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2-Segal objects and algebras in spans. (English) Zbl 07370434

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This paper is concerned with consideration on the relation between 2-Segal objects in an ∞-category $\mathcal{C}$ and algebra objects in an ∞-category $\mathrm{Span}(\mathcal{C})$ whose morphisms are spans in $\mathcal{C}$. Precursors in this direction were *T. Dyckerhoff* and *M. Kapranov* [Higher Segal spaces. Cham: Springer (2019; Zbl 1459.18001)], who constructed monads and algebra objects in $(\infty, 2)$-categories of spans from 2-Segal objects, and *M. D. Penney* ["Simplicial spaces, lax algebras and the 2-Segal condition", Preprint, `arXiv:1710.02742`], who defined lax algebras in spans coming from simplicial objects, demonstrating that the associativity of these lax algebras was equivalent to the 2-Segal condition. Restricting to ∞-categories of spans, the former half of this paper establishes

Theorem 2.25. Let $\mathcal{C}$ be an ∞-category with small limits. There is an equivalence between ∞-categories

$$\left\{ \begin{array}{c} \text{Algebra objects} \\ \text{in Span}(\mathcal{C}) \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{2-Segal simplicial objects} \\ \text{in } \mathcal{C} \end{array} \right\}$$

T. Dyckerhoff and *M. Kapranov* [Contemp. Math. 643, 37–110 (2015; Zbl 1373.18015), §V.2] constructed invariants $X(S, M)$ of stable marked surface (S, M) with boundary, associated to a 2-Segal cyclic object $X : \Lambda^{\mathrm{op}} \to \mathcal{C}$. On top of that, the $X(S, M)$ comes equipped with coherent actions of the mapping class group. It is consequently natural to ask whether the invariants $X(S, M)$ form an open, oriented, ∞-categorical topological field theory in $\mathrm{Span}(\mathcal{C})$. *K. Costello* [Adv. Math. 210, No. 1, 165–214 (2007; Zbl 1171.14038)] considered open oriented theories equipped with a set of D-branes and valued in the dg-category of chain complexes, showing that such field theories are equivalent to Calabi-Yau $\mathcal{A}_\infty$ categories. A similar classification was that of Lurie.

Theorem 4.2.11. [*J. Lurie*, in: Current developments in mathematics, 2008. Somerville, MA: International Press. 129–280 (2009; Zbl 1180.81122)]. Let $\mathcal{C}$ be a symmetric monoidal ∞-category. The following types of data are equivalent:

1. Open oriented topological field theories in $\mathcal{C}$.
2. Calabi-Yau algebra objects in $\mathcal{C}$.

Based on this theorem, the latter half of this paper seeks to related cyclic 2-Segal objects to Calabi-Yau algebras, demonstrating that

Theorem 3.29. Let $\mathcal{C}$ be an ∞-category with small limits. There is an equivalence between ∞-categories

$$\left\{ \begin{array}{c} \text{Calabi-Yau Algebra objects} \\ \text{in Span}(\mathcal{C}) \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{2-Segal cyclic objects} \\ \text{in } \mathcal{C} \end{array} \right\}$$

Once the correspondence in Theorem 3.29 is established, a wealth of avenues to construct topological field theories open up.

- The Waldhausen S-construction gives rise to many cyclic 2-Segal spaces [*T. Dyckerhoff* and *M. Kapranov*, J. Eur. Math. Soc. (JEMS) 20, No. 6, 1473–1524 (2018; Zbl 1403.18011)].
- 1-Segal cyclic objects provide a zoo of interesting exmaples.
- An intriguing incarnation of the cyclic Čech nerve construction is its application to a morphism $f : * \to X$ into a connected space X, where the Čech nerve has the loop space ΩX based at as its space of 1-simplices, and the author expects the resulting surface invariants to relate to string topology.

Theorem 2.25 and Theorem 3.29 bear an intriguing relation to another construction in the literature. Following [*D.-C. Cisinski* and *I. Moerdijk*, J. Topol. 6, No. 3, 675–704 (2013; Zbl 1291.55004)], *T. Walde* [Algebr. Geom. Topol. 21, No. 1, 211–246 (2021; Zbl 1469.18031)] defined a notion of a cyclic ∞-operad, showing that there are equivalences of ∞-categories

$$\left\{ \begin{array}{c} \text{invertible cyclic} \\ \infty\text{-operads} \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{2-Segal cyclic objects} \\ \text{in } \mathcal{S} \end{array} \right\}$$

and

$$\left\{ \begin{array}{c} \text{invertible} \\ \infty\text{-operads} \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{2-Segal simplicial objects} \\ \text{in } \mathcal{S} \end{array} \right\}$$

which now has the immediate implications of relating invertible (cyclic) ∞-operads to (Calabi-Yau) algebras in Span $(\mathcal{S})$.

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MSC:

18N60 $(\infty, 1)$-categories (quasi-categories, Segal spaces, etc.); ∞-topoi, stable ∞-categories

Full Text: DOI arXiv

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