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Modalities in homotopy type theory. (English) Zbl 07155169

Log. Methods Comput. Sci. 16, No. 1, Paper No. 2, 79 p. (2020)

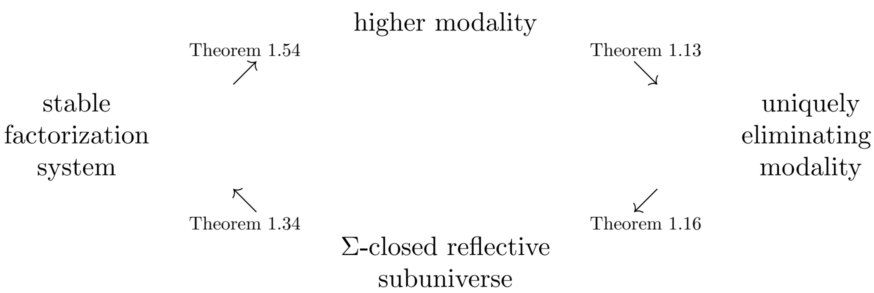
In traditonal modal logic, a *modality* is a unary operation on propositions. In type theory and particularly dependent type theory such as homotopy type theory, where propositions are viewed as certain types, it is natural to extend the notion of modality to a unary operation on types. This paper attempts to take a first step towards the study of higher modalities in homotopy type theory, restricting to idempotent, monadic ones.

A synopsis of the paper goes as follows.

§1 introduces the following four notions of modality, establishing their equivalence.

- higher modalities
- uniquely eliminating modalities
- Σ-closed reflective modalities
- stable orthogonal factorization systems

In each case, being modal is a family of mere propositions indexed by the universe, i.e., a subuniverse. It is shown in Theorems 1.12, 1.15, 1.18 and 1.53 that each kind of structure is completely determined by the subuniverse, so that the type of all modalities of each kind is a subset of the set $\mathrm{U} \to \mathrm{Prop}$ of all subuniverses, and in particular is a set. The equivalence is established cyclically.



§2 gives a general construction of modalities using a higher inductive *localization* type. Localization is the process of inverting a specified class of maps. The authors are concerned not with the universal property of the localized category but with construction of reflective subcategories of local objects. Localizations are much better-studied in classical homotopy theory. Modern calculational homotopy theory often works in subuniverses that are localized at a prime number of a cohomology theory.

§3 studies a very significant class of left exact or *lex* modaiities. When homotopy type theory is regarded as an internal language for higher toposes, lex modalities correspond to reflective subtoposes. In the traditional internal logic of 1-toposes, subtoposes are represented by *Lawvere-Tierney operators* on the subobject classifier, which generate a subtopos by internal sheafification. *R. Goldblatt* [J. Log. Comput. 21, No. 6, 1035–1063 (2011; Zbl 1247.03028)] provided an overview of the modal logic perspective on these operators on propositions. Dependent type theory allows of speaking directly about the subtoposes as an operation on a type universe (the lex modality), demonstrating internally that any Lawvere-Tierney operator on the universe of propositions gives rise to a lex modality. While every lax modality arises from a Lawvere-Tierney operator in 1-topos theory or, more generally, in n-topos theory for any $n < \infty$, this is no longer true in ∞-topos theory. The subtoposes determined by their behavior on propositions were called *topological* in [*J. Lurie*, Higher topos theory. Princeton, NJ: Princeton University Press (2009; Zbl 1175.18001)], the name being appropriated for lex modalities of this sort as well. The authors construct lex modalities by nullifying any familiy of propositions (Corollary 3.12). These correspond categorically to the *topological* localization. Lex and topological modalities are to be viewed as a contribution to the program of giving an elementary (first-order) definition of an ∞-topos as a purported model of homotopy type theory, being a higher-categorical analogue of the standard theory of Lawvere-Tierney operators in 1-topos theory, which are the usual way to internalize the notion of subtopos.

Appendix sketches something of how the syntactic description of modalities in the text correspond to semantic structures in higher category theory, working at the level of comprehension categories and their corresponding model categories. The authors are sketchy about coherent theorems, expecting that the methods of [*P. Lefanu Lumsdaine* and *M. A. Warren*, "The local universes model: an overlooked coherence construction for dependent type theories", ACM Trans. Comput. Logic 16, No. 3, Paper No. 23, 31 p. (2015; doi:10.1145/2754931)] will apply. The appendix is divided into four parts.

A.1. Judgement modalities

A.2. Modalities in model categories

A.3. Modalities in syntactic categories

A.4. Localizations in model categories

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MSC:

03B70 Logic in computer science
68-XX Computer science

Full Text: arXiv

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