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Faithfully flat descent of almost perfect complexes in rigid geometry. (English) Zbl 07455913

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Let K be a complete nonarchimedian field, and let $A \to A'$ be a faithfully flat map of K-affinoid algebras. In ordinary algebra, faithfully flat descent [*M. Raynaud* and *A. Grothendieck* (ed.), Séminaire de géométrie algébrique du Bois Marie 1960/61 (SGA 1), dirigé par Alexander Grothendieck. Augmenté de deux exposés de M. Raynaud. Revêtements étales et groupe fondamental. Exposés I à XIII. (Seminar on algebraic geometry at Bois Marie 1960/61 (SGA 1), directed by Alexander Grothendieck. Enlarged by two reports of M. Raynaud. Ètale coverings and fundamental group). Springer, Cham (1971; Zbl 0234.14002), Exp. VIII] states that the category of -modules is to be described as the category of -modules with descent data taking place over the tensor products $A' \otimes_A A'$, $A' \otimes_A A' \otimes_A A'$. A similar conclusion obtains in rigid geometry with the tensor products replaced by completed tensor products and only for finitely generated modules.

Given a K-affinoid algebra B, $\mathrm{Coh}(B)$ denotes the category of finitely generated B-modules.

Theorem [*S. Bosch* and *U. Görtz*, J. Reine Angew. Math. 495, 119–134 (1998; Zbl 0884.14009)]. Let $A \to A'$ be a faithfully flat map of K-affinoid algebras. Then we have an equivalence of categories

$$\mathrm{Coh}(A) \simeq \varprojlim(\mathrm{Coh}(A') \rightrightarrows \mathrm{Coh}(A' \otimes_A A') \substack{\to\\ \to\\ \to} \cdots)$$

Let $\mathcal{O}_K \subset K$ be the ring of integers, and let $\pi \in \mathcal{O}_K$ denote a nonzero nonunit. Let $\mathrm{Alg}^{\mathrm{b}}_{\mathcal{O}_K}$ denote the category of $\mathcal{O}_K$-algebras R which are π-torsion-free and π-adically complete. A map $R \to R'$ in $\mathrm{Alg}^{\mathrm{b}}_{\mathcal{O}_K}$ is said to be π-completely faithfully flat if $R/\pi \to R'/\pi$ is faithfully flat, which defines the π-complete topology on $(\mathrm{Alg}^{\mathrm{b}}_{\mathcal{O}_K})^{\mathrm{op}}$. For any ring A, $\mathrm{Vect}(A)$ denotes the category of finitely generated projective A-modules. Drinfeld [*V. Drinfeld*, Prog. Math. 244, 263–304 (2006; Zbl 1108.14012), Theorem 3.11] has observed that

Theorem. The construction

$$R \mapsto \mathrm{Vect}(R[1/\pi])$$

is a sheaf of categories on $\mathrm{Alg}^{\mathrm{b}}_{\mathcal{O}_K}$. That is to say, given

$$R \to R'$$

in $\mathrm{Alg}^{\mathrm{b}}_{\mathcal{O}_K}$, which is π-completely faithful flat, the natural functor

$$\mathrm{Vect}(R[1/\pi]) \to \varprojlim(\mathrm{Vect}(R'[1/\pi]) \rightrightarrows \mathrm{Vect}(\widehat{R' \otimes_R R'}[1/\pi]) \substack{\to\\ \to\\ \to} \cdots)$$

is an equivalence of categories.

The principal result in this paper is a version of faithfully flat descent in rigid analytic geometry for almost perfect complexes and without finiteness assumptions on the rings involved (Theorem 7.8), which is an extension of Drinfeld's above theorem. Closely related is [*K. S. Kedlaya* and *R. Liu*, "Relative p-adic Hodge theory. II: Imperfect period rings", Preprint, arXiv:1602.06899, §2], which constructed a category of pseudocoherent sheaves on adic spaces. Theorem 7.8 goes as follows.

Theorem. The construction

$$S \mapsto \mathrm{APerf}(\mathrm{Spec}(\widehat{S}_I) \backslash V(I))$$

defines a hypercomplete sheaf for the I-completely flat topology. Similarly for the subcategories $\mathrm{APerf}_{\geq 0}$ of connective almost perfect modules, Perf of perfect modules, and $\mathrm{Perf}_{[a,b]}$ of perfect modules with

Tor-amplitude in $[a, b]$ for any $a \leq b$.

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MSC:

18G99 Homological algebra in category theory, derived categories and functors
18F20 Presheaves and sheaves, stacks, descent conditions (category-theoretic aspects)

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