Coquand, Thierry; Huber, Simon; Sattler, Christian

Canonicity and homotopy canonicity for cubical type theory. (English) Zbl 07471717

Log. Methods Comput. Sci. 18, No. 1, Paper No. 28, 35 p. (2022)

The principal objective in this paper is to contribute to the analysis of the computational content of the univalence axiom [*V. Voevodsky*, "The equivalence axiom and univalent models of type theory", Preprint, arXiv:1402.5556] and higher inductive types. Various *presheaf models* have been developed in a constructive metatheory [*C. Angiuli* et al., Math. Struct. Comput. Sci. 31, No. 4, 424–468 (2021; Zbl 07460116); *M. Bezem* et al., LIPIcs – Leibniz Int. Proc. Inform. 26, 107–128 (2014; Zbl 1359.03009); *C. Cohen* et al., LIPIcs – Leibniz Int. Proc. Inform. 69, Article 5, 34 p. (2018; Zbl 1434.03036); *T. Coquand* et al., in: Proceedings of the 2018 33rd annual ACM/IEEE symposium on logic in computer science, LICS 2018, Oxford, UK, July 9–12, 2018. New York, NY: Association for Computing Machinery (ACM). 255–264 (2018; Zbl 1452.03036); *I. Orton* and *A. M. Pitts*, LIPIcs – Leibniz Int. Proc. Inform. 62, Article 24, 19 p. (2016; Zbl 1370.03016); *I. Orton* and *A. M. Pitts*, Log. Methods Comput. Sci. 14, No. 4, Paper No. 23, 33 p. (2018; Zbl 07003193)], where the notion of *fibrant type* is stated as a refinement of the *path lifting operation*, being a way to state a homotopy extension property and having been recognized, as early as [*S. Eilenberg*, Fundam. Math. 32, 167–175 (1939; Zbl 0021.16204)], as a key concept for an abstract development of algebraic topology. The axiom of univalence is then captured by a suitable *equivalence extension operation*, which states that we can extend a partially defined equivalence of a given total codomain to a total equivalence. These presheaf models admit of possible extensions of type theory where one can manipulate higher dimensional objects. Therein, one can define a notion of reduction, establishing canonicity [*S. Huber*, J. Autom. Reasoning 63, No. 2, 173–210 (2019; Zbl 1477.03036); *T. Coquand* et al., Log. Methods Comput. Sci. 18, No. 1, Paper No. 28, 35 p. (2022; Zbl 07471717)], though there are however several *non-canonical* choices when dealing with the path lifting operation by induction on the type. The principal result in this paper is the homotopy canonicity theorem claiming that the value of a term is independent of such non-canonical choices.

The proof of homotopy canonicity is to be seen as a proof-relevant extension of the *reducibility* or *computability* method going back to [*K. Gödel*, Dialectica 12, 280–287 (1958; Zbl 0090.01003)] and [*W. W. Tait*, J. Symb. Log. 32, 198–212 (1967; Zbl 0174.01202)], though it is expressed in an algebraic setting. The authors define a notion of model, called cubical category with families, as a category with families [*P. Dybjer*, Lect. Notes Comput. Sci. 1158, 120–134 (1996; Zbl 1434.03149)] with certain special operations internal presheves over a category $\mathcal{C}$ with respect to the parameters of an interval $\mathbb{I}$ and a cofibration classifier $\mathbb{F}$, describing the term model and how to re-interpret the cubical presheaf models as cubical categories with families. The computability method is to be seen as a general opeation, called *sconing*, which, applied to an arbitrary model $\mathcal{M}$, produces a new model $\mathcal{M}^*$ with a strict morphism $\mathcal{M}^* \to \mathcal{M}$. Homotopy canonicity is obtained by applying this general operation to the initial model.

The authors then explain how a similar method can be used to establish *canonicity* or *strict canonicity* when computation rules of filling at type formers are added.

M. Shulman [Math. Struct. Comput. Sci. 25, No. 5, 1203–1277 (2015; Zbl 1362.03008)] established homotopy canonicity for homotopy type theory with a truncatedness assumption using the sconing technique, which inspired this paper.

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

03B70 Logic in computer science
68-XX Computer science

Full Text: arXiv Link

References:

[1] P. Aczel. On relating type theories and set theories. In T. Altenkirch, B. Reus, and W. Naraschewski, editors,Types for Proofs

and Programs, volume 1657 ofLecture Notes in Computer Science, pages 1-18. Springer Verlag, Berlin, Heidelberg, New York, 1999. doi:10.1007/3-540-48167-2_1. · Zbl 0944.03056

[2] M. Bezem, T. Coquand, and S. Huber. A model of type theory in cubical sets. In R. Matthes and A. Schubert, editors,19th International Conference on Types for Proofs and Programs (TYPES · Zbl 1359.03009

[3] C. Cohen, T. Coquand, S. Huber, and A. M¨ortberg. Cubical type theory: A constructive interpretation of the univalence axiom. In T. Uustalu, editor,21st International Conference on · Zbl 1434.03036

[4] T. Coquand, S. Huber, and A. M¨ortberg. On higher inductive types in cubical type theory. In Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS '18, pages 255-264, New York, NY, USA, 2018. ACM.doi:10.1145/3209108.3209197. · Zbl 1452.03036

[5] T. Coquand. A survey of constructive presheaf models of univalence.ACM SIGLOG News, 5(3):54-65, 2018.doi:10.1145/3242953.3242962.

[6] T. Coquand. Canonicity and normalization for dependent type theory.Theoretical Computer Science, 777:184-191, 2019.doi:10.1016/j.tcs.2019.01.01 · Zbl 1454.03019

[7] P. Dybjer. Internal Type Theory. InLecture Notes in Computer Science, pages 120-134. Springer Verlag, Berlin, Heidelberg, New York, 1996.doi:10.1007/3-540-61780-9_66. · Zbl 1434.03149

[8] T. Ehrhard.Une s´emantique cat´egorique des types d´ependants: Application au Calcul des Constructions. Phd thesis, University Paris VII, 1988.

[9] S. Eilenberg. On the relation between the fundamental group of a space and the higher homotopy groups.Fundamenta Mathematicae, 32(1):169-175, 1939.doi:10.4064/fm-32-1-167-175. · Zbl 65.0865.01

[10] N. Gambino and M. Hyland. Wellfounded trees and dependent polynomial functors. InInternational Workshop on Types for Proofs and Programs, pages 210-225. Springer, 2003. · Zbl 1100.03055

[11] N. Gambino and J. Kock. Polynomial functors and polynomial monads. InMathematical Proceedings of the Cambridge Philosophical Society, volume 154, pages 153-192. Cambridge University Press, 2013.doi:10.1017/S0305004112000394. · Zbl 1278.18013

[12] N. Gambino and C. Sattler. The Frobenius condition, right properness, and uniform fibrations. Journal of Pure and Applied Algebra, 221(12):3027-3068, 2017.doi:10.1016/j.jpaa.2017.02. · Zbl 1378.18002

[13] P. Gabriel and M. Zisman.Calculus of fractions and homotopy theory, volume 35 ofErgebnisse der Mathematik und ihrer Grenzgebiete. Springer, 1967.doi:10.1007/978-3-642-85844-4. · Zbl 0186.56802

[14] M. Hofmann. Syntax and semantics of dependent types. In A.M. Pitts and P. Dybjer, editors, Semantics and logics of computation, volume 14 ofPubl. Newton Inst., pages 79-130. Cambridge University Press, Cambridge, 1997.doi:10.1007/978-1-4471-0963-1_2. · Zbl 0919.68083

[15] S. Huber. Canonicity for cubical type theory.Journal of Automated Reasoning, 63:172-210, 2019. doi:10.1007/s10817-018-9469-1. · Zbl 1477.03036

[16] A. Joyal. Notes on clans and tribes, 2017.

[17] G.M. Kelly. A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on.Bulletin of the Australian Mathematical Society, 22(1):1-83, 1980. doi:10.1017/S0004972700006353. · Zbl 0437.18004

[18] K. Kapulkin and P. LeFanu Lumsdaine. The simplicial model of univalent foundations (after Voevodsky).Journal of the European Mathematical Society, 23:2071-2126, 2012.doi:10.4171/ JEMS/1050. · Zbl 1471.18025

[19] K. Kapulkin and V. Voevodsky. A cubical approach to straightening.Journal of Topology, 13:1682-1700, 12 2020.doi:10.1112/topo.12173. · Zbl 1470.18028

[20] D. R. Licata, I. Orton, A. M. Pitts, and B. Spitters. Internal universes in models of homotopy type theory. In H´el'ene Kirchner, editor,3rd International Conference on Formal Structures for · Zbl 1462.03010

[21] P. LeFanu Lumsdaine and M. Shulman. Semantics of higher inductive types.Mathematical Proceedings of the Cambridge Philosophical Society, 169(1):159-208, 2020.doi:10.1017/ · Zbl 1470.18007

[22] I. Orton and A. M. Pitts. Axioms for modelling cubical type theory in a topos. In25th EACSL Annual Conference on Computer Science Logic (CSL 2016), volume 62 ofLeibniz International · Zbl 1370.03016

[23] E. Palmgren and S. J. Vickers. Partial Horn logic and cartesian categories.Annals of Pure and Applied Logic, 145(3):314-353, 2007.doi:10.1016/j.apal.2006.10.001. · Zbl 1109.03022

[24] E. Riehl and M. Shulman. A type theory for synthetic⊠-categories.Higher Structures, 1(1), 2018. · Zbl 1437.18016

[25] M. Shulman. Univalence for inverse diagrams and homotopy canonicity.Mathematical Structures in Computer Science, 25(5):1203-1277, 2015.doi:10.1017/S0960129514000565. · Zbl 1362.03008

[26] J. Sterling. Algebraic type theory and universe hierarchies.arXiv preprint arXiv:1902.08848, 2019.

[27] T. Streicher.Semantics of type theory: correctness, completeness, and independence results. Progress in Theoretical Computer Science. Birkh¨auser Basel, 1991.doi:10.2307/2275776. · Zbl 0790.68068

[28] A. Swan. An algebraic weak factorisation system on 01-substitution sets: A constructive proof. Journal of Logic \& Analysis, 8(1):1-35, 2016.doi:10.4115/jla.2016.8.1. · Zbl 1403.03124

[29] A. Swan. Semantics of higher inductive types, 2017. On the HoTT mailing list.

[30] W. W. Tait. Intensional interpretations of functionals of finite type I.Journal of Symbolic Logic, 32(2):198-212, 1967.doi:10.2307/2271658. · Zbl 0174.01202

[31] The Univalent Foundations Program.Homotopy Type Theory: Univalent Foundations of Mathematics.http://homotopytypetheory.org/book, Institute for Advanced Study, 2013.

[32] V. Voevodsky. The equivalence axiom and univalent models of type theory. (Talk at CMU on February 4, 2010). Preprint arXiv:1402.5556 [math.LO], 2014. URL:https://arxiv.org/pdf/ 1402.5556.pdf.

[33] ⊠) are equivalence classes ofA with Γ‘XAmodulo judgmental equality, and elements are defined similarly as equivalence

[34] ⊠f) (forf:Y→Xandr⊠I(Y)) and elements [u]⊠⊠Elem([Γ]⊠,[Aid,0]⊠) and [v]⊠⊠([Γ]⊠,[Aid,1]⊠), we have to give an element of

[35] ⊠), which we do by the formation rule forPath. Vol. 18:1CANONICITY AND HOMOTOPY CANONICITY FOR CUBICAL TYPE THEORY28:31