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Duality and traces for indexed monoidal categories. (English) Zbl 1275.18019

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It is well known [Zbl 0473.55008; Zbl 0556.55006; Zbl 0845.18005; Zbl 0447.18005] that in any symmetric monoidal category, there are useful intrinsic notions of *duality* and *trace*. This paper presents an abstract framework for constructing traces in *indexed* symmetric monoidal categories, which gives rise to the Reidenmeister trace as a particular example. An indexed monoidal category is a family of symmetric monoidal category $\mathcal{C}^A$, one for each object A of a cartesian monoidal base category $\mathbf{S}$, equipped with base change functors indexed by the morphisms of $\mathbf{S}$. The following three are primary examples in this paper.

1. $\mathbf{S}$ = sets, $\mathcal{C}^A$ = A-indexed families of abelian groups.
2. $\mathbf{S}$ = topological spaces, $\mathcal{C}^A$ = spectra parametrized over A.
3. $\mathbf{S}$ = sets, $\mathcal{C}^A$ = A-indexed families of abelian groups.

In any such context one can consider duality and trace in the individual symmetric monoidal category $\mathcal{C}^A$, i.e., *fiberwise* [Zbl 1119.55001]. The first main result in this paper, which is stated in §6 and is established in §11, goes as follows.

Theorem 1. If $M \in \mathcal{C}^A$ is a fiberwise dualizable and $f : M \to M$ is any endomorphism, then the symmetric monoidal trace of f factors as a composite

$$I_A \to (\pi_A)^* \langle\langle A \rangle\rangle \xrightarrow{\mathrm{tr}(\widehat{f})} I_A$$

Trace-like information such as fixed point invariants can be extracted from an endomorphism $f : M \to M$ in some cartesian monoidal category $\mathbf{S}$ such as sets, groupoids or spaces, where a non-cartesian monoidal category $\mathbf{C}$ such as abelian groups, chain complexes or spectra can be chosen, a functor $\Sigma : \mathbf{S} \to \mathbf{C}$ such as the free abelian group or suspension spectrum is applied, and the symmetric monoidal trace $\Sigma(f)$ in $\mathbf{C}$ is considered. In most examples where this is done, there is actually an indexed symmetric monoidal category over $\mathbf{S}$ such that $\mathbf{C} = \mathcal{C}^*$ is the category indexed by the terminal object of $\mathbf{S}$, and $\Sigma(A)$ is the pushforward to $\mathcal{C}^*$ of the unit object of $\mathcal{C}^A$. The second main result in this paper, which is established in §8, goes as follows.

Theorem 2. If I_A is totally dualizable [Zbl 1362.55001; Zbl 1119.55001] and $f : M \to M$ is an endomorphism in $\mathbf{S}$, then the symmetric monoidal trace of $\Sigma(f)$ factors as a composite

$$I_* \xrightarrow{\mathrm{tr}(\check{f})} \langle\langle A_f \rangle\rangle \to I_*$$

The *transfer* of f, which is a map $I_* \to \Sigma(A)$, is the trace of the composite

$$\Sigma(A) \xrightarrow{\Sigma(f)} \Sigma(A) \xrightarrow{\Sigma(\Delta_A)} \Sigma(A) \otimes \Sigma(A)$$

The third main result in this paper, which is established in §8, goes as follows.

Theorem 3. In the situation of the above theorem, the transfer of f factors as a composite

$$I_* \xrightarrow{\mathrm{tr}(\check{f})} \langle\langle A_f \rangle\rangle \to \Sigma(A)$$

§§9–10 are devoted to string diagram calculus for indexed monoidal categories, which is a Poincaré dual way of drawing composition in categorical structures making complicated computations much more visually evident. String diagrams for monoidal categories and bicategories are described in [Zbl 0738.18005; Zbl 0845.18005; Zbl 0216.43502; Zbl 1217.18002; Zbl 1436.81004; Zbl 1405.81001].

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