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THH and traces of enriched categories. (English) Zbl 07487081

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The topological Hochschild homology (THH) of a ring or ring spectrum R is an object much studied in recent years. If $\mathcal{C}$ is a spectral ∞-category, then $\mathrm{THH}(\mathcal{C})$ is the geometric realization of the simplicial spectrum

$$\mathrm{THH}(\mathcal{C})_n = \coprod_{X_0,\dots,X_n\in\mathcal{C}} \mathrm{Hom}(X_0,X_1)\otimes\cdots\otimes\mathrm{Hom}(X_n,X_0)$$

It is known [*J. A. Campbell* and *K. Ponto*, Algebr. Geom. Topol. 19, No. 2, 965–1017 (2019; Zbl 1460.16010); *D. Kaledin*, Contemp. Math. 643, 227–262 (2015; Zbl 1346.18027)] that THH is a trace functor.

The principal objective in this paper is

1. to present a combinatorial, model-independent framework for enriched higher category theory, describing enriched categories as symmetric monoidal functors

$$\mathrm{Bypass}_S \to \mathcal{V}$$

where $S$is a set of objects, $\mathcal{V}$ is a symmetric monoidal ∞-category, and Bypass_S denotes the category of directed graphs on the fixed set S of vertices as objects and bypass operations as morphisms with the monoidal product $\Gamma\otimes\Gamma'$ of $\Gamma,\Gamma'\in\mathrm{Bypass}_S$ denoting the graph whose edges are the disjoint union of those in Γ and those in Γ',

2. to present a universal construction of THH, using this framework, in which THH is the pushforward along $\mathrm{Bypass}_S\to\mathcal{V}$ of a certain presheaf $\mathcal{O}_{\mathrm{thh}}$ on Bypass_S obtained as a push-pull construction applied to the circle, and

3. to explicitly calculate the presheaf $\mathcal{O}_{\mathrm{thh}}$.

A synopsis of the paper goes as follows.

§2 gives the above combinatorial description of enriched categories. Gepner-Haugseng's enriched categories have underlying spaces of objects, while the author's having underlying sets of objects. Both are equivalent beholden to [*D. Gepner* and *R. Haugseng*, Adv. Math. 279, 575–716 (2015; Zbl 1342.18009), Theorem 5.3.17].

§3 reviews *A. Connes*' [C. R. Acad. Sci., Paris, Sér. I 296, 953–958 (1983; Zbl 0534.18009)] cyclic category Λ or, roughly speaking, the category of cyclically ordered finite sets. It is observed crucially that

Lemma (Lemma 3.4). There are canonical equivalences

$$\Lambda\cong\mathrm{Bypass}_S^{\mathrm{Eul}}\cong\Lambda^{\mathrm{op}}$$

where $\mathrm{Bypass}_S^{\mathrm{Eul}}$ denotes the category of nonempty graphs in Bypass_S with specified Eulerian tour.

§4 defines the presheaf $\mathcal{O}_{\mathrm{thh}}$

§5 identifies $\mathcal{O}_{\mathrm{thh}}$ when $S=*$. Let Ass denote the associative PROP [https://people.math.harvard.edu/~lurie/papers/HA.pdf, 4.1.1 The Associative ∞-Operads]. There is a functor $\Lambda\xrightarrow{k}\mathrm{Ass}$ that forgets the cyclic ordering. The classifying space of Λ is BS^1, so that there is a functor of ∞-categories $\Lambda\xrightarrow{r}BS^1$ formally inverting all the morphisms of Λ. Therefore we have the following diagram

$$\begin{array}{ccccc}\Lambda & \xrightarrow{k} & \mathrm{Ass} & \xrightarrow{\mathcal{C}} & \mathcal{V}\\ {\scriptstyle r}\downarrow & & & & \\ BS^1 & & & & \end{array}$$

It is shown (Theorem 5.3) that

Theorem. If $\mathcal{V}$ is presentable and A is an associative algebra in $\mathcal{V}$, we have

$$\mathrm{THH}(A) \equiv A_* k_* r^* (S^1)$$

where

(i) S^1 is the circle with the free S^1-action, regarded as an -space thus a presheaf $(BS^1)^{\mathrm{op}} \to \mathrm{Top}$;

(ii) $r^* : \mathcal{P}(BS^1) \to \mathcal{P}(\Lambda)$ denotes precomposition of a presheaf by r with $\mathcal{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Top})$;

(iii) $k_* : \mathcal{P}(\Lambda) \to \mathcal{P}(\mathrm{Ass})$ denotes left Kan extension along k;

(iv) $A_* : \mathcal{P}(\mathrm{Ass}) \to \mathcal{V}$ is the unique functor extending A to $\mathcal{P}(\mathrm{Ass}) \supseteq \mathrm{Ass}$, which preserves small colimits.

§6 identifies $\mathcal{O}_{\mathrm{thh}} \in \mathcal{P}(\mathrm{Bypass}_S)$ for an arbitrary set S. It is shown as a generalization of Theorem 5.3 that

Theorem (Theorem 6.3). If $\mathcal{C} : \mathrm{Bypass}_S \to \mathcal{V}$ is a $\mathcal{V}$-enriched category with set S of objects, then we have

$$\mathrm{THH}(A) \equiv \mathcal{C}_* k_* r^* (S^1)$$

with maps as in the diagram

$$\begin{array}{ccccc} \mathrm{Bypass}_S^{\mathrm{Eul}} & \xrightarrow{k} & \mathrm{Bypass}_S & \xrightarrow{A} & \mathcal{V} \\ {\scriptstyle r}\downarrow & & & & \\ BS^1 & & & & \end{array}$$

where the functor $r : \mathrm{Bypass}_S^{\mathrm{Eul}} \to BS^1$ is that which exhibits as the unifying space of $\mathrm{Bypass}_S^{\mathrm{Eul}}$, and $k : \mathrm{Bypass}_S^{\mathrm{Eul}} \to \mathrm{Bypass}_S$ is the forgetful functor.

§7 aims to establish that the following data are equivalent.

- a graph $\Gamma \in \mathrm{Bypass}_S$ with a chosen Eulerian tour;
- a cyclically ordered set of edges $E \in \Lambda$ with a labeling of its vertices in S.

§8 identifies the presheaf $\mathcal{O}_{\mathrm{thh}} \in \mathcal{P}(\mathrm{Bypass}_S)$ explicitly, restricted away from the empty graph (Theorem 8.6).

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MSC:

18D20 Enriched categories (over closed or monoidal categories)
18N99 Higher categories and homotopical algebra

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