joining strategy of a population-(i) agent upon a state s is based on the expected waiting time upon arrival, that is, $T_i(|s|,0)$.
By induction, it is easy to determine that $T_i(u,v)\geq \frac{u}{\lambda_{\tilde{i}}}$. Intuitively, $\frac{u}{\lambda_{\tilde{i}}}$ is the expected waiting time of a population-(i) agent at position u, in an "ideal" scenario that population-$(\tilde{i})$ is not strategic (that is, population-$(\tilde{i})$ agents always join the system with probability 1). Therefore, when population-$(\tilde{i})$ is strategic, a population-(i) agent should expect a longer waiting time.
In equilibrium, let $s_1=\min\left\{s:-N_1\leq s\leq 0,\pi\left(\bar{\mathbf{X}},s\right)>0\right\}$ and $s_2=\max\left\{s:0\leq s\leq N_2,\pi\left(\bar{\mathbf{X}},s\right)>0\right\}$.
Consider the following 4 cases.

Case 1: $s_1<0$ and $s_2>0$. Induced from **Lemma 1**, the following results can be obtained:

- $\bar{\sigma}_s^{(2)}=1$ and $\bar{\sigma}_{s+1}^{(1)}>0$ for all $s_1\leq s<0$, and
- $\bar{\sigma}_s^{(1)}=1$ and $\bar{\sigma}_{s-1}^{(2)}>0$ for all $0<s\leq s_2$, and
- any state s satisfying $s_1\leq s\leq s_2$ is recurrent.

From (6), we have $\bar{\sigma}_{s_1-1}^{(2)}\lambda_2\pi(\bar{\mathbf{X}},s_1-1)=\bar{\sigma}_{s_1}^{(1)}\lambda_1\pi(\bar{\mathbf{X}},s_1)$, which implies that $\bar{\sigma}_{s_1}^{(1)}=0$ (as $\pi(\bar{\mathbf{X}},s_1)>0$ and $\pi(\bar{\mathbf{X}},s_1-1)=0$).
Similarly, $\bar{\sigma}_{s_2}^{(2)}=0$ can be obtained.
For any $s_1<s\leq 0$, the condition $R_1-C_1T_1(|s|+1,0)\geq 0$ must be satisfied because if $R_1-C_1T_1(|s|+1,0)<0$, then an arbitrary population-(1) agent who adopts a strategy $\boldsymbol{\sigma}^{(1)}$, where $\sigma_s^{(1)}=0$, and $\sigma_j^{(1)}=\bar{\sigma}_j^{(1)}$ for $j\neq s$, would find a payoff of

$$\begin{aligned}
U^{(1)}\left(\boldsymbol{\sigma}^{(1)}|\bar{\mathbf{X}}\right)&=\sum_{j\neq s}\pi(\bar{\mathbf{X}},j)U^{(1)}\left(\boldsymbol{\sigma}^{(1)}|\bar{\mathbf{X}},j\right)+\pi(\bar{\mathbf{X}},s)U^{(1)}\left(\boldsymbol{\sigma}^{(1)}|\bar{\mathbf{X}},s\right)\\
&=\sum_{j\neq s}\pi(\bar{\mathbf{X}},j)U^{(1)}\left(\boldsymbol{\sigma}^{(1)}|\bar{\mathbf{X}},j\right)\\
&>\sum_{j\neq s}\pi(\bar{\mathbf{X}},j)U^{(1)}\left(\boldsymbol{\sigma}^{(1)}|\bar{\mathbf{X}},j\right)+\pi(\bar{\mathbf{X}},s)\bar{\sigma}_s^{(1)}(R_1-C_1T_1(|s|+1,0))\ (\text{as } \bar{\sigma}_s^{(1)}>0 \text{ and } \pi(\bar{\mathbf{X}},s)>0),\\
&=U^{(1)}\left(\bar{\boldsymbol{\sigma}}^{(1)}|\bar{\mathbf{X}}\right),
\end{aligned}$$

which contradicts the definition of "best response" and "equilibria" in (2) and (3). Furthermore, if $R_1-C_1T_1(|s|+1,0)>0$, then it is easily implied that $\sigma_s^{(1)}=1$. If $R_1-C_1T_1(|s|+1,0)=0$, then $\sigma_s^{(1)}\in(0,1)$.
Consequently,

$$|s_i|\leq\mathcal{N}_i. \tag{8}$$

Consider the case in which $i=1$. The above condition is obvious when $N_1\leq N_1^e$ (which is equivalent to $\mathcal{N}_1=N_1$) because s_1 cannot exceed the buffer capacity of the population-(1) queue (N_1). When $N_1>N_1^e$ (that is, $\mathcal{N}_1=N_1^e$), this can be proved by contradiction. Assume that $|s_1|>N_1^e$, then

$$R_1-C_1T_1(N_1^e+1,0)\leq R_1-C_1\frac{N_1^e+1}{\lambda_2}<0,$$

which contradicts the definition of N_1^e.
Similarly, for any $s_2\leq\mathcal{N}_2$ and $0\leq s<s_2$, the condition $R_2-C_2T_2(s+1,0)\geq 0$ must be satisfied. If $R_2-C_2T_2(s+1,0)>0$, then $\bar{\sigma}_s^{(2)}=1$. If $R_2-C_2T_2(s+1,0)=0$, then $\bar{\sigma}_s^{(2)}\in(0,1)$.
Furthermore, the condition $R_i-C_iT_i(|s_i|+1,0)\leq 0$ $(i=1,2)$ must also be satisfied. Assuming that $R_i-C_iT_i(|s_i|+1,0)>0$, then an arbitrary population-(i) agent who adopts a strategy $\boldsymbol{\sigma}^{(i)}$, where $\sigma_{s_i}^{(i)}>0$ and $\sigma_j^{(i)}=\bar{\sigma}_j^{(i)}$ for $j\neq s_i$, would find a payoff of

$$\begin{aligned}
U^{(i)}\left(\boldsymbol{\sigma}^{(i)}|\bar{\mathbf{X}}\right)&=\sum_{j\neq s}\pi(\bar{\mathbf{X}},j)U^{(i)}\left(\boldsymbol{\sigma}^{(i)}|\bar{\mathbf{X}},j\right)+\pi(\bar{\mathbf{X}},s)U^{(i)}\left(\boldsymbol{\sigma}^{(i)}|\bar{\mathbf{X}},s_i\right)\\
&=\sum_{j\neq s_i}\pi(\bar{\mathbf{X}},j)U^{(i)}\left(\boldsymbol{\sigma}^{(i)}|\bar{\mathbf{X}},j\right)+\pi(\bar{\mathbf{X}},s_i)\sigma_{s_i}^{(i)}(R_i-C_iT_i(|s_i|+1,0))\\
&>\sum_{j\neq s_i}\pi(\bar{\mathbf{X}},j)U^{(i)}\left(\boldsymbol{\sigma}^{(i)}|\bar{\mathbf{X}},j\right)\ (\text{as } \sigma_{s_i}^{(i)}>0 \text{ and } \pi(\bar{\mathbf{X}},s_i)>0),\\
&=U^{(i)}\left(\bar{\boldsymbol{\sigma}}^{(i)}|\bar{\mathbf{X}}\right),
\end{aligned}$$

which contradicts the definition of "best response" and "equilibria" in (2) and (3). In summary, the system equilibrium in this case is

$$\bar{\mathbf{X}}=\left(\bar{\boldsymbol{\sigma}}^{(1)},\bar{\boldsymbol{\sigma}}^{(2)}\right),$$

defined with

$$\bar{\boldsymbol{\sigma}}^{(1)} = \left(0, \bar{\sigma}^{(1)}_{-N_1+1}, ..., \bar{\sigma}^{(1)}_{s_1-1}, 0, \bar{\sigma}^{(1)}_{s_1+1}, ..., \bar{\sigma}^{(1)}_{-1}, \bar{\sigma}^{(1)}_{0}, 1, 1, ..., 1, \bar{\sigma}^{(1)}_{s_2+1}, ..., \bar{\sigma}^{(1)}_{N_2-1}, \bar{\sigma}^{(1)}_{N_2}\right),$$

and

$$\bar{\boldsymbol{\sigma}}^{(2)} = \left(\bar{\sigma}^{(i)}_{-N_1}, \bar{\sigma}^{(i)}_{-N_1+1}, ..., \bar{\sigma}^{(i)}_{s_1-1}, 1, 1, ..., 1, \bar{\sigma}^{(i)}_{0}, \bar{\sigma}^{(i)}_{1}, ..., \bar{\sigma}^{(i)}_{s_2-1}, 0, \bar{\sigma}^{(i)}_{s_2+1}, ...\bar{\sigma}^{(i)}_{N_2-1}, 0\right),$$

where $0 < |s_i| \leq \mathcal{N}_i$, and the unidentified $\bar{\sigma}^{(i)}_s$ take an arbitrary value in $[0, 1]$ and satisfy the following conditions.

- $\bar{\sigma}^{(1)}_s > 0$ for all $s_1 < s \leq 0$; and $\bar{\sigma}^{(2)}_s > 0$ for all $0 \leq s < s_2$;
- $T_1(|s|+1, 0) \leq \frac{R_1}{C_1}$; if $T_1(|s|+1, 0) < \frac{R_1}{C_1}$, then $\sigma^{(1)}_s = 1$; otherwise, $\sigma^{(1)}_s \in (0, 1)$, for all $s_1 < s \leq 0$; $T_2(|s|+1, 0) \leq \frac{R_2}{C_2}$; if $T_2(|s|+1, 0) < \frac{R_2}{C_2}$, then $\sigma^{(2)}_s = 1$; otherwise, $\sigma^{(2)}_s \in (0, 1)$, for all $0 \leq s < s_2$;
- $T_i(|s_i|+1, 0) \geq \frac{R_i}{C_i}$;

This pattern of equilibria can be illustrated in *Figure 2*.

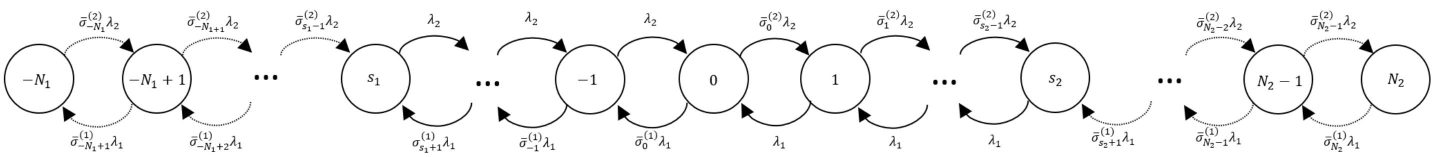

Figure 2: Equilibrium pattern 1.

Case 2: If $s_1 = s_2 = 0$, then the only existing recurrent state, $s = 0$, becomes the only absorbing state. This implies that $\pi(\bar{\mathbf{X}}, 0) = 1$ and $\pi(\bar{\mathbf{X}}, s) = 0$ for all $s \neq 0$. Now, the solutions of $\bar{\sigma}^{(1)}_0$ and $\bar{\sigma}^{(2)}_0$ must be acquired to obtain the equilibrium strategies of agents in the only recurrent state of the system. From (6), $\bar{\sigma}^{(2)}_0 \lambda_2 \pi(\bar{\mathbf{X}}, 0) = \bar{\sigma}^{(1)}_1 \lambda_1 \pi(\bar{\mathbf{X}}, 1)$ can be obtained, which is equivalent to $\bar{\sigma}^{(2)}_0 \lambda_2 = 0$ (because $\pi(\bar{\mathbf{X}}, 1) = 0$), implying that $\bar{\sigma}^{(2)}_0 = 0$. Similarly, $\bar{\sigma}^{(1)}_0 = 0$ can be obtained. Now, the payoff to a population-(i) who follows the crowd and adopts strategy $\bar{\boldsymbol{\sigma}}^{(i)}$ is

$$U^{(i)}\left(\bar{\boldsymbol{\sigma}}^{(i)}|\bar{\mathbf{X}}\right) = \sum_s \pi(\bar{\mathbf{X}}, s) U^{(i)}\left(\bar{\boldsymbol{\sigma}}^{(i)}|\bar{\mathbf{X}}, s\right) = \pi(\bar{\mathbf{X}}, 0) U^{(i)}\left(\bar{\boldsymbol{\sigma}}^{(i)}|\bar{\mathbf{X}}, 0\right) = 0.$$

Consider an arbitrary population-(i) who adopts a strategy $\boldsymbol{\sigma}^{(i)}$ where $\sigma^{(i)}_0 > 0$. The payoff to this focal population-(i) agent is

$$U^{(i)}\left(\boldsymbol{\sigma}^{(i)}|\bar{\mathbf{X}}\right) = \sum_s \pi(\bar{\mathbf{X}}, s) U^{(i)}\left(\boldsymbol{\sigma}^{(i)}|\bar{\mathbf{X}}, s\right) = \pi(\bar{\mathbf{X}}, 0) U^{(i)}\left(\boldsymbol{\sigma}^{(i)}|\bar{\mathbf{X}}, 0\right) = \sigma^{(i)}_0 \left(R_i - C_i T_i(1, 0)\right).$$

For $\bar{\mathbf{X}}$ to be a state of equilibrium, the condition (5) is necessary, meaning that the focal population-(i) agent should find a non-positive payoff. Thus, the set of $\bar{\sigma}^{(i)}_s$ $(s \neq 0)$ must satisfy $T_i(1, 0) \geq \frac{R_i}{C_i}$. Otherwise, $R_i - C_i T_i(1, 0) > 0$, then $\sigma^{(i)}_0 \in \arg\max_{\bar{\sigma}^{(i)}_0} \sigma^{(i)}_0 \left(R_i - C_i T_i(1, 0)\right) = \{1\}$, which contradicts $\sigma^{(i)}_0 = 0$.
In summary, the system equilibrium in this case is

$$\bar{\mathbf{X}} = \left(\bar{\boldsymbol{\sigma}}^{(1)}, \bar{\boldsymbol{\sigma}}^{(2)}\right),$$

where

$$\bar{\boldsymbol{\sigma}}^{(1)} = \left(0, \bar{\sigma}^{(1)}_{-N_1+1}, ..., \bar{\sigma}^{(1)}_{-1}, 0, \bar{\sigma}^{(1)}_{1}, ..., \bar{\sigma}^{(1)}_{N_2-1}, \bar{\sigma}^{(1)}_{N_2}\right),$$

and

$$\bar{\boldsymbol{\sigma}}^{(2)} = \left(\bar{\sigma}^{(2)}_{-N_1}, \bar{\sigma}^{(2)}_{-N_1+1}, ..., \bar{\sigma}^{(2)}_{-1}, 0, \bar{\sigma}^{(2)}_{1}, ..., \bar{\sigma}^{(2)}_{N_2-1}, 0\right),$$

with all $\bar{\sigma}^{(i)}_s$ $(s > 0)$ satisfying $T_2(1, 0) \geq \frac{R_2}{C_2}$, and all $\bar{\sigma}^{(i)}_s$ $(s < 0)$ satisfying $T_1(1, 0) \geq \frac{R_1}{C_1}$. This equilibrium pattern is illustrated in *Figure 3*.

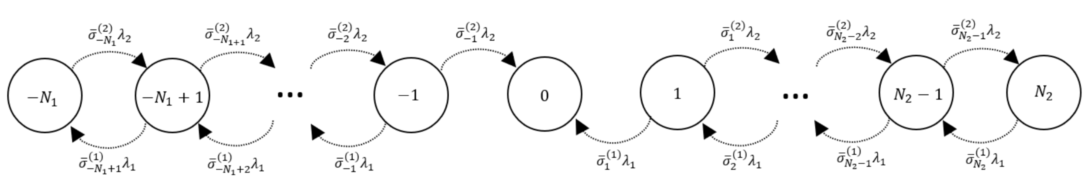

Figure 3: Equilibrium pattern 2.

Case 3&4: $s_i = 0$ and $s_{\tilde{i}} \neq 0$. These cases can be treated similarly to *Case 1* and *Case 2*. The patterns of equilibria are illustrated in *Figure 4* and *Figure 5*. In these equilibrium patterns, there exists only one population of agents in the queue.

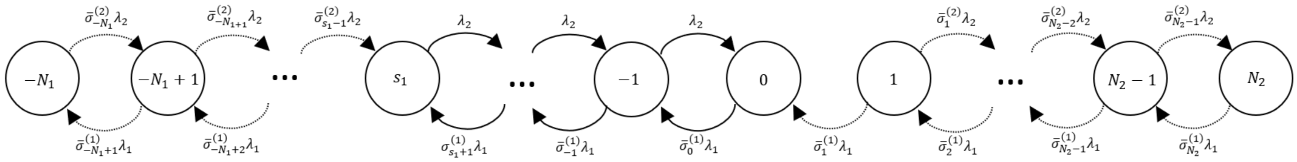

Figure 4: Equilibrium pattern 3.

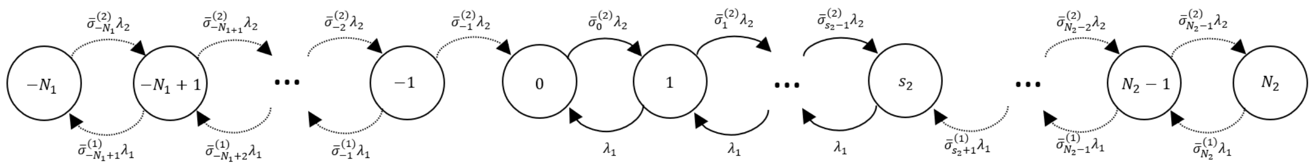

Figure 5: Equilibrium pattern 4.

In summary, in all four cases, the system may end up at a Nash equilibrium at which the length of each population's buffer does not exceed a certain threshold (prescribed by $|s_i|$, in the case of population-(i)), and such thresholds do not exceed Naor's threshold. A social profile $\bar{\mathbf{X}} = \left(\bar{\boldsymbol{\sigma}}^{(1)}, \bar{\boldsymbol{\sigma}}^{(2)}\right)$ is in equilibrium if all of the following conditions are satisfied:

(i) There exist states $s_1 \leq 0$ and $s_2 \geq 0$ such that $\bar{\sigma}^{(i)}_{s_i} = 0$, $0 \leq |s_i| \leq \mathcal{N}_i$.
If $s_1 \neq 0$, $\bar{\sigma}^{(1)}_s > 0$ for all $s_1 < s \leq 0$. If $s_2 \neq 0$, $\bar{\sigma}^{(2)}_s > 0$ for all $0 \leq s < s_2$.

(ii) $T_1(|s|+1, 0) \leq \frac{R_1}{C_1}$; if $T_1(|s|+1, 0) < \frac{R_1}{C_1}$, then $\sigma^{(1)}_s = 1$; otherwise, $\sigma^{(1)}_s \in (0,1)$, for all $s_1 < s \leq 0$;
$T_2(|s|+1, 0) \leq \frac{R_2}{C_2}$; if $T_2(|s|+1, 0) < \frac{R_2}{C_2}$, then $\sigma^{(2)}_s = 1$; otherwise, $\sigma^{(2)}_s \in (0,1)$, for all $0 \leq s < s_2$.

(iii) $T_i(|s_i|+1, 0) \geq \frac{R_i}{C_i}$.

Proposition 1 (Sensitivity of agents' strategies against buffer capacity)**.** *An arbitrary social profile under the setting of buffer capacity $(N_i, N_{\tilde{i}})$ $(i = 1, 2)$, is denoted by $\boldsymbol{X}^{(N_i, N_{\tilde{i}})} = \left(\boldsymbol{\sigma}^{(i),(N_i,N_{\tilde{i}})}, \boldsymbol{\sigma}^{(\tilde{i}),(N_i,N_{\tilde{i}})}\right)$, defined over $[0,1]^{N_1+N_2+1} \times [0,1]^{N_1+N_2+1}$. Let $\boldsymbol{X}^{(N_i+k, N_{\tilde{i}})} = \left(\boldsymbol{\sigma}^{(i),(N_i+k,N_{\tilde{i}})}, \boldsymbol{\sigma}^{(\tilde{i}),(N_i+k,N_{\tilde{i}})}\right)$, defined over $[0,1]^{N_1+N_2+k+1} \times [0,1]^{N_1+N_2+k+1}$, be a social profile with buffer capacity $(N_i + k, N_{\tilde{i}})$, where k is an arbitrary positive integer. All vector elements are indexed by the corresponding system states.*

If $\sigma^{(i),(N_i,N_{\tilde{i}})}_s = \sigma^{(i),(N_i+k,N_{\tilde{i}})}_s$ for all $s = -N_1, -N_1+1, ..., N_2$ and $\boldsymbol{X}^{(N_i,N_{\tilde{i}})}$ is not an equilibrium under the setting $(N_i, N_{\tilde{i}})$ by violating condition (i) when $N^e_i \leq N_i$, or violating condition (ii), then $\boldsymbol{X}^{(N_i+k,N_{\tilde{i}})}$ is not an equilibrium under the setting $(N_i + k, N_{\tilde{i}})$.

Proof. Consider the following two cases.

- A state s_i satisfying $\sigma^{(i),(N_i,N_{\tilde{i}})}_{s_i} = 0$ and $|s_i| \leq \mathcal{N}_i$ does not exist (violation of condition (i)) and $N^e_i \leq N_i$. As a result, in the setting $(N_i + k, N_{\tilde{i}})$, a state s_i that satisfies $\sigma^{(i),(N_i+k,N_{\tilde{i}})}_{s_i} = 0$ and $|s_i| \leq N^e_i = \min\{N^e_i, N_i + k\}$ does not exist, which implies that $\boldsymbol{X}^{(N_i+k,N_{\tilde{i}})}$ is not an equilibrium.
- Two states s_1 and s_2 satisfying $\sigma^{(i),(N_i,N_{\tilde{i}})}_{s_i} = 0$ and $|s_i| \leq \mathcal{N}_i$ exist; however, at least one of the equilibrium conditions concerning the expected waiting times is violated (violation of condition (ii)). It is implied from (7) that expected waiting times at all states within the these two states do not change when the buffer size increases from N_i to $N_i + k$. Therefore, the corresponding equilibrium conditions for the expected waiting times in the $(N_i + k, N_{\tilde{i}})$ setting are violated, which implies that $\boldsymbol{X}^{(N_i+k,N_{\tilde{i}})}$ is not an equilibrium state.

□

Remark 2 (Discussion on the case of infinite buffer size)**.** *Consider a system with an infinite buffer size on both sides of the queue. Similarly to the results in Proposition 1, a social profile denoted*

$$\mathbf{X} = \left(\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)}\right) = \left(\left(..., \bar{\sigma}^{(1)}_{-1}, \bar{\sigma}^{(1)}_0, \bar{\sigma}^{(1)}_1, ...\right), \left(..., \bar{\sigma}^{(2)}_{-1}, \bar{\sigma}^{(2)}_0, \bar{\sigma}^{(2)}_1, ...\right)\right)$$

is not in equilibrium if:

- *A state s_i satisfying $\sigma^{(i)}_{s_i} = 0$ and $|s_i| \le N^e_i$ does not exist (violation of condition (i)), or*
- *Two states s_1 and s_2 satisfying $\sigma^{(i)}_{s_i} = 0$ and $|s_i| \le N^e_i$ exist; however, at least one of the equilibrium conditions concerning the expected waiting times is violated (violation of condition (ii)). The recursion in the case of infinite buffer size is slightly different from (7) with regard to the bound. Without a bounding state N_i, the recursion for the calculation of the expected waiting times (under the same notations as (7)) is*

$$T_i(u,v) = \frac{1}{\sigma^{(i)}_w \lambda_i + \sigma^{(\tilde{i})}_w \lambda_{\tilde{i}}} + \frac{\sigma^{(i)}_w \lambda_i}{\sigma^{(i)}_w \lambda_i + \sigma^{(\tilde{i})}_w \lambda_{\tilde{i}}} T_i(u, v+1) + \frac{\sigma^{(\tilde{i})}_w \lambda_{\tilde{i}}}{\sigma^{(i)}_w \lambda_i + \sigma^{(\tilde{i})}_w \lambda_{\tilde{i}}} T_i(u-1, v).$$

*However, if there exists a state s_i satisfying $\sigma^{(i)}_{s_i} = 0$, it is implied that expected waiting times at all states bounded by s_i are calculable and insensitive to the buffer size (s_i becomes the bound of the recursion over all states within this state). Therefore, the result in **Proposition 1** similarly holds when $k \to +\infty$, which is equivalent to the case of an infinite buffer size on both sides.*

On another note, whether a state $s < 0$ (if $s > 0$ can be similarly analyzed) is recurrent or transient is directly controlled by population-(1) agents' decisions (because the queue length of population-(1) agents is the consequence of their decision to join or balk); however, such decisions depend on the strategy of population-(2) agents. Because a transient state may never be observed (it occurs with probability 0), the strategy of population-(2) agents in transient states does not affect their expected payoff; therefore, they can choose to join at those states with arbitrary probabilities. Meanwhile, the strategy of population-(1) agents in those transient states would depend on their beliefs about the strategy of population-(2) agents. If population-(1) agents hold the belief that population-(2) agents join the queue at such transient states $s < 0$ with probabilities smaller than 1, the system may end up at an equilibrium at which the maximal buffer of population-(1) agents is smaller than $\mathcal{N}_1$. When population-(1) agents believe that population-(2) agents optimize their payoff at every state (including transient states) and vice versa, the outcome of the system is a subgame perfect Nash equilibrium derived in the next section.

4. Subgame perfect Nash equilibrium

In this section, the subgame perfect Nash equilibria of this game is derived, defined as $\bar{\mathbf{X}}^* = \left(\bar{\boldsymbol{\sigma}}^{*(1)}, \bar{\boldsymbol{\sigma}}^{*(2)}\right)$, where

$$\bar{\boldsymbol{\sigma}}^{*(i)} = \left(\bar{\sigma}^{*(i)}_{-N_1}, \bar{\sigma}^{*(i)}_{-N_1+1}, ..., \bar{\sigma}^{*(i)}_{-1}, \bar{\sigma}^{*(i)}_{0}, \bar{\sigma}^{*(i)}_{1}, ..., \bar{\sigma}^{*(i)}_{N_2-1}, \bar{\sigma}^{*(i)}_{N_2}\right).$$

It immediately follows that $\bar{\sigma}^{*(1)}_{-N_1} = \bar{\sigma}^{*(2)}_{N_2} = 0$.
At any state $s > 0$, we have

$$\bar{\sigma}^{*(1)}_s \in \arg\max_{\bar{\sigma}^{*(1)}_s} U^{(1)}\left(\bar{\sigma}^{*(1)}_s | \mathbf{X}^*, s\right) = \arg\max_{\bar{\sigma}^{*(1)}_s} \bar{\sigma}^{*(1)}_s R_1 = \{1\}.$$

In other words, $\bar{\sigma}^{*(1)}_s = 1$ for all $0 < s \le N_2$.
Similarly, at any state $s < 0$, we have

$$\bar{\sigma}^{*(2)}_s \in \arg\max_{\bar{\sigma}^{*(2)}_s} U^{(2)}\left(\bar{\sigma}^{*(2)}_s | \mathbf{X}^*, s\right) = \arg\max_{\bar{\sigma}^{*(2)}_s} \bar{\sigma}^{*(2)}_s R_2 = \{1\}.$$

In other words, $\bar{\sigma}^{*(2)}_s = 1$ for all $-N_1 \le s < 0$. It then easily follows from (7) by induction that $T_i(u,0) = \frac{u}{\lambda_{\tilde{i}}}$. Therefore, at any state $s \le 0$, we have

$$\begin{aligned}\bar{\sigma}^{*(1)}_s \in \arg\max_{\bar{\sigma}^{*(1)}_s} U^{(1)}\left(\bar{\sigma}^{*(1)}_s | \mathbf{X}^*, s\right) &= \arg\max_{\bar{\sigma}^{*(1)}_s} \left(\bar{\sigma}^{*(1)}_s \left(R_1 - C_1 \frac{|s|+1}{\lambda_2}\right)\right) \\ &= \begin{cases} \{0\} & \text{if } R_1 - C_1 \frac{|s|+1}{\lambda_2} < 0, \\ [0,1] & \text{if } R_1 - C_1 \frac{|s|+1}{\lambda_2} = 0, \\ \{1\} & \text{if } R_1 - C_1 \frac{|s|+1}{\lambda_2} > 0, \end{cases}\end{aligned}$$

which is equivalent to

$$\bar{\sigma}^{*(1)}_s = \begin{cases} 0 & \text{if } -N_1 \le s \le -\mathcal{N}_1, \\ p_1 & \text{if } s = -\mathcal{N}_1 + 1, \\ 1 & \text{if } -\mathcal{N}_1 + 2 \le s \le 0, \end{cases}$$

where $p_1 = 1$ if $R_1 - C_1\frac{\mathcal{N}_1}{\lambda_2} > 0$, and p_1 takes any value on $[0,1]$ if $R_1 - C_1\frac{\mathcal{N}_1}{\lambda_2} = 0$. Similarly, at any state $s \geq 0$,

$$\bar{\sigma}_s^{*(2)} = \begin{cases} 0 & \text{if } \mathcal{N}_2 \leq s \leq N_2, \\ p_2 & \text{if } s = \mathcal{N}_2 - 1, \\ 1 & \text{if } 0 \leq s \leq \mathcal{N}_2 - 2, \end{cases}$$

where $p_2 = 1$ if $R_2 - C_2\frac{\mathcal{N}_2}{\lambda_1} > 0$, and p_2 takes any value on $[0,1]$ if $R_2 - C_2\frac{\mathcal{N}_2}{\lambda_1} = 0$. This subgame perfect Nash equilibrium can be illustrated in *Figure 6*.

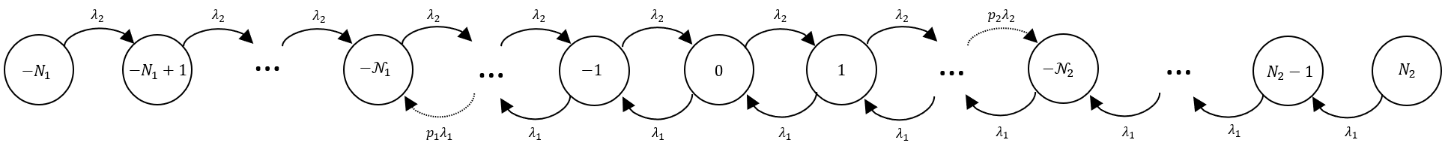

Figure 6: Subgame perfect Nash equilibrium.

This is a special case of the equilibrium pattern in *Case 1* considered in the previous section, where $s_i = \mathcal{N}_i$ for $i = 1, 2$.

Remark 3 (Discussion on the case of infinite buffer size)**.** *Under an infinite buffer size setting on both sides of the queue, the subgame perfect Nash equilibrium can still be derived with the same method. If an assumption that agents choose to join when expecting a zero payoff is added, then the conclusion on the subgame perfect Nash equilibrium becomes identical to that in Naor's setting: population-(i) agents join the queue if they observe a queue of population-(i) agents with the length of* $N_i^e - 1$ *or less, and balk otherwise.*

5. Concluding remarks

The comparison between the results in this study and Naor's classical results is briefly discussed in this section. First, in terms of system setting, Naor considered an infinite buffer size; however, the threshold level did not change even if a large finite buffer limit was considered. In the current study, we considered an arbitrary finite buffer size on both sides of the queue. When such a buffer limit is small owing to physical constraints, the maximum length of the agent queues might be identical to the buffer limit. We also extended part of the results to the case where the buffer size was infinite. Second, in terms of game setting, Naor considered a one-population game, and the threshold strategy is a subgame perfect equilibrium and the only Nash equilibrium of the game. Strategies adopted by agents in transient states do not affect their expected payoffs. Meanwhile, in this two-population game setting, different patterns exist that generalize multiple Nash equilibria. Strategies of agents in transient states in a population did not affect their expected payoff but affected agents' strategies in the other population. This results in different possible queue buffer thresholds that can be equal to or smaller than Naor's threshold level. The existence of a unique subgame perfect Nash equilibrium at which the system is most likely to end up in reality was also demonstrated (often referred to as "equilibrium threshold strategy" in existing literature). However, a subgame imperfect Nash equilibrium may also be the solution of the game in special cases where agents hold some initial beliefs. For example, the extreme equilibrium at which no agents choose to join the system may occur when the system is terminated. Knowing all the possibilities for different outcomes may help social planners in planning the system. The problem of equilibrium selection, however, falls outside the scope of this research and may be a topic for future work.
The extended game theoretical framework under the scope of multi-population games may also be similarly applicable in queuing systems with two different populations of strategic agents under different settings, such as unobservable systems. Further studies may consider the application of this framework to other queuing systems.

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References

[1] A. Economou. *The Impact of Information Structure on Strategic Behavior in Queueing Systems*, pages 137–169. Wiley-ISTE, 2021. ISBN 9781119755234.

[2] R. Hassin. *Rational queueing.* Chapman and Hall/CRC, New York, 1st edition, 2016.

[3] R. Hassin and M. Haviv. Nash equilibrium and subgame perfection in observable queues. *Annals of Operations Research*, 113:15–26, 2002.

[4] R. Hassin and M. Haviv. *To queue or not to queue: Equilibrium behavior in queueing systems*. Kluwer Academic Publishers, Boston, 2003.

[5] P. Naor. The regulation of queue size by levying tolls. *Econometrica*, 37(1):15–24, 1969.

[6] Y. Shi and Z. Lian. Optimization and strategic behavior in a passenger-taxi service system. *European Journal of Operational Research*, 249(3):1024–1032, 2016.