INVERSE SPECTRAL THEORY

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ABSTRACT. Many self-adjoint operators appearing in mathematical physics and geometry have their spectral data : eigenvalues, informations of eigenvectors, scattering matrices. A natural attempt is the reconstruction of the original operator in terms of its spectral data. The precursor of this inverse spectral problem goes back at least to the Sturm-Liouville theory of differential operators. The systematic study of inverse spectral problems has become active from early 20th century, and the interest on this subject is unceasingly growing up since then.

The aim of this article is to give a brief survey of the inverse spectral problem for self-adjoint differential operators : boundary value problems and scattering problems for Schrödinger opeartors, Laplace-Beltrami operators on Riemannian manifolds. Both of the 1-dimensional and the multi-dimensional problems are discussed. There is so extensive literature on the inverse problem that our arguments must be restricted to limited aspects of the subject. The basic feature I would like to stress is :

One-dimensional spectral problems are smoothly deformable like C^∞-functions, while multi-dimensional problems are rigid like analytic functions (at least in Euclidean spaces).

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Part I 1-dimensional Problem

1. Inverse eigenvalue problem

Let us begin with this lecture by the 1-dimensional inverse eigenvalue problem. In §1 and §2, we follow Pöschel-Trubowitz [66] and Deift [13].

1.1 *Theorem of Borg-Levinson.* Let us consider the simplest case of the Dirichlet boundary value problem on $(0, 1)$:

$$-y'' + V(x)y = \lambda y, \quad 0 < x < 1, \tag{1.1}$$

$$y(0) = y(1) = 0. \tag{1.2}$$

If $V(x)$ is real-valued, this problem has a set of eigenvalues

$$\lambda_1(V) < \lambda_2(V) < \cdots < \lambda_n(V) < \cdots .$$

The first question of the inverse eigenvalue problem is

Question *If $\lambda_n(V_1) = \lambda_n(V_2)$ for all $n \geq 1$, does V_1 coincide with V_2?*

The answer is easily seen to be *negative.* You have only to take $V_2(x) = V_1(1 - x) \neq V_1(x)$ as a counter example. A potential $V(x)$ is said to be *even* if $V(x) = V(1-x)$. This parity is the only symmetry that the above Dirichlet problem has. The following theorem due to Borg and Levinson is the first uniqueness result up to symmetry.

THEOREM 1.1 *Suppose V_1 and V_2 are even and $\lambda_n(V_1) = \lambda_n(V_2)$ $\forall n \geq 1$. Then $V_1 = V_2$.*

If the potential is not even, one needs some auxiliary condition to prove the uniqueness. The second theorem, also due to Borg and Levinson, shows that the values of the derivatives at the boundary of eigenfunctions serve for this purpose.

Let $y(x, \lambda)$ satisfy (1.1) and

$$y(0, \lambda) = 0, \quad y'(0, \lambda) = 1. \tag{1.3}$$

We put

$$k_n(V) = y'(1, \lambda_n(V)). \tag{1.4}$$

THEOREM 1.2 *Suppose $\lambda_n(V_1) = \lambda_n(V_2), k_n(V_1) = k_n(V_2), \forall n \geq 1$. Then $V_1 = V_2$.*

1.2 *Global structure of isospectral potentials.* The set of potentials having the same eigenvalues is parametrized by $\{k_n(V)\}_{n=1}^{\infty}$, which makes it possible to deform continuously the potential keeping the eigenvalues fixed. Let us formulate it rigorously.

Let $L^2_{\mathbf{R}}(0, 1)$ be the set of all real-valued L^2-functions on $(0, 1)$. For $V \in L^2_{\mathbf{R}}(0, 1)$, $\lambda_n(V)$ has an asymptotic expansion

$$\lambda_n(V) = n^2\pi^2 + \int_0^1 V(x)dx + \mu_n(V), \quad \sum_{n=1}^{\infty} (\mu_n(V))^2 < \infty.$$

With this in mind, we put

$$\mu_0(V) = \int_0^1 V(x)dx, \tag{1.5}$$

and let $\mu(V) = (\mu_0(V), \mu_1(V), \mu_2(V), \cdots)$. Then μ is a map

$$\mu : L^2_{\mathbf{R}}(0,1) \to \mathbf{R} \times l^2 =: S. \tag{1.6}$$

We also put $\kappa(V) = (\kappa_1(V), \kappa_2(V), \cdots)$, where

$$\kappa_n(V) = \log(-1)^n k_n(V) \tag{1.7}$$

and let l^2_1 be defined by

$$l^2_1 \ni \alpha = (\alpha_1, \alpha_2, \cdots) \Longleftrightarrow \sum_{n=1}^{\infty} (n\alpha_n)^2 < \infty. \tag{1.8}$$

THEOREM 1.3 *$\kappa \times \mu : L^2_{\mathbf{R}}(0,1) \to l^2_1 \times S$ is a real analytic isomorphism.*

Here a map between real Hilbert spaces is said to be real analytic if it is continuouly Fréchet differentiable on the complexification of the real Hilbert spaces.

Theorem 1.3 characterizes the Dirichlet spectral data, the eigenvalues and the derivatives at the boundary of the eigenfunctions.

For $q \in L^2_{\mathbf{R}}(0,1)$, let

$$M(q) = \{V \in L^2_{\mathbf{R}}(0,1) : \lambda_n(q) = \lambda_n(V), \forall n \geq 1\}. \tag{1.9}$$

THEOREM 1.4 *$M(q)$ is a real analytic submanifold of $L^2_{\mathbf{R}}(0,1)$ and κ is a global coordinate system on $M(q)$.*

Therefore one can deform $V(x)$ continuously keeping eigenvalues fixed, by varying $k_1(V)$ for instance.

2. Isospectral deformation

It is well-known that for two bounded operators A and B

$$\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\}, \tag{2.1}$$

where $\sigma(A)$ denotes the spectrum of A. The proof of (2.1) is elementary for the finite dimensional case, and the infinite dimensional case uses the equation

$$\lambda(AB + \lambda)^{-1} + A(BA + \lambda)^{-1}B = 1. \tag{2.2}$$

For unbounded operators, (2.1) is also true if A is a densely defined closed operator and $B = A^*$. This is particularly useful in the application to 1-dimensional problems. In fact, Crum [11] used this method, by a straightforward computation, to deform and remove eigenvalues of Strum Liouville operators.

The commutation method has a long history, precursors of which are seen in the works of Jacobi [41] and Darboux [12]. We elucidate it here formally.

Let $V \in L^2_{\mathbf{R}}(0,1)$ and φ_n be a Dirichlet eigenfunction of $-d^2/dx^2 + V(x)$ with eigenvalue λ_n. Ignoring the question of domain of all relevant operators, we put formally

$$A = \varphi_n \frac{d}{dx}(\frac{1}{\varphi_n}\cdot), \quad A^* = -\frac{1}{\varphi_n}\frac{d}{dx}(\varphi_n \cdot). \tag{2.3}$$

Using $-\varphi_n'' + V\varphi_n = \lambda_n \varphi_n$, we have

$$A^*A = -\frac{d^2}{dx^2} + V(x) - \lambda_n, \tag{2.4}$$

$$AA^* = -\frac{d^2}{dx^2} + V(x) - 2\frac{d^2}{dx^2}\log\varphi_n(x) - \lambda_n. \tag{2.5}$$

Therefore we will get $\sigma(A^*A)\setminus\{0\} = \sigma(AA^*)\setminus\{0\}$. However, the potential of AA^* is not in L^2 in neighbourhoods of the zeros of $\varphi_n(x)$. This causes troubles.

The remedy comes from the *double commutation*. Namely by taking

$$u = \frac{1}{\varphi_n}(a + b\int_0^x \varphi_n(t)^2 dt) \tag{2.6}$$

and putting

$$B = u\frac{d}{dx}(\frac{1}{u}\cdot), \quad B^* = -\frac{1}{u}\frac{d}{dx}(u\cdot), \tag{2.7}$$

we have

$$AA^* = B^*B, \tag{2.8}$$

$$BB^* = -\frac{d^2}{dx^2} + V(x) - 2\frac{d^2}{dx^2}\log(u\varphi_n) - \lambda_n. \tag{2.9}$$

Furthermore, we have

$$BB^*\frac{1}{u} = 0. \tag{2.10}$$

This shows that $V(x)$ and $V(x) - 2\frac{d^2}{dx^2}\log(u\varphi_n)$ have the same Dirichlet eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$.

Let us note that this commutation method does not work in multi-dimension. We show in §4 that there is no analogue of isospectral deformation of $-\Delta + V$ like Theorem 1.4.

3. Inverse scattering

3.1 *Scattering problem.* The concept of scattering experiment is as follows. One puts a target and projects a beam of particles. By observing the scattered particles, one tries to investigate the target. In the case of potential scattering in quantum mechanics, this process is described by the following Schrödinger equation in $\mathbf{R}^3$

$$(-\Delta + V(x))\varphi = E\varphi, \tag{3.1}$$

where $E > 0$ denotes the energy of scattering particles and $V(x)$ is a real function, which is assumed to be rapidly decreasing.

The equation (3.1) has an infinite number of solutions. However, the solution corresponding to the above scattering process can be chosen uniquely under a suitable boundary condition at infinity, and has the following asymptotic expansion

$$\varphi \simeq e^{i\sqrt{E}\omega\cdot x} + \frac{e^{i\sqrt{E}r}}{r}f(E;\theta,\omega) \tag{3.2}$$

as $r = |x| \to \infty, \theta = x/r$. Here the first term of the right-hand side of (3.2) represents the incident plane wave having direction $\omega \in S^2$, and the second term represents the scattered spherical wave. $f(E;\theta,\omega)$ is called the *scattering amplitude*. $|f(E;\theta,\omega)|^2$ is called the *differential cross section* and denotes the ratio of number of particles reflected to the θ direction to number of incident particles with direction ω. This is the physically observed quantity.

Time-dependent picture visualizes the scattering process more clearly. Let $H = -\Delta + V(x)$. Then the behavior of the particle is described by a solution of the time-dependent Schrödinger eqiuation $i\partial_t u = Hu$, namely $u(t) = e^{-itH}u$. In the remote past and the remote future, this particle runs very far from the scattering center. Since the potential $V(x)$ decays rapidly, the behavior of the particle is approximately goverened by $H_0 = -\Delta$ as $t \to \pm\infty$. More precisely, there exist $u_\pm$

such that $\|e^{-itH}u - e^{-itH_0}u_\pm\| \to 0$ as $t \to \pm\infty$. This means that the incoming free particle $e^{-itH_0}u_-$ is scattered into $e^{-itH_0}u_+$ after the collision. The operator

$$S : u_- \to u_+ \tag{3.3}$$

is called the *scattering operator*. The structure of S is seen more explicitly by passing to the Fourier transformation

$$\mathcal{F}_0 u(\xi) = \hat{u}(\xi) = (2\pi)^{-3/2} \int_{\mathbf{R}^3} e^{-ix\cdot\xi} u(x)dx. \tag{3.4}$$

In fact, we have

$$\hat{u}_+(\sqrt{E}\theta) = \hat{u}_-(\sqrt{E}\theta) - \frac{\sqrt{E}}{2\pi i} \int_{S^2} f(E;\theta,\omega)\hat{u}_-(\sqrt{E}\omega)d\omega, \tag{3.5}$$

where $f(E;\theta,\omega)$ is the scattering amplitude which appeared in (3.2). This means that the operator

$$\hat{S} = \mathcal{F}_0 S \mathcal{F}_0^* \tag{3.6}$$

acts only on the variable $\omega \in S^2$. This is physically natural, since $\mathcal{F}_0 H_0 \mathcal{F}_0^* = |\xi|^2$ and the energy is conserved during the scattering process.

Let $\hat{S}(E)$ be the integral operator on $L^2(S^2)$:

$$\hat{S}(E)\psi(\theta) = \psi(\theta) - \frac{\sqrt{E}}{2\pi i} \int_{S^2} f(E;\theta,\omega)\psi(\omega)d\omega. \tag{3.7}$$

Then $\hat{S}$ is written as

$$(\hat{S}u)(\sqrt{E}\theta) = (\hat{S}(E)u(\sqrt{E}\cdot)(\theta). \tag{3.8}$$

$\hat{S}(E)$ is a unitary operator on $L^2(S^2)$ and is called the *S-matrix*.

The inverse problem of scattering is now formulated as follows :

Given the scattering amplitude $f(E;\theta,\omega)$, reconstruct the potential $V(x)$.

Physically, the differential cross section is the only observable quantity. However, since $\hat{S}(E)$ is unitary, $f(E;\theta,\omega)$ satisfies an integral equation. Using this equation one can construct $f(E;\theta,\omega)$ from the square of its modulus when $|f(E;\theta,\omega)|$ is sufficiently small. See Martin [52]. However, when the differential cross section is not small, this is no longer true. See Newton [60]

3.2 *Spherically symmetric potentials.* When the potential $V(x)$ is spherically symmetric, $V(x) = V(|x|)$, the above problem is reduced to the one on the interval $(0,\infty)$ by the well-known procedure of partial wave expansion.

Let $k = \sqrt{E}$ and $r = |x|$. Then the solution φ of (3.1) depends only on r and the angle between ω and x. Letting θ denote this angle, we expand φ as

$$\varphi = \frac{1}{kr} \sum_{l=0}^{\infty} i^l (2l+1) u_l(r,k) P_l(\cos\theta), \tag{3.9}$$

where P_l is the Legendre polynomial. Then by (3.1) and (3.2), $u_l(r,k)$ satisfies

$$-\frac{d^2}{dr^2}u_l(r,k) + \left[V(r) + \frac{l(l+1)}{r^2}\right] u_l(r,k) = k^2 u_l(r,k), \tag{3.10}$$

$$u_l(0,k) = 0, \tag{3.11}$$

$$u_l(r,k) \simeq \sin(kr - \frac{l\pi}{2} + \delta_l) \quad \text{as} \quad r \to \infty \tag{3.12}$$

for some $\delta_l \in \mathbf{R}$. This is called the *phase shift* and the scattering amplitude is written as

$$f(k,\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos\theta). \tag{3.13}$$

In the following we pick up the case $l = 0$.

3.3 *Gel'fand-Levitan theory.* Let us consider the scattering theory on the half line described by the following equation

$$-\frac{d^2}{dx^2}\psi(x,k) + V(x)\psi(x,k) = k^2\psi(x,k), \quad 0 < x < \infty, \tag{3.14}$$

$$\psi(0,k) = 0, \quad k \in \mathbf{R}. \tag{3.15}$$

If $V(x)$ decays sufficiently rapidly at infinity, the solution $\psi(x,k)$ has the following asymptotics

$$\begin{aligned}\psi(x,k) &\simeq C(k)\sin(kx-\delta(k)) \\ &= \frac{C(k)}{2i}(e^{-i\delta(k)}e^{ikx} - e^{i\delta(k)}e^{-ikx})\end{aligned}$$

as $x \to \infty$. The term e^{ikx} represents the outgoing wave and the term e^{-ikx} represents the incoming wave. The correspondence $e^{i\delta(k)} \to e^{-i\delta(k)}$ then defines the S-matix. Hence we put

$$S(k) = e^{-2i\delta(k)}. \tag{3.16}$$

The inverse scattering problem is rephrased as

Given $\delta(k)$, reconstruct $V(x)$.

The first complete solution of the 1-dimensional inverse problem was given by Gel'fand-Levitan [26]. They reconstructed $V(x)$ from the spectral function, which is the density function in the generalized eigenfunction expansion theory for Sturm-Liouville operators. Krein and Marchenko completed the inverse scattering by showing the passage from $S(k)$ to the spectral function. The completed theory contains a characterization of the scattering matrix and the reconstruction procedure. It is summarized in the folllowing theorem.

THEOREM 3.1. *In order that a $\mathbf{C}$-valued function $S(k)$ defined on $\mathbf{R}$ be the scattering matrix of a Schrödinger operator $H = -d^2/dx^2 + V(x)$ on $(0,\infty)$ with Dirichlet boundary condition at 0 and a real-valued potential $V(x)$ satisfying*

$$\int_0^\infty x|V(x)|dx < \infty,$$

it is necessary and sufficient that

(1)

$$S(k) \in C(\mathbf{R}), \quad |S(k)| = 1, \quad \overline{S(k)} = S(-k),$$

(2)

$$S(k) - 1 = \int_{-\infty}^{\infty} e^{-ikt}(F_1(t) + F_2(t))dt,$$

where $F_1(t) \in L^1(\mathbf{R})$, $F_2(t) \in L^2(\mathbf{R}) \cap L^\infty(\mathbf{R})$ and

$$\int_0^\infty t|F_1'(t) + F_2'(t)|dt < \infty,$$

(3)

$$\arg S(+0) - \arg S(+\infty) + \frac{\pi}{2}(S(0) - 1) = 2\pi m,$$

where m is a non-negative integer.

If $m = 0$, the potential $V(x)$ is uniquely determined. If $m > 0$, there exists an m-parameter family of potentials having the same $S(k)$ as the S-matrix.

In the above theorem, m is the number of discrete eigenvalues of $-d^2/dx^2 + V(x)$. The deformation of the potential keeping the S-matrix fixed is carried out by the method of commutation in §2.

For the proof, see e.g. Faddeev [20], or Marchenko [51].

3.4 *Generalized sine transformation.* To study the spectral structure of H and the properties of S-matrix, the generalized sine transformation plays a key role. Let us recall that for a self-adjoint operator A, the absolutely continuous subspace for A, $\mathcal{H}_{ac}(A)$, is the set of all u such that $(E(\lambda)u, u)$ is absolutely continuous with respect to $d\lambda$, where $A = \int_{-\infty}^{\infty} \lambda dE(\lambda)$.

Now let $\varphi(x, k)$ be the solution of the equation

$$-\varphi''(x, k) + V(x)\varphi(x, k) = k^2\varphi(x, k), \quad x > 0, \tag{3.17}$$

$$\varphi(0, k) = 0, \quad \varphi'(0, k) = 1, \tag{3.18}$$

where $' = d/dx$. Then $\varphi(x, k)$ behaves like

$$\varphi(x, k) = \frac{A(k)}{k} \sin(kx - \delta(k)) + o(1) \tag{3.19}$$

as $x \to \infty$. We define

$$\mathcal{F}u(k) = \int_0^{\infty} u(x)\psi^{(+)}(x, k)dx, \tag{3.20}$$

$$\psi^{(+)}(x, k) = \varphi(x, k)/M(k), \quad M(k) = A(k)e^{i\delta(k)}. \tag{3.21}$$

Then $\mathcal{F}$ is a unitary from $\mathcal{H}_{ac}(H)$ to $L^2((0, \infty); \frac{2k^2}{\pi}dk)$, and we have the inversion formula for $u \in \mathcal{H}_{ac}(H)$

$$u = \frac{2}{\pi} \int_0^{\infty} \overline{\psi^{(+)}(x, k)} \mathcal{F}u(k)k^2 dk. \tag{3.22}$$

Moreover $\mathcal{F}$ diagonalizes H :

$$(\mathcal{F}Hu)(k) = k^2(\mathcal{F}u)(k). \tag{3.23}$$

When $V(x) = 0$, $\mathcal{F}$ reduces to the sine transformation

$$\mathcal{F}_0 u(k) = \int_0^{\infty} u(x) \frac{\sin kx}{k} dx. \tag{3.24}$$

3.5 *The core of Gel'fand-Levitan theory.* Let us explain the essence of Gel'fand-Levitan theory. Let $\varphi(x, k)$ be as in (3.17), (3.18). Then $\varphi(x, k)$ is an even and entire function of $k \in \mathbf{C}$ satisfying

$$\varphi(x, k) = \frac{1}{k} \sin kx + o(\frac{e^{|\mathrm{Im}\ k|x}}{|k|}), \quad |k| \to \infty. \tag{3.25}$$

Here we recall the Paley-Wiener theorem. An entire function $F(z)$ is said to be of exponential type σ if for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that $|F(z)| \leq C_\epsilon e^{(\sigma+\epsilon)|z|}$, $z \in \mathbf{C}$.

Theorem 3.2 *(Paley-Wiener). $F(x) \in L^2(\mathbf{R})$ is extended to an entire function of exponential type σ if and only if there exists $h \in L^2(-\sigma, \sigma)$ such that*

$$F(z) = \int_{-\sigma}^{\sigma} h(\xi) e^{iz\xi} d\xi.$$

By virtue of Paley-Wiener theorem and (3.25), $\varphi(x,k)$ has the following representation

$$\varphi(x,k) = \frac{\sin kx}{k} + \int_0^x K(x,y) \frac{\sin ky}{k} dy. \tag{3.26}$$

We insert this expression to the equation (3.17). Then K is shown to satisfy the equation

$$(\partial_y^2 - \partial_x^2 + V(x))K(x,y) = 0. \tag{3.27}$$

The crucial fact is

$$2\frac{d}{dx} K(x,x) = V(x). \tag{3.28}$$

One can further derive the following equation

$$K(x,y) + \Omega(x,y) + \int_0^x K(x,t)\Omega(t,y)dt = 0, \quad x > y, \tag{3.29}$$

where $\Omega(x,y)$ is a function constructed from the S-matrix and informations of bound states. This is called the *Gel'fand-Levitan equation*.

Thus the scenario of the reconstruction of $V(x)$ is as follows. From the scattering matrix and the bound states, one constructs $\Omega(x,y)$. Solving (3.29) one gets $K(x,y)$. The potential $V(x)$ is obtained by (3.28).

3.6 *What is the hidden mechanism?* This is truely an ingenious trick and it is not easy to find the key fact behind their theory. It is Kay and Moses [43] who studied an algebraic aspect of the Gel'fand-Levitan method.

Let H_0 and H be two self-adjoint operators. A unitary operator U from $\mathcal{H}_{ac}(H_0)$ to $\mathcal{H}_{ac}(H)$ is said to *intertwine* H_0 and H if it satisfies

$$HU = UH_0. \tag{3.30}$$

An important example of intertwining operator for $H_0 = -\Delta$ and $H = -\Delta + V(x)$ is the wave operator

$$W_\pm = \mathrm{s} - \lim_{t\to\pm\infty} e^{itH} e^{-itH_0}. \tag{3.31}$$

Another important example of intertwining operator is the so called *spectral representation*. Let A be salf-adjoint and I be its absolutely continuous spectrum, $I = \sigma(A|_{\mathcal{H}_{ac}(A)})$. A unitary operator T from $\mathcal{H}_{ac}(A)$ to $L^2(I;\mathbf{h})$, $\mathbf{h}$ being an auxiliaray Hilbert space, is called a spectral representation of A if

$$(TAu)(\lambda) = \lambda(Tu)(\lambda), \quad \lambda \in I, \quad u \in \mathcal{H}_{ac}(A).$$

For $H_0 = -d^2/dx^2$ and $H = -d^2/dx^2 + V(x)$ on $(0,\infty)$, the above (generalized) sine transformations $\mathcal{F}_0$ and $\mathcal{F}$ are spectral representations. It is obvious that $U = \mathcal{F}^*\mathcal{F}_0$ intertwines H_0 and H.

It is also evident that if U intertwines $H_0 = -d^2/dx^2$ and $H = -d^2/dx^2 + V(x)$, its kernel $U(x,y)$ satisfies

$$(-\partial_x^2 + V(x))U(x,y) = -\partial_y^2 U(x,y).$$

The spectral representation for the 1-dimensional Schrödinger operator H has the following distinguished property. Let $\psi^{(+)}(x,k)$ be as in (3.21). Let

$$U(x,y) = \int_0^\infty \psi^{(+)}(x,k)\frac{\sin ky}{k}\rho(k)dk, \quad \rho(k) = \frac{2k^2}{\pi} \tag{3.32}$$

be the integral kernel of the intertwining operator $\mathcal{F}^*\mathcal{F}_0$. In view of the formula (3.26), we have

$$U(x,y) = \delta(x-y) + \eta(x-y)K(x,y),$$

where $\eta(t)$ is the Heaviside function. Namely, U is an integral operator of *Volterra* type. The key fact discovered by Kay-Moses is the following theorem.

THEOREM 3.3 *Let $H_0 = -d^2/dx^2$ be defined on $(0,\infty)$ with Dirichlet boundary condition, and let $H = H_0 + Q$ be a self-adjoint perturbation of H_0. Suppose $U = I + K$ intertwines H_0 and H and that U is Volterra, i.e. $K(x,y) = 0$ if $x < y$. Then Q is an operator of multiplication by*

$$q(x) = 2\frac{d}{dx}K(x,x). \tag{3.33}$$

For $x > y$, the following equation holds

$$(\partial_x^2 - \partial_y^2)K(x,y) = q(x)K(x,y). \tag{3.34}$$

Proof. Let us content ourselves by the formal proof, since we are mainly interested in the algebraic or operator theoritical aspect of the Gel'fand-Levitan theory.

Since $U(x,y) = \delta(x-y) + \eta(x-y)K(x,y)$ satisfies the wave equation

$$(\partial_x^2 - \partial_y^2)U(x,y) = \int_0^\infty Q(x,z)U(z,y)dz,$$

we have

$$\int_0^\infty Q(x,z)U(z,y)dz = 2\delta(x-y)\frac{d}{dx}K(x,x) + \eta(x-y)(\partial_x^2 - \partial_y^2)K(x,y).$$

Since U is Volterra, so is U^{-1}. Hence by multiplying U^{-1} to the above equation we have

$$Q(x,y) = 2\delta(x-y)\frac{d}{dx}K(x,x) + \eta(x-y)C(x,y).$$

However, since Q is self-adjoint, $C(x,y) = 0$, which proves

$$Q(x,y) = 2\delta(x-y)\frac{d}{dx}K(x,x).$$

This means that Q is the operator of multiplication by $2\dfrac{d}{dx}K(x,x)$.

Let us summarize the above arguments. In the 1-dimensional case, the generalized eigenfunction $\varphi(x,k)$ of the Schrödinger operator $H = -d^2/dx^2 + V(x)$ has the triangular expression (3.26). This makes the intertwining operator $\mathcal{F}^*\mathcal{F}_0$ into Volterra type. The potential $V(x)$ is reconstructed from the kernel of this Volterra operator.

An excellent exposition of the 1-dimensional inverse scattering is the one given by Faddeev [20]. A historical survey by Newton in the foreword of the monograph of Chadan-Sabatier [8] contains an extensive literature up to 1977.

Part II Multi-dimensional Problem

4. n-dimensional Borg-Levinson theorem

The simplest multi-dimensional spectral problem is the Dirichlet problem. Let Ω be a bounded domain in $\mathbf{R}^n, n \geq 2$, with smooth boundary S. Consider the boundary value problem

$$(-\Delta + V(x))u = \lambda u \quad \text{in } \Omega, \tag{4.1}$$

$$u|_S = 0, \tag{4.2}$$

where $V(x)$ is a real-function. Then there exists an infinite number of eigenvalues

$$\lambda_1(V) < \lambda_2(V) \leq \cdots .$$

The smallest eigenvalue is always simple.

As can be proved easily, even if $\lambda_i(V_1) = \lambda_i(V_2)$ for all $i \geq 1$, V_1 is not necessary equal to V_2. The 1-dimensional Borg-Levinson theorem suggests that the set of normal derivatives of eigenfunctions are a candidate of the auxiliary condition to guarantee the uniqueness of the potential. Let us mention parenthetically that Gel'fand raised this problem in 1954 [25]. If one considers the Neumann problem, the auxiliary condition will be the values of eigenfunctions at the boundary. However we have to be careful in choosing the eigenfunctions, since the eigenvalues are not simple in general.

Let m be the multiplicity of $\lambda_i(V)$ and $u_1, \cdots, u_m$ be a syetem real-valued orthnormal eigenfunctions associted with $\lambda_i(V)$. We set

$$E_i(V) = \left\{ \Big(\frac{\partial u_1}{\partial \nu}, \cdots, \frac{\partial u_m}{\partial \nu}\Big)\Big|_S \right\}, \tag{4.3}$$

ν being the outer unit normal to S. It is easy to see that for two such system of eigenfunctions $\{u_1, \cdots, u_m\}, \{v_1, \cdots, v_m\}$, there exists an orthgonal matrix T such that

$$\left(\frac{\partial u_1}{\partial \nu}, \cdots, \frac{\partial u_m}{\partial \nu}\right)\Big|_S = \left(\frac{\partial v_1}{\partial \nu}, \cdots, \frac{\partial v_m}{\partial \nu}\right)\Big|_S T.$$

This defines an equivalent relation $\sim$ in the space of functions on the boundary S, and for the set $E_i(V)$, there corresponds only one equivalence class, which we denote by $W_i(V)$:

$$W_i(V) = E_i(V)/\sim . \tag{4.4}$$

The following theorem is due to Nachman-Sylvester-Uhlman [58].

Theorem 4.1. *Suppose* $V_1, V_2 \in C^\infty(\overline{\Omega})$ *and*

$$\lambda_i(V_1) = \lambda_i(V_2), \quad W_i(V_1) = W_i(V_2), \quad \forall i \geq 1.$$

Then $V_1 = V_2$.

Apparently, this theorem is a first step of generalization of the 1-dimensional inverse spectral theory to the multi-dimensional case. However, unlike the 1-dimensional problem, neither the map : $V \to \{\lambda_i(V)\} \times \{W_i(V)\}$ is an isomorphism, nor $\{W_i(V)\}$ is a system of coordinates of the manifold of isospectral potentials. This can be seen most easily, perhaps, in the following theorem [32].

Theorem 4.2. *Let $V_1, V_2 \in C^\infty(\overline{\Omega})$. Suppose there exists $N > 0$ such that*

$$\lambda_i(V_1) = \lambda_i(V_2), \quad W_i(V_1) = W_i(V_2), \quad \forall i \geq N.$$

Then $V_1 = V_2$.

In the 1-dimensional case, it is possible to deform continuouly the potential $V(x)$ by varying $W_1(V)$ keeping all $\lambda_i(V)$ fixed. This is not true in the multi-dimensional case.

Let us breifly explain the proof of Theorem 4.2. We use the Dirichlet-Neumann map $N(\lambda, V)$:

$$N(\lambda, V)f = \frac{\partial u}{\partial \nu}\Big|_S, \tag{4.5}$$

where u is a solution to the Dirichlet problem

$$(-\Delta + V - \lambda)u = 0 \quad \text{in } \Omega, \tag{4.6}$$

$$u|_S = f. \tag{4.7}$$

We are assuming that $\lambda \notin \sigma_p(-\Delta_D + V)$. $N(\lambda, V)$ has, formally, the integral kernel

$$N(\lambda, V; x, y) = \sum_{i=1}^{\infty} \frac{1}{\lambda_i - \lambda} \frac{\partial u_i}{\partial \nu}(x) \frac{\partial u_i}{\partial \nu}(y), \tag{4.8}$$

u_i being the eigenfunctions associated with λ_i.

Let $(\ ,\ )$ and $\langle\ ,\ \rangle$ be the inner product of $L^2(\Omega)$ and $L^2(S)$, respectively. Letting $\varphi_{\lambda,\omega}(x) = e^{i\sqrt{\lambda}\omega\cdot x}, \omega \in S^{n-1}$, we put

$$S(\lambda, \theta, \omega; V) = \langle N(\lambda, V)\varphi_{\lambda,\omega}, \overline{\varphi_{\lambda,-\theta}}\rangle. \tag{4.9}$$

Then by integration by parts one can show

$$S(\lambda, \theta, \omega; V) = -\frac{\lambda}{2}(\theta - \omega)^2 \int_\Omega e^{-i\sqrt{\lambda}(\theta-\omega)\cdot x} dx$$

$$+ \int_\Omega e^{-i\sqrt{\lambda}(\theta-\omega)\cdot x} V(x) dx - (R(\lambda) V\varphi_{\lambda,\omega}, V\overline{\varphi_{\lambda,-\theta}}), \tag{4.10}$$

where $R(\lambda) = (-\Delta_D + V - \lambda)^{-1}$. From $S(\lambda, \theta, \omega; V)$, one can reconstruct $V(x)$. In fact for $0 \neq \xi \in \mathbf{R}^n$, choose $\eta \in S^{n-1}$ such that $\eta \perp \xi$. For a large parameter N, we put

$$\begin{aligned} \theta_N &= C_N\eta + \xi/(2N), \quad C_N = (1 - |\xi|^2/(4N^2))^{1/2}, \\ \omega_N &= C_N\eta - \xi/(2N), \\ \sqrt{t_N} &= N + i. \end{aligned}$$

Then we have

$$\lim_{N\to\infty} S(t_N, \theta_N, \omega_N; V) = -\frac{|\xi|^2}{2} \int_\Omega e^{-ix\cdot\xi} dx + \int_\Omega e^{-ix\cdot\xi} V(x) dx. \tag{4.11}$$

This proves Theorem 4.1, since $S(\lambda, \theta, \omega; V)$ is determined by $\{\lambda_i(V)\}$ and $\{W_i(V)\}$. Moreover from the formal formula (4.8), one can show that if V_1 and V_2 satisfy the assumption of Theorem 4.2,

$$\|N(\lambda, V_1) - N(\lambda, V_2)\|_{\mathbf{B}(L^2(\partial\Omega))} \leq C/|\lambda|,$$

for large $|\lambda|$. This and (4.11) imply $\hat{V}_1(\xi) = \hat{V}_2(\xi)$, which proves Theorem 4.2.

The expression (4.10) is very close to the S-matrix in potential scattering, and the above proof is inspired by the Born approximation in scattering theory, which will be discussed later.

5. Generalized Gel'fand problem

Let us recall the problem raised by Gel'fand.

Gel'fand problem. *Do the eigenvalues and the eigenfunctions at the boundary determine the potential part of the following Neumann problem uniquely ?*

$$\begin{cases} (-\Delta + V)u = \lambda u & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

As has been seen in the previous section, the answer is affirmative. Belishev and Kurylev observed this problem from a more general view point.

Let M be an m-dimensional Riemannian manifold with boundary $S = \partial M$, equipped with Riemannian metric $g_{jk}dx^j dx^k$. Let $g = \det(g_{jk})$. In $L^2(M, \sqrt{g}dx)$, consider the following differential operator

$$A = -\frac{1}{\sqrt{g}}(\partial_j + ib_j)\sqrt{g}g^{jk}\mu(\partial_k + ib_k) + q \tag{5.1}$$

with boundary condition

$$(\partial_\nu + ib_\nu + \sigma)u|_S = 0, \tag{5.2}$$

where $b = (b_1, \cdots, b_m), \mu, q, \sigma$ are real-valued and $\mu \geq \mu_0$ for a constant $\mu_0 > 0$. The ∂_ν is the normal derivative and b_ν is the normal component of b at S. In the application to electromagnetism, μ is a conductivity, σ is a boundary impedance. In quantum mechanics, b and q are megnetic and electric potentials.

The operator A has the discrete spectrum

$$\lambda_1 \leq \lambda_2 \leq \cdots \to \infty.$$

Let $\varphi_k(x)$ be the associated orthonormal eigenfunctions. The boundary spectral data (BSD) of A is the triple $(S, \{\lambda_k\}, \{\varphi_k|_S\})$. Belishev-Kurylev posed the following question.

Generalized Gel'fand problem. *Does BSD $(S, \{\lambda_k\}, \{\varphi_k|_S\})$ determine the manifold M and the operator A?*

Let us call the triple $(M, (g_{jk}), A)$ the operator system. Belishev and Kurylev gave an affirmative answer to this question by a constructive procedure, which they call BC (boundary controll) method [45], [2]. Their starting point is to allow the equivalence by conformal diffeomorphism.

Definition 5.1. Two BSD $(S, \{\lambda_k\}, \{\varphi_k|_S\})$ and $(R, \{\mu_k\}, \{\psi_k|_R\})$ of the operator systems (M, g, A) and (N, h, B) are said to be equivalent if

(1) $\lambda_k = \mu_k, \quad \forall k \geq 1$,
(2) there exists a conformal diffeomorphism $\phi : S \to R$ and a real-valued function $\kappa_0 > 0$ on S such that

$$\phi^*(\psi_k|_R) = \kappa_0 \varphi_k|_S, \tag{5.3}$$

$$\phi^*(h|_R) = \kappa_0^{4/m} g|_S. \tag{5.4}$$

The conditions (5.3), (5.4) are assumed to allow the generalized gauge transformation $u \to \alpha e^{i\beta} u$.

Theorem 5.2. *The BSD's $(S, \{\lambda_k\}, \{\varphi_k|_S\})$ and $(R, \{\mu_k\}, \{\psi_k|_R\})$ of the operator systems (M, g, A) and (N, h, B) are equivalent if and only if there exists a conformal diffeomorphism $\Phi : M \to N$ and a complex-valued function $\kappa \neq 0$ such that*

$$B = \Phi \circ (\kappa A \kappa^{-1}) \circ \Phi^{-1},$$

$$\Phi|_S = \phi, \quad \kappa|_S = \kappa_0.$$

If $\mu = 1$ in (5.1), it is natural to call A a Schrödinger operator on M. The Schrödinger operator A adimits a gauge transormation, i.e. $e^{ic} A e^{-ic}$ is again a Schrödinger operator with magnetic field $b_k + \partial_k c$.

Theorem 5.3. *For a BSD $(S, \{\lambda_k\}, \{\varphi_k|_S\})$, there exists a unique operator system (M, g, A), where A is a Schrödinger operator on M, up to gauge transformation.*

The original interest of Gel'fand was the determination of potential term from the Dirichlet-Neumann map of the Schrödinger operator. Let us return to this problem in the anisotropic case.

Let Ω be a bounded domain in $\mathbf{R}^m$ with smooth boundary S and consider the following boundary value problem

$$\partial_i (a^{ij}(x) \partial_j u) = 0 \quad \text{in} \quad \Omega,$$

$$u = f \quad \text{on} \quad S,$$

where $A = (a^{ij}(x))$ is a positive definite C^∞-matrix and $n = (n_1, \cdots, n_m)$ is the outer unit normal to S. Let Λ_A be the Dirichlet-Neumann map

$$\Lambda_A f = \rho n_i a^{ij} \partial_j u|_S,$$

where ρ is a positive function determined by the metric and S. If $\Phi : \Omega \to \Omega$ is a diffeomorphism which fixes the boundary and we define

$$\Phi_* A = \frac{{}^t(D\Phi) A (D\Phi)}{\det(D\Phi)} \circ \Phi^{-1},$$

then $\Lambda_{\Phi_* A} = \Lambda_A$. The converse is also true.

Theorem 5.4. *Let Ω be a bounded open set in $\mathbf{R}^2$ with C^3-boundary. Suppose two C^3-metrics A and B satisfy $\Lambda_A = \Lambda_B$. Then there exists a diffeomorphism $\Phi : \Omega \to \Omega$ such that*

$$B = \Phi_* A \quad \text{and} \quad \Phi|_{\partial\Omega} = I.$$

If $\dim \Omega \geq 3$, this theorem is proved under the additional assumption of analyticity.

Theorem 5.5. *Let Ω be a compact, simply connected real-analytic manifold of dimension ≥ 3 with real-analytic boundary. Let A and B be real-anlytic metrics on Ω such that $\Lambda_A = \Lambda_B$. Suppose one of the following condition holds :*

(a) Both of the metrics have the following property. For any two points $p, q \in \Omega$, there is a unique minimal geodesic joining p and q whose interior lies in Ω.

(b) Either A or B extends to a complete real analytic metric on a non-compact real-anlytic manifold without boundary containing Ω.

Then there exists a real-analytic diffeomorphism $\Phi : \Omega \to \Omega$ such that

$$B = \Phi_* A \quad \text{and} \quad \Phi|_{\partial\Omega} = I.$$

For the proof, see [49], [70] and [56]. More recent results are seen in [48] and [47].

6. Kac problem

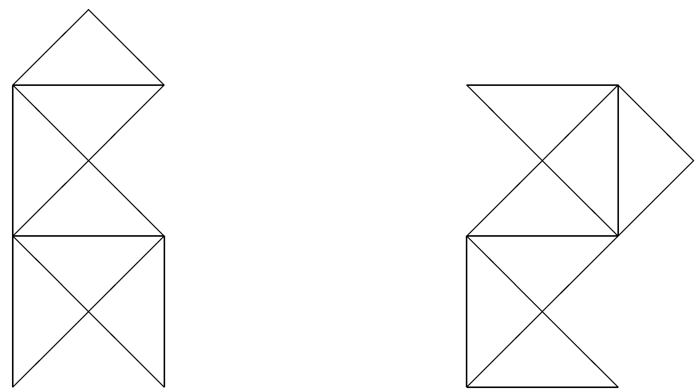
In 1966 M. Kac raised the following problem with an impressive title "Can one hear the shape of a drum?" and an interesting episode of D.Hilbert and H.Weyl [42]

Problem. *What we can know about the geometry of a Riemannian manifold from the knowledge of eigenvalues of its Laplace-Beltrami operator?*

Suppose we are given two compact Riemaniann manifolds whose all eigenvalues of Laplace-Beltrami operators coincide. Are they isometric? The answer is negative. There is a counter example of 16-dimensional tori due to Milnor. Thus the next aim is to extract geometric properties as much as possible from the knowledge of eigenvalues. Kac computed the asymptotic expansion of the trace of the heat kernel of a planar domain and showed that one can know the area of the domain, the length of the boundary and in the case of polygonal region the number of holes in the domain. Kac's paper stimulated two directions of research, isospectral manifolds and spectral invariants.

6.1 *Isospectral manifolds.* These problems attracted so many people that an extensive literature has been devoted to them. Let us cite important contributions of Japanese mathematicians. Ikeda-Yamamoto [30] proved that 3-dimensional isospectral lens spaces are isometric. Ikeda [29] constructed non isometric isospectral lens spaces of dimension ≥ 5. Urakawa [72] constructed non congruent regions in Euclidean space with the same Dirichlet (and Neumann) eigenvalues. Sunada [69] developed a general method of constructing isospectral manifolds. An exposition of Sunada's theory is given by Bérard [4]. For more details on this subject, see a review article of Urakawa [73].

Let us cite an example of two non-congruent polygonal regions having same Dirichlet and Neumann eigenvalues due to Chapman [9].



Ω_1 Ω_2

6.2 *Spectral invariants*. Spectral invariants are the quantities determined only by the spectrum. The trace of the heat kernel is often employed to compute them. Let M be a compact n-dimensional Riemannian manifold without boundary. Let

$$Z(t) = \mathrm{tr}\, e^{t\Delta} = \sum_{j=1}^{\infty} e^{-t\lambda_j},$$

where $\lambda_1 \leq \lambda_2 \leq \cdots$ are eigenvalues of the Laplace-Beltrami operator $-\Delta$ on M. Mckean-Singer [50] proved that as $t \to 0$

$$(4\pi t)^{n/2} Z(t) = \mathrm{Vol}(M) + \frac{t}{3}\int_M K + \frac{t^2}{180}\int_M (10A - B + 2C) + \cdots,$$

where K is the scalar curvature and A, B, C are polynomials of the curvature tensor. If $n = 2$, this fomrula reads

$$Z(t) = \frac{|M|}{4\pi t} + \frac{E}{t} + \frac{\pi t}{60}\int_M K^2 + \cdots,$$

where $|M|$ = the area of M, $E = \frac{1}{2\pi}\int_M K$ = the Euler characteristics of M.

If M has a boundary S, the above formula is modified as follows. Let Δ_D and Δ_N be the Dirichlet and Neumann Laplacians on M. Let

$$Z_+(t) = \mathrm{tr}\, e^{t\Delta_N}, \quad Z_-(t) = \mathrm{tr}\, e^{t\Delta_D}.$$

Then as $t \to 0$

$$(4\pi t)^{n/2} Z_\pm(t) = \mathrm{Vol}(M) \pm \frac{\sqrt{\pi t}}{2}|S| + \frac{t}{3}\int_M K - \frac{t}{6}\int_S J + \cdots,$$

where $|S|$ = the surface area of S and J = the mean curvature.

We also have the complete asymptotic expansion. Let M be a compact manifold and let $H = -\Delta + V$. Then

$$(4\pi t)^{n/2}\mathrm{tr}\, e^{-tH} \simeq \sum_{j=0}^{\infty} t^j a_j(V),$$

where $a_j(V)$'s are integrals of certain functions over M which are universal polynomials in the covariant derivatives of V and of the curvature tensor of M.

By this asymptotic expansion one can prove (see e.g. Brüning [6], Brooks-Perry-Petersen [5]) that the set of Riemaniann metrics on M isospectral to (M, g) is compact with respect to C^∞-topology on Riemaniann metrics. See also a survey of Perry [64].

7. Overdeterminacy

7.1 *High energy Born approximation.* Let us return to the inverse scattering problem in $\mathbf{R}^n, n \geq 2$, for the Schrödinger operator $H = -\Delta + V(x)$. If the potential $V(x)$ satisfies

$$|V(x)| \leq C(1 + |x|)^{-1-\epsilon}, \quad \epsilon > 0, \tag{7.1}$$

the scattering operator S in (3.3) is well-defined and unitary on $L^2(\mathbf{R}^n)$. One can also allow certain local singularities for V, which is omitted for the sake of simplicity. To write down the scattering amplitude, we need the limit of $R(z) = (H - z)^{-1}$ when z approaches to $\lambda \in \sigma_{cont}(H) = [0, \infty)$. For $s \in \mathbf{R}$, let $L^{2,s}$ be the function space defined by

$$u \in L^{2,s} \Longleftrightarrow \|u\|_s^2 = \int_{\mathbf{R}^n} (1 + |x|)^{2s}|u(x)|^2 dx < \infty. \tag{7.2}$$

Then if $s > 1/2$ and $\lambda > 0$, there exists a limit

$$R(\lambda \pm i0) = \lim_{\epsilon \downarrow 0} R(\lambda \pm i\epsilon) \in \mathbf{B}(L^{2,s}; L^{2,-s}). \tag{7.3}$$

Here for Banach spaces X and Y, $\mathbf{B}(X; Y)$ denotes the set of all bounded operators from X to Y.

We basically assume that

$$|V(x)| \leq C(1 + |x|)^{-n-\epsilon}, \quad \epsilon > 0. \tag{7.4}$$

Then the scattering amplitude is a continuous function of $\lambda > 0, \theta, \theta' \in S^{n-1}$ and is written as, up to a constant depending only on $\lambda > 0$

$$\begin{aligned} A(\lambda; \theta, \theta') &= \int_{\mathbf{R}^n} e^{-i\sqrt{\lambda}(\theta - \theta')\cdot x} V(x) dx \\ &- \int_{\mathbf{R}^n} e^{-i\sqrt{\lambda}\theta\cdot x} V(x) R(\lambda + i0)(V(\cdot)e^{i\sqrt{\lambda}\theta'\cdot}) dx. \end{aligned} \tag{7.5}$$

The first rigorous mathematical result for the multi-dimensional inverse scattreing is due to Faddeev [19]. For $0 \neq \xi \in \mathbf{R}^n$, choose $\eta \in S^{n-1}$ such that $\eta \perp \xi$ and put

$$\theta = (1 - \frac{|\xi|^2}{4\lambda})^{1/2}\eta + \frac{\xi}{2\sqrt{\lambda}}, \tag{7.6}$$

$$\theta' = (1 - \frac{|\xi|^2}{4\lambda})^{1/2}\eta - \frac{\xi}{2\sqrt{\lambda}}. \tag{7.7}$$

Theorem 7.1. *Let $n \geq 2$ and θ, θ' be as above. Then we have*

$$\lim_{\lambda\to\infty} A(\lambda;\theta,\theta') = \int_{\mathbf{R}^n} e^{-ix\cdot\xi} V(x)dx.$$

In fact, this theorem folows from (7.5) and the high-energy resolvent estimate

$$\|R(\lambda \pm i0)\|_{\mathbf{B}(L^{2,s};L^{2,-s})} \leq C/\sqrt{\lambda}, \tag{7.8}$$

for $s > 1/2$ and $\lambda > \lambda_0 > 0$.

This is usually called the Born approximation, since it neglects the second-term of the right-hand side of (7.5). This method is a powerful tool to derive the uniqueness of the perturbation term with given scattering amplitude for large $\lambda > 0$. It is extended to general short-range potentials ($V(x) = O(|x|^{-1-\epsilon})$) by Saito [68], long-range potentials by Isozaki-Kitada [38] ($V(x) = O(|x|^{-1/2-\epsilon})$), magnetic Schrödinger operators by Nicoleau [62], Dirac operators by Ito [39] and even for N-body Schrödinger operators by Wang [74].

7.2 *Time-dependent inverse scattering.* Enss [14] invented a beautiful method for proving the asymptotic completeness of wave operators by localizing solutions of Schrödinger equations along the orbit of scattering particles in classical mechanics. He also used this idea to prove the uniqueness of the potential with a given scattering operator S. This method is simple, has a wide range of applicability and can be extended easily to N-body problems as well as time-dependent potentials. See [15], [77], [40].

7.3 *Overdeterminacy.* The above Theorem 7.1 already reveals the characteristic feature of the multi-dimensional inverse scattering problem. Unlike the 1-dimensional case, the potential is determined by the scattering matrix, moreover only its high-energy part is necessary to reconstruct $V(x)$. This apparently convenient fact is at the same time a cause of difficulties in multi-dimensional inverse problem.

In the 1-dimensonal case, both of the potential and the S-matrix are functions of one variable. The Gel'fand-Levitan-Marchenko theory gives a necessary and sufficient condition for a function $S(k)$ to be the scattering matrix of a Schrödinger operator $-d^2/dx^2 + V(x)$.

In the n-dimensional case, the scattering amplitude $A(\lambda;\theta,\theta')$ is a function of $2n - 1$ parameters, while the potential $V(x)$ is a function of n variables. This overdeterminacy requires a sort of compatibility condition for a function $A(\lambda;\theta,\theta')$ of $2n-1$ parameters to be the scattering amplitude of associated with a Schrödinger operator $-\Delta + V(x)$. To find this compatibility condition or at least to seek a necessary condition which weakens this overdeterminacy is the very question of multi-dimensional inverse problems.

7.4 *Inverse back scattering.* The back scattering amplitude $A(\lambda;\theta,-\theta)$ is a function of n parameters. Therefore to reconstruct the potential $V(x)$ from $A(\lambda;\theta,-\theta)$ seems to be a natural attempt. This was studied by Moses [54] and Prosser [65] in a rather formal manner. Eskin and Ralston [17] continued this direction and studied the bijectivity of the map

$$V(x) \to A(\frac{|\xi|}{2};\hat{\xi},-\hat{\xi}), \quad \hat{\xi} = \xi/|\xi|.$$

They show that this map is a local diffeomorphism on a certain open set of the Banach space of real potentials.

Part III Inverse scattering in n-dimensions

8. Key idea of Faddeev

8.1 *Spectral representation.* The remaining part of this paper is devoted to discussing the Faddeev theory of inverse scattering for multi-dimensional Schrödinger operators and its developments. We basically assume (7.4).

Consider the equation

$$(-\Delta + V - \lambda)u = f \in L^{2,s}, \tag{8.1}$$

for $s > 1/2, \lambda > 0$. A solution u of (8.1) is said to satisfy the outgoing radiation condition if for some $0 < \alpha < 1/2 < s$

$$u \in L^{2,-s}, \quad (\frac{\partial}{\partial r} - i\sqrt{\lambda})u \in L^{2,-\alpha}. \tag{8.2}$$

An outgoing spherical wave $r^{-(n-1)/2}e^{i\sqrt{\lambda}r}$ satisfies (8.2). It is known that the solution of (8.1) satisfying (8.2) is unique and is given by $u = R(\lambda + i0)f$. Moreover this u admits an asymptotic expansion

$$u \simeq r^{-(n-1)/2}e^{i\sqrt{\lambda}r}\psi_0(\hat{x}), \quad \hat{x} = x/r \tag{8.3}$$

as $r \to \infty$. Therefore the solution $\varphi(x, E, \omega)$ of (3.1) having the asymptotic expansion (3.2) is given by

$$\varphi(x, E, \omega) = e^{i\sqrt{E}\omega\cdot x} - R(E + i0)(V(\cdot)e^{i\sqrt{E}\omega\cdot}). \tag{8.4}$$

Let us call this $\varphi(x, E, \omega)$ the physical eigenfunction of H. We also put

$$\Phi(x, \xi) = \varphi(x, |\xi|^2, \xi/|\xi|). \tag{8.5}$$

Using this Φ, we define a spectral representation for H by

$$(\mathcal{F}u)(\xi) = (2\pi)^{-n/2}\int_{\mathbf{R}^n} \overline{\Phi(x, \xi)}u(x)dx. \tag{8.6}$$

When $V = 0$, this reduces to the usual Fourier transformation

$$(\mathcal{F}_0 u)(\xi) = (2\pi)^{-n/2}\int_{\mathbf{R}^n} e^{-ix\cdot\xi}u(x)dx. \tag{8.7}$$

The operator $\mathcal{F}$ defined for $u \in C_0^\infty(\mathbf{R}^n)$ is uniquely extended to a bounded operator on $L^2(\mathbf{R}^n)$. Let $\mathcal{H}_{ac}(H)$ be the absolutely continuous subspace for H, and let $\mathcal{H}_{pp}(H)$ the closure of the linear hull of eigenvectors of H. We then have the orthogonal decomposition

$$L^2(\mathbf{R}^n) = \mathcal{H}_{ac}(H) \oplus \mathcal{H}_{pp}(H). \tag{8.8}$$

For $u \in \mathcal{H}_{pp}(H)$, $\mathcal{F}u = 0$. $\mathcal{F}$ is unitary from $\mathcal{H}_{ac}(H)$ onto $L^2(\mathbf{R}^n)$. For $u \in \mathcal{H}_{ac}(H) = E((0, \infty))L^2(\mathbf{R}^n)$, where $E(\lambda)$ denotes the spectral measure for H, the following inversion formula holds

$$u = \mathrm{s} - \lim_{N\to\infty}(2\pi)^{-n/2}\int_{\frac{1}{N}<|\xi|<N} \Phi(x, \xi)\mathcal{F}u(\xi)d\xi. \tag{8.9}$$

8.2 *Higher dimensional Volterra operator.* The Volterra integral operator played an important role in the 1-dimensional problem. Therefore it would be useful to generalize the notion of Volterra operator to higher dimension. Let $\gamma \in S^{n-1}$ be arbitrarily fixed. An integral operator K is said to be triangular with respect to γ if its kernel $K(x,y)$ satisfies

$$K(x,y) = 0 \quad \text{if} \quad x\cdot\gamma < y\cdot\gamma. \tag{8.10}$$

$T = I + K$ is called the Volterra type with respect to γ if K is triangular with respect ro γ.

The following theorem is proved in the same way as in Theorem 3.3.

Theorem 8.1. *Let $L_0 = -\Delta$ on $\mathbf{R}^n$, and let $L = L_0 + Q$ be a self-adjoint perturbation of L_0. Suppose $U = I + K$ intertwines L_0 and L. If K is triangular with respect to $(1,0,\cdots,0)$, Q is the multiplication operator with respect to x_1, and its kernel is give by*

$$Q(x_1,x',y') = 2\frac{d}{dx_1}K(x_1,x',x_1,y').$$

Moreover for $x_1 > y_1$

$$(\Delta_x - \Delta_y)K(x,y) = \int_{\mathbf{R}^{n-1}} Q(x_1,x',z')K(x_1,z',y_1,y')dz'$$

holds.

We have now arrived at a crucial point of Gel'fand-Levitan theory. In the 1-dimensional case, we constructed the generalized sine transformation $\mathcal{F}$ by using the solution of (3.17), (3.18). Then the intertwining operator $\mathcal{F}^*\mathcal{F}_0$ becomes the Volterra type, and the potential $V(x)$ is reconstructed from its kernel.

This is not the case for the multi-dimensional problem. If one constructs the generalized Fourier transformation by using the physical eigenfunction, the intertwining operator $\mathcal{F}^*\mathcal{F}_0$ does not have a direction with respect to which it becomes Volterra. One can easily convince oneself by thinking of the fact that in the 1-dimension one is on the real line, the straight line, which has clearly a distinguished direction, while in the 3-dimensions there is no special direction.

What shall we do? The idea is to abandan the physical eigenfunction and to look for non physical eigenfunction by which the intertwining operator becomes Volterra. Kay and Moses tried to find a good non physical eigenfunction. But they could not succeed in getting it. It is Faddeev who found an essential idea ([21], [22], [23]).

8.3 *Faddeev's Green operator.* For a solution φ of $(-\Delta + V - E)\varphi = 0$, take $\zeta = (\zeta_1,\cdots,\zeta_n) \in \mathbf{C}^n$ such that $\zeta^2 = \sum_{j=1}^n \zeta_j^2 = E$ and let $\varphi = e^{ix\cdot\zeta}(1+v)$. Then v satisfies

$$(-\Delta - 2i\zeta\cdot\nabla + V)v = -V(x).$$

With this in mind let us start with the equation

$$(-\Delta - 2i\zeta\cdot\nabla + V)u = f. \tag{8.11}$$

We first consider the case $V = 0$. Among infinite number of solutions of (8.11), we select the one written by the Fourier transformation

$$u(x) = (2\pi)^{-n/2}\int_{\mathbf{R}^n} \frac{e^{ix\cdot\xi}}{\xi^2 + 2\zeta\cdot\xi}\hat{f}(\xi)d\xi =: \widetilde{G}(\zeta)f. \tag{8.12}$$

This is the Green operator introduced by Faddeev. Here we note that the integrand of (8.12) is absolutely convergent if $f \in \mathcal{S}$ = the Schwartz space of rapidly decreasing functions, and Im $\zeta \neq 0$. In fact the zeros of the denominator form a sphere of codimension 2 and hence the denominator can be rewritten near zeros in the form $\xi_1 + i\xi_2$ if we choose suitable coordinates. This yields the absolute convergence.

For the moment, let us proceed formally. Let $u = (1 + \widetilde{G}(\zeta)V)^{-1}\widetilde{G}(\zeta)f$, which is a solution to (8.11). Any $\zeta \in \mathbf{C}^n$ with Im $\zeta \neq 0$ can be uniquely written as $\zeta = \eta + z\gamma$, where $\eta \in \mathbf{R}^n, \gamma \in S^{n-1}, \eta \cdot \gamma = 0$ and $z \in \mathbf{C}_+ = \{\text{Im } z > 0\}$. This z will play an important role. We let z tend to $t \in \mathbf{R}$. Then $\widetilde{G}(\zeta)$ converges to $\widetilde{G}(k)$, where $k = \eta + t\gamma \in \mathbf{R}^n$ with $k^2 = E$. We have thus obtained the following solution

$$\psi = e^{ik\cdot x} - e^{ik\cdot x}(1 + \widetilde{G}(k)V)^{-1}\widetilde{G}(k)V \tag{8.13}$$

of $(-\Delta + V - E)\psi = 0$. This is a non-physical eigenfunction. This is different from φ which behaves like (3.2) and was utilized to define the S-matrix. An important feature of Faddeev's Green operator is that $\widetilde{G}(\eta + z\gamma)$ is analytic in $z \in \mathbf{C}_+$. Therefore the second term of the right-hand side of (8.13) is the boundary value of a function analytic in the upper-half plane. Therefore by the Paley-Wiener theorem, it has the following expression

$$\psi = e^{ik\cdot x} - \int_{\gamma\cdot x}^{\infty} A_\gamma(x, \eta, s)e^{ist}ds. \tag{8.14}$$

THEOREM 8.2 *(Paley-Wiener). A function $f(x) \in L^2(\mathbf{R})$ is a boundary value of an analytic function $f(x + iy)$ in the upper-half plane such that*

$$\sup_{y>0} \int_{-\infty}^{\infty} |f(x + iy)|^2 dy < \infty,$$

if and only if

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \hat{f}(\lambda)e^{i\lambda x}d\lambda.$$

.

The formula (8.14) is a multi-dimensional counter part of (3.26). With this eigenfunction ψ, one can construct a spectral representation of H and the associated intertwining operator for H_0 and H. Since ψ has the triangular expression, this intertwining operator will be Volterra. One can thus expect an analogy to the 1-dimensional Gel'fand-Levitan theory.

Faddeev proceeds as follows. The basic strategy is to replace the usual Green operator by the direction dependent Green operator as above. Let $R_\gamma(E, t)$ be the direction dependent Green operator for $H = -\Delta + V$. (We shall explain it precisely later.) Using the non-physical eigenfunction

$$\Psi_\gamma(x, E, \theta) = e^{i\sqrt{E}\theta\cdot x} - R_\gamma(E, \sqrt{E}\theta\cdot\gamma)(V(\cdot)e^{i\sqrt{E}\theta\cdot}), \tag{8.15}$$

we define a new scattering amplitude

$$\widetilde{A}_\gamma(E, \theta, \theta') = (2\pi)^{-n}2^{-1}E^{(n-2)/2}\int_{\mathbf{R}^n} e^{-i\sqrt{E}\theta\cdot x}V(x)\Psi_\gamma(x, E, \theta')dx. \tag{8.16}$$

We put

$$Q_\gamma^{(\pm)}(E, \theta, \theta') = 2\pi i F(\pm\gamma\cdot(\theta - \theta') \geq 0)\widetilde{A}_\gamma(E, \theta, \theta'), \tag{8.17}$$

where $F(\cdots)$ denotes the characteristic function of the set $\{\cdots\}$. Let $Q_\gamma^{(\pm)}(E)$ be the integral operator with kernel $Q_\gamma(E, \theta, \theta')$. Then the main results are as follows :

(I) *Factorization of the S-matrix.*

$$\hat{S}(E) = (1 - Q_\gamma^{(-)}(E))(1 + Q_\gamma^{(+)}(E))^{-1}. \tag{3.18}$$

(II) *Volterra operator.* Let $\Phi_\gamma(x,\xi) = \Psi_\gamma(x, |\xi|^2, \xi/|\xi|)$ and put

$$U_\gamma(x,y) = (2\pi)^{-n} \int_{\mathbf{R}^n} \Phi_\gamma(x,\xi) e^{-iy\cdot\xi} d\xi. \tag{8.19}$$

Then we have

$$U_\gamma(x,y) = \delta(x-y) - K_\gamma(x,y), \tag{8.20}$$

$$K_\gamma(x,y) = 0 \quad \text{if} \quad x\cdot\gamma > y\cdot\gamma. \tag{8.21}$$

(III) *Gel'fand-Levitan equation.* For $x\cdot\gamma < y\cdot\gamma$, we have

$$K_\gamma(x,y) + \Omega_\gamma(x,y) + \int_{(x-y)\cdot\gamma<0} K_\gamma(x,z)\Omega_\gamma(z,y)dz = 0, \tag{8.22}$$

where $\Omega_\gamma(x,y)$ is a function constructed from the scattering data.

The scenario of the reconstruction of $V(x)$ will be as follows. From the scattering matrix $\hat{S}(E)$, construct $Q_\gamma^{(\pm)}(E)$, and then $\Omega_\gamma(x,y)$. Solve the Gel'fand-Levitan equation to get $K_\gamma(x,y)$. Since $1 - K_\gamma$ is a Volterra operator intertwining H_0 and H, the potential $V(x)$ will be obtained from $K_\gamma(x,y)$.

Faddeev goes further. He observes that the analyticity of $\widetilde{G}(\eta + z\gamma)$ plays a key role to guarantee that $V(x)$, reconstructed from $\widetilde{A}_\gamma$, is independent of the artificially introduced direction γ. This will be crucial in the characterization of the physical scattering amplitude.

9. Changing Green operators

In the following sections, we examine the Faddeev theory in detail. The first question we address is "What occurs when one changes the Green operator in scattering theory?" The stationary scatteing theory is composed of the usual Green operator $(-\Delta - E - i0)^{-1}$ of the Laplacian. Since there are many Green operators, it would be worthwhile to consider the effect of changing Green operators.

We shall discuss in a general setting. Let H_0 be a self-adjoint operator in a Hilbert space $\mathcal{H}$. Let I be the spctrum of H_0. Suppose there are two Banach spaces $\mathcal{H}_\pm$ such that

$$\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-, \tag{9.1}$$

and the inclusions from $\mathcal{H}_+$ to $\mathcal{H}$, from $\mathcal{H}$ to $\mathcal{H}_-$ are dense and continuous. Moreover we assume that the inner product (,) on $\mathcal{H}$ is uniquely extended to a sesqui-linear form on $\mathcal{H}_- \times \mathcal{H}_+$. We assume that for any $\lambda \in I_{int} = \{$ the interior points of I $\}$, there exists a strong limit

$$R_0(\lambda + i0) = (H_0 - \lambda \mp i0)^{-1} \in \mathbf{B}(\mathcal{H}_+; \mathcal{H}_-) \tag{9.2}$$

and $R_0(\lambda \pm i0)$ is a $\mathbf{B}(\mathcal{H}_+; \mathcal{H}_-)$-valued continuous function of $\lambda \in I_{int}$. Suppose there exists an auxiliary Hilbert space h such that for any $\lambda \in I_{int}$, there exists a bounded operator $\mathcal{F}_0(\lambda) \in \mathbf{B}(\mathcal{H}_+; h)$ satisfying

$$\mathcal{F}_0(\lambda) H_0 u = \lambda \mathcal{F}_0(\lambda) u, \quad \forall u \in D(H_0) \cap \mathcal{H}_+. \tag{9.3}$$

We finally assume that the operator

$$\mathcal{F}_0 u(\lambda) = \mathcal{F}_0(\lambda) u$$

defined on $\mathcal{H}_+$ is uniquely extended to be a unitary operator from $\mathcal{H}$ to $L^2(I,h;d\lambda)$.

Example. Let $H_0 = -\Delta$ in $\mathcal{H} = L^2(\mathbf{R}^n)$, and $\mathcal{H}_\pm = L^{2,\pm s}$ with $s > 1/2$. Let $I = [0,\infty), h = L^2(S^{n-1})$ and

$$\mathcal{F}_0(\lambda)u = (2\pi)^{-n/2}2^{-1/2}\lambda^{(n-2)/4}\int_{\mathbf{R}^n} e^{-i\sqrt{\lambda}\omega\cdot x}u(x)dx. \tag{9.4}$$

Let V be a bounded self-adjoint operator on $\mathcal{H}$ such that $V \in \mathbf{B}(\mathcal{H}_+;\mathcal{H}_+)$, and let $H = H_0 + V$.

Definition 9.1. Let $E \in I_{int}$. An operator $G^{(0)} \in \mathbf{B}(\mathcal{H}_+;\mathcal{H}_-)$ is said to be a Green operator of $H_0 - E$ if

$$(H_0 - E)G^{(0)} = I \quad \text{on} \quad \mathcal{H}_+. \tag{9.5}$$

An operator $G \in \mathbf{B}(\mathcal{H}_+;\mathcal{H}_-)$ is called a perturbed Green operator associated with $G^{(0)}$ if it satisfies

$$G = G^{(0)} - G^{(0)}VG = G^{(0)} - GVG^{(0)}. \tag{9.6}$$

Note that $(H - E)G = I$ on $\mathcal{H}_+$ by virtue of (9.5) and (9.6).

With a perturbed Green operator G, we define the scattering amplitude associated with G by

$$A = \mathcal{F}_0(E)(V - VGV)\mathcal{F}_0(E)^*. \tag{9.7}$$

Now suppose that we are given two Green operators $G_1^{(0)}, G_2^{(0)}$ for $H_0 - E$. Let G_1, G_2 be the associated perturbed Green operators. Let A_1, A_2 be the scattering amplitudes associated with G_1, G_2. What is the relationship between A_1 and A_2?

Theorem 9.2. *Suppose there exists $M \in \mathbf{B}(h;h)$ such that*

$$G_2^{(0)} - G_1^{(0)} = \mathcal{F}_0(E)^*M\mathcal{F}_0(E). \tag{9.8}$$

Then we have

$$A_2 = A_1 - A_1MA_2. \tag{9.9}$$

The above theorem follows from the equation

$$G_2 = G_1 + (1 - G_1V)T(1 - VG_2), \tag{9.10}$$

where $T = G_2^{(0)} - G_1^{(0)}$.

Let us next discuss the solvability of the equation (9.9). We need the following assumptions :

$$G_1^{(0)}V, G_2^{(0)}V : \mathcal{H}_- \to \mathcal{H}_- \text{ are compact,}$$

$$A_1 : h \to h \text{ is compact.}$$

Theorem 9.3. *Under the above assumptions, the equation (9.9) is uniquely solvable with respect to A_2 :*

$$A_2 = (1 + A_1M)^{-1}A_1.$$

The proof again follows from simple algebraic manipulations. Letting $\tilde{K} = (1 - G_1V)TV$, we have

$$1 + G_2^{(0)}V = (1 + G_1^{(0)}V)(1 + \tilde{K}). \tag{9.11}$$

The existence of the perturbed Green operator G_j is equivalent to $-1 \notin \sigma_p(G_j^{(0)}V)$. Here for an operator A in a Hilbert space $\mathcal{H}$, $\sigma_p(A)$ means the set of eigenvalues of A. Putting

$$S_1 = (1 - G_1V)\mathcal{F}_0(E)^*M, \quad S_2 = \mathcal{F}_0(E)V,$$

we have

$$\tilde{K} = S_1S_2, \quad A_1M = S_2S_1.$$

Since $\sigma_p(S_1S_2) \setminus \{0\} = \sigma_p(S_2S_1) \setminus \{0\}$, we have

$$-1 \notin \sigma_p(\tilde{K}) \Longleftrightarrow -1 \notin \sigma_p(A_1M). \tag{9.12}$$

Theorem 9.3 follows from (9.11) and (9.12).

Theorem 9.3 means that for a certain pair of Green operators there is a linear equation between the corresponding scattering amplitudes, which is solvable.

Now the question is : What kind of property of Green operator is useful in inverse scattering? According to Faddeev, it is *analyticity.* We elucidate it in the next sections.

10. Direction dependent Green operators

We summarize various properties of Faddeev's Green operator in this section. It is a little simpler to consider the following operator

$$G_{\gamma,0}(\lambda, z)f(x) = (2\pi)^{-n/2}\int_{\mathbf{R}^n} \frac{e^{ix\cdot\xi}}{\xi^2 + 2z\gamma\cdot\xi - \lambda^2}\hat{f}(\xi)d\xi, \tag{10.1}$$

where $\gamma \in S^{n-1}, \lambda > 0, z \in \mathbf{C}_+$. If $\zeta = \eta + z\gamma, \eta \in \mathbf{R}^n, \eta\cdot\gamma = 0$, we have

$$e^{-ix\cdot\eta}G_{\gamma,0}(|\eta|^2, z)e^{ix\cdot\eta} = \widetilde{G}(\zeta). \tag{10.2}$$

Theorem 10.1. *Let* $s > 1/2$.
(1) As a $\mathbf{B}(L^{2,s}; L^{2,-s})$*-valued function,* $G_{\gamma,0}(\lambda, z)$ *is continuous with respect to* $\lambda \geq 0, \gamma \in S^{n-1}, z \in \overline{\mathbf{C}_+}$ *except for* $(\lambda, z) = (0, 0)$.
(2) $G_{\gamma,0}(\lambda, z)$ *is a* $\mathbf{B}(L^{2,s}; L^{2,-s})$*-valued analytic function of* $z \in \mathbf{C}_+$.
(3) For any $\epsilon_0 > 0$ *there exists* $C > 0$ *such that for* $\lambda + |z| \geq \epsilon_0$,

$$\|G_{\gamma,0}(\lambda, z)\|_{\mathbf{B}(L^{2,s};L^{2,-s})} \leq C(\lambda + |z|)^{-1}.$$

(4) For $t \in \mathbf{R}$, *let* $\widetilde{R}_{\gamma,0}(\lambda, t) = e^{it\gamma\cdot x}G_{\gamma,0}(\lambda, t)e^{-it\gamma\cdot x}$. *Then*

$$(-\Delta - \lambda^2 - t^2)\widetilde{R}_{\gamma,0}(\lambda, t) = 1.$$

For the proof, see [75].

The most important features of the Green operator of Faddeev are the analyticity in $z \in \mathbf{C}_+$ and the following formula

$$\widetilde{R}_{\gamma,0}(\lambda, t) = R_0(E - i0)M_\gamma^{(+)}(t) + R_0(E + i0)M_\gamma^{(-)}(t), \tag{10.3}$$

where $E = \lambda^2 + t^2$, $R_0(E \pm i0) = (-\Delta - E \mp i0)^{-1}$, and

$$M_\gamma^{(\pm)}(t) = (\mathcal{F}_{x\to\xi})^{-1}F(\pm\gamma\cdot(\xi - t\gamma) \geq 0)\mathcal{F}_{x\to\xi}. \tag{10.4}$$

The equation (10.4) is intuitively obvious, since

$$\widetilde{R}_{\gamma,0}(\lambda,t)f=(2\pi)^{-n/2}\int_{\mathbf{R}^n}\frac{e^{ix\cdot\xi}}{\xi^2+2i0\gamma\cdot(\xi-t\gamma)-E}\hat{f}(\xi)d\xi,$$

$$R_0(E\pm i0)f=(2\pi)^{-n/2}\int_{\mathbf{R}^n}\frac{e^{ix\cdot\xi}}{\xi^2-(E\pm i0)}\hat{f}(\xi)d\xi,$$

Namely, $\widetilde{R}_{\gamma,0}(\lambda,t)$ is outgoing in the half-space $\gamma\cdot\xi<t$ and incoming in the half-space $\gamma\cdot\xi>t$.

The analyticity of $G_{\gamma,0}(\lambda,z)$ in $z\in\mathbf{C}_+$ follows from the following lemma (see Lemma 3.2 of [36]).

Lemma 10.2. *Let D be an open set in $\mathbf{C}$. Let $p(\xi,z)$ be a $\mathbf{C}$-valued function which is smooth in $\xi\in\mathbf{R}^n$ and analytic in $z\in D$. Let $M_z=\{\xi\in\mathbf{R}^n;p(\xi,z)=0\}$ and assume that for $z\in D$, $\nabla_\xi\mathrm{Re}\,p(\xi,z)$ and $\nabla_\xi\mathrm{Im}\,p(\xi,z)$ are linearly independent on M_z. Then the distribution $S(z)$ defined by*

$$S(z)f=\int_{\mathbf{R}^n}\frac{f(\xi)}{p(\xi,z)}d\xi,\quad f(\xi)\in C_0^\infty(\mathbf{R}^n)$$

satisfies

$$\overline{\partial_z}S(z)f=\pi\int_{\mathbf{R}^n}f(\xi)\overline{\partial_z p(\xi,z)}\delta(p(\xi,z))d\xi,$$

where

$$\int_{\mathbf{R}^n}g(\xi)\delta(p(\xi,z))d\xi=\int_{M_z}g(\xi)dM_z,$$

and dM_z is the induced measure on M_z.

Let $E=\lambda^2+t^2$. Then the integral kernel of $G_{\gamma,0}(\lambda,t)$ is formally written as

$$(2\pi)^{-n}\int\frac{e^{i(x-y)\cdot\xi}}{\xi^2+2(t+i0)\gamma\cdot\xi+t^2-E}d\xi.$$

To find an analytic continuation with respect to t of this kernel is an important problem. Apparently it should be

$$(2\pi)^{-n}\int\frac{e^{i(x-y)\cdot\xi}}{\xi^2+2z\gamma\cdot\xi+z^2-E}d\xi.$$

However this is not analytic! Eskin-Ralston [18] found the analytic continuation of the above operator. They formulated it by passing to the Fourier transformation. Let us show its expression in x-space.

The Green operator of Eskin-Ralston consists of two parts :

$$U_{\gamma,0}(E,z)=V_{\gamma,0}(E,z)+W_{\gamma,0}(E,z).\tag{10.5}$$

Let us first explain $V_{\gamma,0}(E,z)$. Let

$$D_\epsilon=\{z\in\mathbf{C}_+;|\mathrm{Re}z|<\epsilon/2\}.$$

Let $\varphi_1(t)\in C^\infty(\mathbf{R})$ be such that $\varphi_1(t)=1$ if $|t|>2\epsilon$, $\varphi_1(t)=0$ if $|t|<\epsilon$. For $z\in D_\epsilon$, $V_{\gamma,0}(E,z)$ is defined by

$$V_{\gamma,0}(E,z)f=(2\pi)^{-n/2}\int_{\mathbf{R}^n}\frac{e^{i(x-y)\cdot\xi}\varphi_1(\gamma\cdot\xi)}{\xi^2+2z\gamma\cdot\xi+z^2-E}\hat{f}(\xi)d\xi.\tag{10.6}$$

This is $\mathbf{B}(L^2;L^2)$-valued analytic in $z\in D_\epsilon$.

To explain $W_{\gamma,0}(E,z)$, let $\gamma = (1, 0, \cdots, 0)$. We write $x \in \mathbf{R}^n$ as $x = (x_1, x')$, $x' \in \mathbf{R}^{n-1}$, and let $\Delta' = \sum_{j=2}^n (\partial/\partial x_j)^2$. For $a \in \mathbf{R}$, let

$$\mathcal{H}'_a = \{f; \int_{\mathbf{R}^{n-1}} e^{2a|x'|}|f(x')|^2 dx' < \infty\},$$

$$\mathcal{H}_a = \{f; \int_{\mathbf{R}^n} e^{2a|x|}|f(x)|^2 dx < \infty\}.$$

It is well-known that for any $\delta > 0$, $(-\Delta' - z)^{-1}$ defined on $\mathbf{C}_\pm$ has an analytic continuation across the positive real axis $(0, \infty)$ into the region $\{z; \pm \mathrm{Im}\sqrt{z} > -\delta\}$ as a $\mathbf{B}(\mathcal{H}'_\delta; \mathcal{H}'_{-\delta})$-valued function. Denoting this operator by $r_\pm(z)$, we define

$$\begin{aligned} W_{\gamma,0}(E,z) = (\mathcal{F}_{x_1\to\xi_1})^{-1}\{&r_+(E - (\xi_1 + z)^2)F(\xi_1 < 0) \\ &+ r_-(E - (\xi_1 + z)^2)F(\xi_1 > 0)\}\varphi_0(\xi_1)\mathcal{F}_{x_1\to\xi_1}, \end{aligned} \tag{10.7}$$

where $\varphi_0(t) = 1 - \varphi_1(t)$.

Theorem 10.3. *Let $E > 0$.*
(1) For any $\delta > 0$, there exists $\epsilon > 0$ such that $U_{\gamma,0}(E,z)$ is $\mathbf{B}(\mathcal{H}_\delta; \mathcal{H}_{-\delta})$-valued analytic on D_ϵ.
(2) $U_{\gamma,0}(E,z)$ has a continuous boundary value for $z \in \overline{D_\epsilon} \cap \mathbf{R}$, and for $t \in (-\epsilon/2, \epsilon/2)$,

$$U_{\gamma,0}(E,t) = G_{\gamma,0}(\sqrt{E - t^2}, t).$$

(3) For $\tau > 0$

$$U_{\gamma,0}(E,i\tau) = G_{\gamma,0}(\sqrt{E + \tau^2}, i\tau).$$

(4) For $0 < s < 1$

$$\|U_{\gamma,0}(E,i\tau)\|_{\mathbf{B}(L^{2,s};L^{2,s-1})} \le C/\tau, \quad \tau > 1.$$

(5) Let $R_{\gamma,0}(E,t) = e^{it\gamma\cdot x}U_{\gamma,0}(E,t)e^{-it\gamma\cdot x}$. Then

$$(-\Delta - E)R_{\gamma,0}(E,t) = 1.$$

For the proof see Isozaki [34]. As is seen from (2) and (3) an important role of the Green operator of Eskin-Ralston is to connect the Green operator of Faddeev on the real axis $G_{\gamma,0}(\sqrt{E - t^2}, t)$ to that on the imaginary axis $G_{\gamma,0}(\sqrt{E + \tau^2}, i\tau)$ via the analytic continuation.

11. Inverse scattering at a fixed energy

We shall discuss in this section the reconstruction of the potential $V(x)$ from the scattering amplitude at a fixed energy $E > 0$. This is a physically reasonable problem, and demonstrates the utility of direction dependent Green operators.

11.1 *Perturbed Green operators.* We first construct the direction dependent Green operator for $H = -\Delta + V$. Assume that

$$|V(x)| \le Ce^{-\delta_0|x|}, \tag{11.1}$$

for some $\delta_0, C > 0$. Then for $\delta < \delta_0/2$, $U_{\gamma,0}(E,z)V$ is compact on $\mathcal{H}_{-\delta}$. We define the set of *exceptional points*, $\mathcal{E}_\gamma(E)$, to be the set of $z \in \overline{D_\epsilon}$ such that $-1 \in \sigma_p(U_{\gamma,0}(E,z)V)$. It is easy to see that $\mathcal{E}_\gamma(E)$ is independent of $\delta < \delta_0/2$.

Lemma 11.1. *$\mathcal{E}_\gamma(E) \cap \mathbf{C}_+$ is discrete and there exists $C > 0$ such that $\{i\tau \in \mathcal{E}_\gamma(E); \tau > C\} = \emptyset$. Moreover $\mathcal{E}_\gamma(E) \cap \mathbf{R}$ is a closed set of measure zero.*

Since $R_{\gamma,0}(E,t) = e^{it\gamma\cdot x}U_{\gamma,0}(E,t)e^{-it\gamma\cdot x}$, $t \in \mathcal{E}_\gamma(E)$ is eqivalent to $-1 \in \sigma_p(R_{\gamma,0}(E,t)V)$. We define for $E > 0$ and $t \in (-\epsilon/2, \epsilon/2) \setminus \mathcal{E}_\gamma(E)$,

$$R_\gamma(E,t) = (1 + R_{\gamma,0}(E,t)V)^{-1}R_{\gamma,0}(E,t). \tag{11.2}$$

11.2 *Faddeev scattering ampliude.* The scattering matrix has the following expression

$$\hat{S}(E) = 1 - 2\pi i A(E),$$
$$A(E) = \mathcal{F}_0(E)(V - VR(E+i0)V)\mathcal{F}_0(E)^*, \tag{11.3}$$

where $\mathcal{F}_0(E)$ is defined by (9.4), and $R(E+i0) = (H - E - i0)^{-1}$. Replacing $R(E+i0)$ by $R_\gamma(E,t)$, we define

$$A_\gamma(E,t) = \mathcal{F}_0(E)(V - VR_\gamma(E,t)V)\mathcal{F}_0(E)^*, \tag{11.3}$$

It follows from (10.3) and Theorem 10.3 (2), $R_{\gamma,0}(E,t)$ satisfies

$$R_{\gamma,0}(E,t) = R_0(E+i0) - T_\gamma, \tag{11.4}$$

where

$$T_\gamma = 2\pi i\mathcal{F}_0(E)^*F_\gamma(t)\mathcal{F}_0(E), \quad F_\gamma(t) = F(\gamma\cdot\theta > \frac{t}{\sqrt{E}}). \tag{11.5}$$

Therefore by Theorem 9.2, we have the following equation

$$A_\gamma(E,t) = A(E) + 2\pi i A(E)F_\gamma(t)A_\gamma(E,t). \tag{11.6}$$

For $t \in (-\epsilon/2, \epsilon/2)\setminus\mathcal{E}_\gamma(E)$, the perturbed Green operator $R_\gamma(E,t)$ exists. Therefore by virtue of Theorem 9.3, the equation (11.6) is solvable with respect to $A_\gamma(E,t)$:

$$A_\gamma(E,t) = (1 - 2\pi i A(E)F_\gamma(t))^{-1}A(E), \quad t \in (-\epsilon/2, \epsilon/2) \setminus \mathcal{E}_\gamma(E). \tag{11.7}$$

Thus one can construct the Faddeev scattering amplitude from the physical scattering amplitude.

11.3 *Inverse scattering at a fixed energy.* One can now reconstruct the potential from the scattering amplitude.

Theorem 11.2. *Let $n \geq 3$. Suppose $V(x)$ satisfy (11.1). Then one can reconstruct $V(x)$ uniquely from the scattering matrix of an arbitrarily fixed energy $E > 0$.*

Proof. From the physical scattering amplitude $A(E)$, construct the Faddeev scattering amplitude $A_\gamma(E,t)$. Up to a constant, it has the following integral kernel

$$\int e^{-i\sqrt{E}(\theta-\theta')\cdot x}V(x)dx - \int e^{-i\sqrt{E}\theta\cdot x}V(x)R_\gamma(E,t)(V(\cdot)e^{i\sqrt{E}\theta'\cdot})dx.$$

We take $\sqrt{E}\theta = \sqrt{E-t^2}\omega + t\gamma$, $\sqrt{E}\theta' = \sqrt{E-t^2}\omega' + t\gamma$, where $\omega, \omega' \in S^{n-1}$ and $\omega\cdot\gamma = \omega'\cdot\gamma = 0$. Then the above kernel is written as

$$\begin{aligned} B_\gamma(\omega,\omega',t) &= \int e^{-i\sqrt{E-t^2}(\omega-\omega')\cdot x}V(x)dx \\ &- \int e^{-i\sqrt{E-t^2}\omega\cdot x}V(x)U_\gamma(E,t)(V(\cdot)e^{i\sqrt{E-t^2}\omega'\cdot})dx, \end{aligned}$$

where

$$U_\gamma(E,t) = e^{-it\gamma\cdot x}R_\gamma(E,t)e^{it\gamma\cdot x} = (1 + U_{\gamma,0}(E,t)V)^{-1}U_{\gamma,0}(E,t).$$

$U_\gamma(E,t)$ has a unique meromorphic extension to D_ϵ and for large $\tau > 0$

$$\|U_\gamma(E,i\tau)\|_{\mathbf{B}(L^{2,s};L^{2,s-1})} \leq C/\tau, \quad 0 < s < 1.$$

Therefore as $\tau \to \infty$

$$B_\gamma(\omega, \omega', i\tau) \simeq \int e^{-i\sqrt{E+\tau^2}(\omega-\omega')\cdot x} V(x) dx.$$

We now use the assumption $n \geq 3$. Take any $\xi \in \mathbf{R}^n$. Take $\gamma, \eta \in S^{n-1}$ such that $\xi \cdot \gamma = \xi \cdot \eta = \gamma \cdot \eta = 0$. We put

$$\omega = \omega(\tau) = (1 - \frac{\xi^2}{4\tau^2})^{1/2}\eta + \frac{\xi}{2\tau},$$

$$\omega' = \omega'(\tau) = (1 - \frac{\xi^2}{4\tau^2})^{1/2}\eta - \frac{\xi}{2\tau}.$$

Then $\sqrt{E+\tau^2}(\omega - \omega') \to \xi$. Therefore

$$B_\gamma(\omega(\tau), \omega'(\tau), i\tau) \to \hat{V}(\xi).$$

11.4 *Slowly decreasing potentials.* When the potential is not exponentially decreasing, it cannot be determined from the scattering amplitude of a fixed energy (see e.g. Regge [67]). However one can construct $\hat{V}(\xi)$ for $|\xi| < 2\sqrt{b}$ from the scattering amplitude for energy interval (a, b). More precisely, we have the following theorem.

THEOREM 11.3 *Let $n \geq 2$. Suppose $V(x)$ satisfies*

$$|\partial^\alpha V(x)| \leq C(1+|x|)^{-3/2-\epsilon-|\alpha|}, \quad |\alpha| \leq n-1$$

for $\epsilon > 0$. Let S be a set of positive measure on $\mathbf{R}$ such that $b = ess.sup\ S > 0$. Suppose we are given the scattering amplitude $A(E)$ for all $E \in S$. Then we can reconstruct $\hat{V}(\xi)$ for all $|\xi| < 2\sqrt{b}$. If S is a half-line : $S = [E_0, \infty)$, for any $\omega, \omega' \in S^{n-1}$ such that $\omega \cdot \gamma = \omega' \cdot \gamma = 0$ and $\omega \neq \omega'$, and for a sufficiently large $\lambda > 0$, one can construct a function $C_\gamma(\lambda, t, \omega, \omega')$ such that

$$\hat{V}(\sqrt{\lambda}(\omega - \omega')) = C_\gamma(\lambda, t_0; \omega, \omega') + \text{p.v.}\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{C_\gamma(\lambda, t; \omega, \omega')}{t - t_0} dt$$

holds for all $t_0 \in \mathbf{R}$.

This theorem is proved by using Faddeev's Green operator instead of Eskin-Ralston's Green operator. For the proof see [44] or [33].

12. Faddeev theory

12.1*Exceptional points.* Let us look closer at the Faddeev theory. By Theorem 10.3 (2), $R_{\gamma,0}(E,t) = \widetilde{R}_{\gamma,0}(\lambda, t)$ if $E = \lambda^2 + t^2$ and $|t| < \epsilon/2$. So, we re-define $R_{\gamma,0}(E,t)$ for $|t| \geq \epsilon/2$ by this formula. The perturbed Green operator is defined by

$$R_\gamma(E,t) = (1 + R_{\gamma,0}(E,t)V)^{-1} R_{\gamma,0}(E,t). \tag{12.1}$$

If we assume that $|V(x)| \leq C(1+|x|)^{-1-\epsilon_0}, \epsilon_0 > 0$, $R_{\gamma,0}(E,t)V = \widetilde{R}_{\gamma,0}(\lambda,t)V$ $= e^{it\gamma\cdot x} G_{\gamma,0}(\lambda, t) V e^{-it\gamma\cdot x}$ is compact on $L^{2,-s}$ for $1/2 < s < (1+\epsilon_0)/2$. For $\lambda \geq 0$, the set of exceptional points $\widetilde{\mathcal{E}}_\gamma(\lambda)$ is defined to be the set of $z \in \overline{\mathbf{C}}_+$ such that $-1 \in \sigma_p(G_{\gamma,0}(\lambda, z)V)$.

Lemma 12.1. *For $\lambda \geq 0$, $\widetilde{\mathcal{E}}_\gamma(\lambda) \cap \mathbf{C}_+$ is a discrete set, and $\widetilde{\mathcal{E}}_\gamma(\lambda) \cap \{|z| > C_0\} = \emptyset$ for large $C_0 > 0$. Moreover $\widetilde{\mathcal{E}}_\gamma(\lambda) \cap \mathbf{R}$ is a closed set of measure zero.*

The existence of real exceptional points is a first barrier of the Faddeev theory. For small potentials, they do not exist. However Lavine-Nachman [46] and Khenkin-Novikov [44] proved that real exceptional points do exist if $-\Delta+V$ has bound states.

Theorem 12.2. *If $\sigma_p(-\Delta+V)\neq\emptyset$, for any $\gamma\in S^{n-1}$, there exist $\lambda\geq 0$ and $t\in\mathbf{R}$ such that $-1\in\sigma_p(G_{\gamma,0}(\lambda,t)V)$.*

Let us continue our arguments under the assumption that

$$-1\notin\sigma_p(R_{\gamma,0}(E,t)V),\quad \forall E>0,\quad -\sqrt{E}\leq\forall t\leq\sqrt{E}. \tag{12.2}$$

This assumption is satisfied when V is sufficiently small. We let

$$\Psi_\gamma(x,E,\theta)=e^{i\sqrt{E}\theta\cdot x}-R_\gamma(E,\sqrt{E}\theta\cdot\gamma)(V(\cdot)e^{i\sqrt{E}\cdot}), \tag{12.3}$$

$$\widetilde{A}_\gamma(E,\theta,\theta')=(2\pi)^{-n}2^{-1}E^{(n-2)/2}\int_{\mathbf{R}^n}e^{-i\sqrt{E}\theta\cdot x}V(x)\Psi_\gamma(x,E,\theta')dx. \tag{12.4}$$

$$Q_\gamma^{(\pm)}(E,\theta,\theta')=2\pi iF(\pm\gamma\cdot(\theta-\theta')\geq 0)\widetilde{A}_\gamma(E,\theta,\theta'). \tag{12.5}$$

Let $Q_\gamma^{(\pm)}(E)$ be the integral operator with kernel $Q_\gamma^{(\pm)}(E,\theta,\theta')$.

12.2 *Factorization of S-matrix*. We show that

$$\hat{S}(E)=(1-Q_\gamma^{(-)}(E))(1+Q_\gamma^{(+)}(E))^{-1}. \tag{12.6}$$

This is proven mainly by algebraic manipulations. In fact, letting

$$L_\gamma^{(\pm)}(t)=2\pi iF(\pm\gamma\cdot(\theta-\frac{t\gamma}{\sqrt{E}})\geq 0)A_\gamma(E,t),$$

we have

$$\hat{S}(E)(1+L_\gamma^{(+)}(t))=1-L_\gamma^{(-)}(t). \tag{12.7}$$

The integral kernel of (12.7) reads

$$\int_{S^{n-1}}\hat{S}(E,\theta,\theta')(\delta(\theta''-\theta')+L_\gamma^{(+)}(t,\theta'',\theta'))d\theta''=\delta(\theta-\theta')-L_\gamma^{(-)}(t,\theta,\theta').$$

Letting $t=\sqrt{E}\theta'\cdot\gamma$ in the above formula, we get

$$\hat{S}(E)(1+Q_\gamma^{(+)}(E))=1-Q_\gamma^{(-)}(E). \tag{12.8}$$

On the other hand, we can show that $(Q_\gamma^{(\pm)}(E))^2=0$. Therefore $1+Q_\gamma^{(+)}(E)$ is invertible, which proves (12.6).

12.3 *Volterra operator*. We shall prove (8.20), (8.21). The proof requires involved computations using resolvent estimates. As far as the author knows, the rigorous proof has not been presented yet, and we also have to omit the detailed proof here due to the lack of space.

Let

$$\Phi_\gamma(x,\xi)=\Psi_\gamma(x,|\xi|^2,\xi/|\xi|). \tag{12.9}$$

Then we have

$$\Phi_\gamma(x,\xi)=e^{ix\cdot\xi}-\widetilde{K}_\gamma(x,\xi), \tag{12.10}$$

$$\widetilde{K}_\gamma(x,\xi)=\widetilde{R}_\gamma(|\xi'|^2,\xi\cdot\gamma)(V(\cdot)e^{i\xi\cdot}),\quad \xi'=\xi-(\xi\cdot\gamma)\gamma. \tag{12.11}$$

Here $\widetilde{R}_\gamma(\lambda,t)$ is defined by

$$\widetilde{R}(\lambda,t)=(1+\widetilde{R}_{\gamma,0}(\lambda,t)V)^{-1}\widetilde{R}_{\gamma,0}(\lambda,t).$$

Let us put $\gamma = (1, 0, \cdots, 0)$ for the sake of simplicity. We assume that $n \geq 3$ and

$$|V(x)| \leq C(1 + |x|)^{-\rho}, \quad \rho > \max\{2, (n+1)/2\}. \tag{12.12}$$

Then one can show that if $|\xi'| \geq \epsilon_0 > 0$

$$|\widetilde{K}_\gamma(x, \xi_1, \xi')| \leq C(1 + |\xi|)^{-1}, \tag{12.13}$$

$$\Big|\frac{\partial}{\partial \xi_1} \widetilde{K}_\gamma(x, \xi_1, \xi')\Big| \leq C|\xi'|(1 + |\xi|)^{-1}. \tag{12.14}$$

We let

$$K_\gamma(x, y_1, \xi') = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-iy_1\xi_1} \widetilde{K}_\gamma(x, \xi_1, \xi') d\xi_1, \tag{12.15}$$

where the integral is taken in the limit in the mean. One can then show that

$$(1 + |y_1|) K_\gamma(x, y_1, \xi') \in L^2(\mathbf{R}_{y_1}).$$

This implies that

$$K_\gamma(x, y_1, \xi') \in L^1(\mathbf{R}_{y_1}). \tag{12.16}$$

Noting that

$$K_\gamma(x, y_1, \xi') = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-i(y_1 - x_1)\xi_1} G_\gamma(|\xi'|^2, \xi_1)(V(x)e^{ix'\cdot\xi'}) d\xi_1,$$

and that $G_\gamma(\lambda, z)$ is analytic in $z \in \mathbf{C}_+$, we have by the Paley-Wiener theorem, $K_\gamma(x, y_1, \xi') = 0$ if $y_1 - x_1 < 0$. Therefore

$$\Psi_\gamma(x, \xi) = e^{ix\cdot\xi} - (2\pi)^{-1/2} \int_{x_1}^{\infty} e^{iy_1\xi_1} K_\gamma(x, y_1, \xi') dy_1. \tag{12.17}$$

This proves (8.20).

12.4 *Gel'fand-Levitan equation.* We have no longer sufficient estimates to guarantee the validity of the remaining arguments. Therefore the following arguments are formal, although very interesting.

Let us introduce three types of integral operaors :

$$T_0 f(\xi) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{-ix\cdot\xi} f(x) dx, \tag{12.18}$$

$$T f(\xi) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} \overline{\Phi(x, \xi)} f(x) dx, \tag{12.19}$$

$$T_\gamma f(\xi) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} \overline{\Phi_\gamma(x, \xi)} f(x) dx, \tag{12.20}$$

where $\Phi(x, \xi) = e^{ix\cdot\xi} - R(|\xi|^2 + i0)(V(\cdot)e^{i\cdot\xi})$. We define

$$Q_\gamma^{(+)} f(\xi) = \left(Q_\gamma^{(+)}(|\xi|^2) f(|\xi|\cdot)\right)(\xi/|\xi|). \tag{12.21}$$

Then we can show

$$T^* = T_\gamma^*(1 - Q_\gamma^{(+)}). \tag{12.22}$$

Now let us derive the Gel'fand-Levitan equation (8.22). For the sake of simplicity we assume that $\sigma_p(-\Delta + V) = \emptyset$. In view of (8.20), we let

$$U_\gamma = T_\gamma^* T_0 = 1 + K_\gamma.$$

From $T^*T = 1$ and (12.22), we have

$$T_\gamma^*(1 + \widetilde{Q}_\gamma) T_\gamma = 1, \tag{12.23}$$

where
$$\widetilde{Q}_\gamma = Q_\gamma^{(+)}{Q_\gamma^{(+)}}^* - Q_\gamma^{(+)} - {Q_\gamma^{(+)}}^*.$$
Replacing T_γ in (12.23) by $T_0 U_\gamma^* = T_0(1 + K_\gamma^*)$, we have
$$1 + K_\gamma + \Omega_\gamma + K_\gamma \Omega_\gamma = (1 + K_\gamma^*)^{-1},$$
where
$$\Omega_\gamma = T_0^* \widetilde{Q}_\gamma T_0.$$
Now $K_\gamma^*(x, y)$ is supported in $\{(x - y) \cdot \gamma \geq 0\}$. Therefore, letting $(1 + K_\gamma^*)^{-1} = 1 + C_\gamma$, we see that $C_\gamma(x, y)$ is also supported in $\{(x - y) \cdot \gamma \geq 0\}$. Then for $(x - y) \cdot \gamma < 0$ we have
$$K_\gamma(x, y) + \Omega_\gamma(x, y) + \int_{(x-y)\cdot\gamma<0} K_\gamma(x, z)\Omega_\gamma(z, y)dz = 0.$$
This is the Gel'fand-Levitan equation.

Little is known about the solvability of this equation. When the scattering amplitude is small, $\Omega_\gamma(x, y)$ is also small. Therefore the above equation might be solved with respect to K_γ. By differentiating $K_\gamma(x, y)$, one might get $V(x)$. However, the estimates of this $V(x)$ (e.g. the spatial decay) would be hard to prove.

Putting these difficulties aside, Faddeev further stepped into the characterization problem of the scattering amplitude. According the above procedure, starting from the scattering amplitude, one can get $K_\gamma(x, y)$. By differentiating this $K_\gamma(x, y)$, one can get an integral kernel depending on γ containing the δ-function (see Theorem 8.1). What guarantees that it is the operator of multiplication independent of γ? Faddeev observed that it is the analyticity of $B_\gamma(\omega, \omega', t)$, which appeared in the proof of Theorem 11.2, with respect to t.

Newton [61] analyzed Faddeev's arguments and encountered the above difficulty. He tried to find another route to obtain $V(x)$ based on the dispersion relation, and pointed out that in this case a sort of miraculous condition must be imposed ([61] p.150).

13. $\overline{\partial}$-approach

In 1980's $\overline{\partial}$-approach was introduced as a new view point of the inverse scattering by Beals-Coifman [1] and Nachman-Ablowitz [57]. In this approach Faddeev's method is rewritten as follows. Let $\widetilde{G}(\zeta)$ be as in (8.12).

Lemma 13.1. *Let $\overline{\partial_j} = \partial/\partial\overline{\zeta_j}$. Then for $Im\, \zeta_j \neq 0$*
$$\overline{\partial_j}\widetilde{G}(\zeta)f = (2\pi)^{1-n/2} \int_{\mathbf{R}^n} e^{ix\cdot\xi} \hat{f}(\xi)\xi_j \delta(\xi^2 + 2\zeta \cdot \xi)d\xi. \tag{13.1}$$
and $\overline{\partial_j}\widetilde{G}(\zeta) \in \mathbf{B}(L^{2,s}; L^{2,-s}), s > 1$.

Let us rewrite the Faddeev scattering amplitude. We define for $\zeta = \eta + i\tau\gamma$
$$\widetilde{G}_V(\zeta) = (1 + \widetilde{G}(\zeta)V)^{-1}\widetilde{G}(\zeta). \tag{13.2}$$
If $i\tau \notin \mathcal{E}_\gamma(E)$ it satisfies
$$\widetilde{G}_V(\eta + i\tau\gamma) = e^{-ix\cdot\eta}U_\gamma(E, i\tau)e^{ix\cdot\eta}. \tag{13.3}$$

For $\omega, \omega' \in S^{n-1}$ satisfying $\omega\cdot\gamma = \omega'\cdot\gamma = 0$, we let $\eta = \sqrt{E+\tau^2}\omega, \eta' = \sqrt{E+\tau^2}\omega'$. Then $B_\gamma(\omega,\omega',i\tau)$ has the follwing expression

$$B_\gamma(\omega,\omega',i\tau) = \int e^{-i(\eta-\eta')\cdot x}V(x)dx - \int e^{-i(\eta-\eta')\cdot x}V(x)\widetilde{G}_V(\zeta)Vdx,$$

where $\zeta = \eta' + i\tau\gamma$. Let us note that $\xi = \eta - \eta'$ satisfies $\xi^2 + 2\zeta\cdot\xi = 0$. Starting from the scattering amplitude $A(E)$, we have thus constructed

$$T(\xi,\zeta) = \int e^{-ix\cdot\xi}V(x)dx - \int e^{-ix\cdot\xi}V(x)\widetilde{G}_V(\zeta)Vdx, \tag{13.4}$$

on the set $\{(\xi,\zeta)\}$ where $\xi\in\mathbf{R}^n$ and $\zeta\in\mathbf{C}^n$ satisfy $\zeta^2 = E, |\zeta| > C, \mathrm{Im}\,\zeta\neq 0$ and $\xi^2+2\zeta\cdot\xi = 0$, C being a large constant. This set has a structure of fibred space and each fibre

$$\mathcal{V}_\xi = \{\zeta\in\mathbf{C}^n; \zeta^2 = E, |\zeta| > C, \mathrm{Im}\,\zeta\neq 0, \xi^2+2\zeta\cdot\xi = 0\}. \tag{13.5}$$

is a complex manifold of dimension $2n-4$. On this complex manifold, the function $T(\xi,\zeta)$ satisfies a $\overline{\partial}$-equation.

Theorem 13.2. *As a 1-form on $\mathcal{V}_\xi$, we have*

$$\overline{\partial}T(\xi,\zeta) = \sum_{j=1}^n A_j(\xi,\zeta)d\overline{\zeta_j},$$

$$A_j(\xi,\zeta) = -(2\pi)^{1-n/2}\int_{\mathbf{R}^n} T(\xi-\eta,\zeta+\eta)T(\eta,\zeta)\eta_j\delta(\eta^2+2\zeta\cdot\eta)d\eta.$$

There are two important applications of Theorem 13.2. Recall the generalized Cauchy formula

$$f(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{\zeta - z}d\zeta - \frac{1}{2\pi i}\int_D \frac{\overline{\partial}f(\zeta)}{\zeta - z}d\overline{\zeta}\wedge d\zeta,$$

where D is a domain in $\mathbf{C}$. Using a similar formula on the complex manifold $\mathcal{V}_\xi$, Nachman [55] derived the following representation formula of the potential by means of $T(\xi,\zeta)$:

$$\hat{V}(\xi) = T(\xi,\zeta_0) + \int_{\partial\mathcal{V}_\xi} T(\xi,\zeta)K(\zeta,\zeta_0) + \int_{\mathcal{V}_\xi}\sum_{j=1}^n A_j(\xi,\zeta)d\overline{\zeta_j}\wedge K(\zeta,\zeta_0), \tag{13.6}$$

where $\zeta_0\in\mathcal{V}_\xi$ and $K(\zeta,\zeta_0)$ is a suitable $2n-5$ form on $\mathcal{V}_\xi$.

Another important application is the characterization of the Faddeev scattering amplitude. Namely the $\overline{\partial}$-equation in Theorem 13.2 gives a necessary and sufficient condition for a function $T(\xi,\zeta)$ defined on the fibred space $\{(\xi,\zeta); \zeta^2 = E, \xi^2+2\zeta\cdot\xi = 0\}$ to be the Faddeev scattering amplitude associated with some Schrödinger operator $-\Delta + V(x)$. This interesting fact was fully discussed by Beals-Coifman [1] and Khekin-Novikov [44]. The advantage compared to Faddeev's characterization is that in the $\overline{\partial}$-approach, one can get the precise estimates of $T(\xi,\zeta)$ and also specify the associated class of potentials. See also Weder [76]. However, the characterization of the physical scattering amplitude itself is still open, though it is linked with the Faddeev scattering amplitude through the equation (11.6).

14. Inverse conductivity problem

It is an interesting fact that inverse eigenvalue problems are reduced to inverse scattering problems. Let Ω be a bounded domain in $\mathbf{R}^n$ and consider the following boundary value problem

$$\nabla \cdot (c(x)\nabla u) = 0 \quad \text{in} \quad \Omega, \tag{14.1}$$

$$u = f \quad \text{on} \quad \partial\Omega,$$

where $c(x) \geq c_0 > 0$. In electromagnetism, $c(x)$ corresponds to the conductivity. Let λ_c be the Dirichlet-Neumann map

$$\Lambda_c f = c(x)\frac{\partial u}{\partial \nu},$$

ν being the outer unit normal to the boundary. A.P. Calderón posed the following problem

Question. *Does Λ_c determine c?*

By the substitution $v = c^{1/2}u$, the above problem is transferred to

$$-\Delta v + qv = 0 \quad \text{in} \quad \Omega, \tag{14.2}$$

$$v = c^{1/2} f \quad \text{on} \quad \partial\Omega,$$

where $q = c^{-1/2}\Delta c^{1/2}$. Let Λ be the D-N map

$$\Lambda g = \frac{\partial v}{\partial \nu},$$

where v is a solution to (14.2) satisfying $v = g$ on $\partial\Omega$.

Theorem 14.1. *Let $n \geq 2$. Suppose $q \in L^\infty(\Omega)$ and $0 \notin \sigma_p(-\Delta_D + q)$. Then q is uniquely determined form the knowledge of the D-N map.*

The above theorem is proved by Sylvester-Uhlmann for $n \geq 3$ ([71]) in the category of pure boundary value problem by using the method of complex geometrical optics. Here we discuss its relation to the scattering problem.

The following observation is useful to see how the interior boundary value problem and the scattering problem are related.

Take a constant $E > 0$ and let

$$V(x) = \begin{cases} q(x) + E, & x \in \Omega \\ 0, & x \notin \Omega. \end{cases}$$

Let ψ^+ be a solution to the Schrödinger equation

$$(-\Delta + V(x) - E)\psi^+ = 0 \quad \text{in} \quad \mathbf{R}^n$$

satisfying

$$\psi^+ \simeq e^{i\sqrt{E}\omega\cdot x} + \frac{e^{i\sqrt{E}r}}{r^{(n-1)/2}} f(E;\theta,\omega) \quad r = |x| \to \infty, \quad \theta = x/r.$$

Then by Green's formula we have

$$\int_{\partial\Omega} e^{-i\sqrt{E}\theta\cdot x}(\frac{\partial}{\partial\nu} + i\sqrt{E}\theta\cdot\nu)\psi^+ dS = \int_\Omega e^{-i\sqrt{E}\theta\cdot x}(\Delta + E)\psi^+ dx = A(E;\theta,\omega),$$

where $A(E;\theta,\omega)$ is the scattering amplitude associated with $-\Delta+V$. Since ψ^+ satisfies (14.2) we have

$$\frac{\partial}{\partial\nu}\psi^+ = \Lambda\psi^+. \tag{14.3}$$

We follow Isakov-Nachman [31]. Let $\Omega_e = \mathbf{R}^n \setminus \Omega$. Then the D-N map Λ_e for the exterior Dirichlet problem is defined in the same way as in the interior problem :

$$\Lambda_e f = \frac{\partial u_e}{\partial\nu},$$

where u_e is a unique solution to

$$(-\Delta - E)u_e = 0 \quad \text{in} \quad \Omega_e,$$

$$u_e = f \in H^{3/2}(\partial\Omega) \quad \text{on} \quad \partial\Omega$$

satisfying the outgoing radiation condition

$$(\frac{\partial}{\partial r} - i\sqrt{E})u_e \in L^{2,-\alpha}(\Omega_e)$$

for some $0<\alpha<1/2$.

Another tool is the simple layer potential associated with $-\Delta+V$. Let $R^{(+)}(E,x,y)$ be the resolvent kernel of $(-\Delta+V-E-i0)^{-1}$, and let

$$K(E)f(x) = \int_{\partial\Omega} R^{(+)}(E,x,y)f(y)dS_y. \tag{14.4}$$

We define $B(E) \in \mathbf{B}(H^{1/2}(\partial\Omega); H^{3/2}(\partial\Omega))$ by

$$B(E)f = K(E)f|_{\partial\Omega}. \tag{14.5}$$

The relations between these operators are summarized in the following lemma.

Lemma 14.2. *Suppose E is not an eigenvalue of the interior problem for $-\Delta+V$. Let Λ_{V-E} and Λ_e be the D-N maps of $-\Delta+V-E$ in Ω and $-\Delta-E$ in Ω_e. Then (1) $\Lambda_{V-E}-\Lambda_e$ is an isomorphism from $H^{3/2}(\partial\Omega)$ to $H^{1/2}(\partial\Omega)$.
(2) $(\Lambda_{V-E}-\Lambda_e)^{-1} = B(E)$.*

Let $\varphi^e_\pm(x,E,\omega)$ be the generalized eigenfunctions for the exterior Dirichlet problem. Namely

$$(-\Delta - E)\varphi^e_\pm = 0 \quad \text{in} \quad \Omega_e,$$

$$\varphi^e_\pm = e^{i\sqrt{E}\omega\cdot x} \quad \text{on} \quad \partial\Omega,$$

$$\varphi^e_\pm \simeq e^{i\sqrt{E}\omega\cdot x} + \frac{e^{\pm i\sqrt{E}r}}{r^{(n-1)/2}} f_\pm(E,\hat{x},\omega) \quad r\to\infty.$$

One can then show

Theorem 14.3.

$$\int_{\partial\Omega} \overline{\frac{\partial}{\partial\nu}\varphi^e_-(x,E,\theta)} B(E)\varphi^e_+(\cdot,E,\omega)dS_x = C(E)(A(E,\theta,\omega) - A_{ext}(E,\theta,\omega)),$$

where $A(E,\theta,\omega)$ and $A_{ext}(E,\theta,\omega)$ are the scattering amplitudes of $-\Delta+V$ in $\mathbf{R}^n$ and $-\Delta$ in Ω_e, respectively.

By the above two lemmas, one can construct the scattering amplitude $A(E,\theta,\omega)$ from the knowledge of the D-N map of the interior boundary value problem. By Theorem 11.2, one can reconstruct $V(x)$ if $n \geq 3$.

Conversely, the inverse scattering problem for the compactly supported potential can be reduced to the inverse boundary value problem. In fact, Nachman [55] directly constructed a counterpart of Faddeev scattering amplitude from the knowledge of the D-N map. The potential $V(x)$ is then reconstructed by the formula (13.6).

The 2-dimensional problem is much harder, because in this case the problem is no longer overdetermined. Nachman [56] proved Theorem 14.1 when q comes from the conductivity problem, i.e. $q = c^{-1/2}\Delta c^{1/2}$. He used the $\overline{\partial}$-operator method and the theory of quasi analytic functions.

15. Other applications

Direction dependent Green operators are now used in various problems other than Schrödinger operators, for example Dirac equations by [34] and Goto [27], Maxwell equations by Ola-Päivärinta-Sommersalo [63] and also elastic equations by Nakamura-Uhlmann [59].

Let us also mention the application to wave equation in layered media

$$\partial_t^2 u = c(x,y)^2 \Delta_{x,y} u, \quad x \in \mathbf{R}^n, \quad y \in \mathbf{R},$$

where $\Delta_{x,y} = \sum_{i=1}^n (\partial/\partial x_i)^2 + (\partial/\partial y)^2$. In this case the unperturbed equation is $c_0(y)^2\Delta_{x,y}$ and the coefficient $c_0(y)$ is piecewise constant with discontinuities at the interfaces. Even in this case one can accomodate Faddeev's apparoach. See Isozaki [35], Weder [78], Guillot-Ralston [28] and Beltita [3].

In [3], the coefficients are assumed to satisfy

$$|c_0(y) - c_\pm| \leq Ce^{-\alpha|y|}, \quad \text{for} \quad \pm\, y > 0,$$

$$|c(x,y) - c_0(y)| \leq Ce^{-\alpha(|x|+|y|)},$$

for a constant $\alpha > 0$. This allows any number of interface for the background medium. One can then reconstruct the sound speed $c(x,y)$ from the knowledge of the scattering matrix at a fixed energy. In the works of [35] and [3], to construct the direction dependent Green operator the commutator calculus is used, which was developed in the study of N-body Schrödinger operators and has now become a basis of spectral and scattering theory.

Let us finally mention that Melin [53] is preparing a multi-dimensional inverse scattering theory based on the ultra-hyperbolic equation $(\Delta_x - \Delta_y)U = V(x)U$. His starting point is a fundamental solution to the free ultra-hyperbolic equation $(\Delta_x - \Delta_y)U = 0$, and has an advantage that it directly constructs the intertwining operator with triangular kernel. We also remark that [37] proposes a new approach to the inverse boundary value problem in Euclidean space by imbedding the problem to hyperbolic manifolds.

Although there remain difficulties in the characterization of scattering amplitudes, Faddeev's method gives a deep insight to the multi-dimensional inverse scattering problem. This theory seems to have much room to accept highly developed tools of partial differential equations and spectral theory for further developments.

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