Van, Hoang Dinh; **Hermans, Lander**; **Lowen, Wendy**
Operadic structure on the Gerstenhaber-Schack complex for prestacks. (English)
Zbl 07498244
Sel. Math., New Ser. 28, No. 3, Paper No. 47, 63 p. (2022)

The deformation theory of algebras by *M. Gerstenhaber* [Ann. Math. (2) 79, 59–103 (1964; Zbl 0123.03101); Proc. Natl. Acad. Sci. USA 55, 690–692 (1966; Zbl 0136.31401); Ann. Math. (2) 84, 1–19 (1966; Zbl 0147.28903); Ann. Math. (2) 88, 1–34 (1968; Zbl 0182.05902); Ann. Math. (2) 99, 257–276 (1974; Zbl 0281.16016); *M. Gerstenhaber* and *C. W. Wilkerson*, Contemp. Math. 227, 89–101 (1999; Zbl 0918.16024)] furnishes the guiding principle for algebraic deformation theory. For an algebra A, the Hochschild complex $\boldsymbol{C}(A)$ is a dg Lie algebra governing the deformation theory of through the Maurer-Cartan formalism and being the shadow of a rather richer operadic structure, which is summarized by saying that $\boldsymbol{C}(A)$ is a homotopy G-algebra [*M. Gerstenhaber* and *A. A. Voronov*, Int. Math. Res. Not. 1995, No. 3, 141–153 (1995; Zbl 0827.18004)]. This structure capturing both the brace operations and the cup product is a special case of a B_∞-structure [*E. Getzler* and *J. D. S. Jones*, "Operads, homotopy algebra and iterated integrals for double loop spaces", Preprint, arXiv:hep-th/9403055], constituting a stepping stone in the proof of the Deligne conjecture and leading to the proof that $\boldsymbol{C}(A)$ is an algebra over the chain little disk operad [*M. Kontsevich* and *Y. Soibelman*, Math. Phys. Stud. 21, 255–307 (2000; Zbl 0972.18005); *J. E. McClure* and *J. H. Smith*, Contemp. Math. 293, 153–193 (2002; Zbl 1009.18009)].

The deformation theory of algebras was later extended to presheves of algebras by *M. Gerstenhaber* and *S. D. Schack* [Contemp. Math. 13, 193–197 (1982; Zbl 0507.18005); Trans. Am. Math. Soc. 279, 1–50 (1983; Zbl 0544.18005); Nato ASI Ser., Ser. C 247, 11–264 (1988; Zbl 0676.16022)] with an introduction of a bicomplex computing the natural bimodule Ext groups. Since this GS-complex $\boldsymbol{C}_{GS}(\mathcal{A})$ of a presheaf $\mathcal{A}$ does not control deformation of $\mathcal{A}$ as a presheaf but as a *twisted* presheaf [*H. D. Van* and *W. Lowen*, Adv. Math. 330, 173–228 (2018; Zbl 1408.18038); *H. D. Van* et al., Sel. Math., New Ser. 28, No. 3, Paper No. 47, 63 p. (2022; Zbl 07498244); *W. Lowen*, Trans. Am. Math. Soc. 360, No. 3, 1631–1660 (2008; Zbl 1135.13008)], it is natural to develop deformation theory at once on the level of twisted presheves or, more generally prestacks, that is to say, pseudofunctors taking values in the 2-category of linear categories over some fixed commutative ground ring. *H. D. Van* and *W. Lowen* [Adv. Math. 330, 173–228 (2018; Zbl 1408.18038)] established a Gerstenhaber-Schack complex for prestacks, involving a differential which features an infinite sequence of higher components in addition to the classical simplicial and Hochschild differentials. On top of that, for a prestack $\mathcal{A}$, they have constructed a homotopy equivalence

$$\boldsymbol{C}_{GS}(\mathcal{A}) \cong \boldsymbol{CC}(\mathcal{A}!)$$

between the Gerstenhaber-Schack complex $\boldsymbol{C}_{GS}(\mathcal{A})$ and the Hochschild complex $\boldsymbol{CC}(\mathcal{A}!)$ of the Grothendieck construction $\mathcal{A}!$ of $\mathcal{A}$, which, through homotopy transfer, endows the GS-complex with an L_∞-structure. The result improves upon the existence of a quasi-isomorphism, which is a consequence of the Cohomoloy Comparison Theorem for presheaves [*M. Gerstenhaber* and *S. D. Schack*, Nato ASI Ser., Ser. C 247, 11–264 (1988; Zbl 0676.16022)] and for prestacks [*W. Lowen* and *M. Van den Bergh*, Trans. Am. Math. Soc. 363, No. 2, 969–986 (2011; Zbl 1268.16009)].

Although the GS-complex does not possess a B_∞-structure, its elements are to be composed in an operadic fashion, so that it makes sense to investigate this higher structure in its own right, using it directly so as to establish an underlying L_∞-structure. For particular types of presheaves, explicit L_∞-structures on the GS-complex have been established [*S. Barmeier* and *Y. Frégier*, J. Noncommut. Geom. 14, No. 3, 1019–1047 (2020; Zbl 1479.16028); *Y. Frégier* et al., New York J. Math. 15, 353–392 (2009; Zbl 1183.14004)].

Let Brace be the brace operad (§2.1) and $\mathrm{F_2S}$ the Gerstenhaber-Voronov operad encoding the homotopy G-algebras (§2.2). *E. Hawkins* ["Operations on the Hochschild bicomplex of a diagram of algebras", Preprint, arXiv:2002.00886] introduced, in the case of a presheaf $(\mathcal{A}, m, f)$, an operad

$$\mathrm{Quilt} \subseteq \mathrm{F_2S} \otimes_H \mathrm{Brace}$$

which he later extended to an operad mQuilt acting on the GS-complex, where these operads are naturally endowed with L_∞-operations, as desired. Nevertheless, the way in which functoriality of f is built into these actions does not allow for an extension to twisted presheaves or prestacks.

The principal objective in this paper is to establish a natural operadic structure with underlying L_∞-structure on $\boldsymbol{C}_{GS}(\mathcal{A})$ in the case of a general prestack $(\mathcal{A}, m, f, c)$ with twist c. A synopsis of the paper goes as follows.

§2 describes the morphisms of operads underlying the results in [*M. Gerstenhaber* and *A. A. Voronov*, Int. Math. Res. Not. 1995, No. 3, 141–153 (1995; Zbl 0827.18004)] which defined a brace-algebra structure on the totalization of a non-symmetric operad $\mathcal{O}$, delineating a homotopy G-algebra on $\mathcal{O}$ with due regard to the cup product and the Gerstenhaber bracket. To this end, the colored operad NSOp encoding non-symmetric operads is recalled, and its natural extension mNSOp adding a multiplication is described (§2.3). Let NSOp_s and mNSOp_{st} be their totalized graded (uncolored) operads with suspended, respectively standard degree (§2.5 and §2.6). The principal objective in this section is the definition of morphisms of dg-operads (Theorems 2.16 and 2.34)

$$\phi : \mathrm{Brace} \to \mathrm{NSOp}_s$$

and

$$\overline{\phi} : \mathrm{F_2S} \to \mathrm{mNSOp}_{st}$$

In these definitions, the authors have to pay particular attention to the choice of signs, for which they make use of morphisms of operads

$$\mathrm{(m)NSOp} \to \mathrm{Multi}\,\Delta$$

landing in the multicategory associated to the simplex category Δ (Proposition 2.11).

§3 captures the higher structure of $\boldsymbol{C}_{GS}(\mathcal{A})$ by introducing the operad (§3.3)

$$\mathrm{Patch} \subseteq \mathrm{mNSOp}_{st} \otimes_H \mathrm{NSOp}_s$$

over which the bicomplex $\boldsymbol{C}^{*,*}(\mathcal{A})$, of which $\boldsymbol{C}_{GS}(\mathcal{A})$ is the totalization, is shown to be an algebra (Theorem 3.24). The authors then construct a morphism (Proposition 3.27)

$$\mathrm{Quilt} \to \mathrm{Patch}_s$$

as a restriction of

$$\overline{\phi} \otimes_H \phi : \mathrm{F_2S} \otimes_H \mathrm{Brace} \to \mathrm{mNSOp}_{st} \otimes_H \mathrm{NSOp}_s$$

The resulting composition (Corollary 3.28)

$$R : \mathrm{Quilt} \to \boldsymbol{End}(s\boldsymbol{C}_{GS}(\mathcal{A}))$$

incorporates the multiplication m and the restrictions f. The auxiliary operad Patch used here is the counterpart of the operad ColorQuilt from [*E. Hawkins*, "Operations on the Hochschild bicomplex of a diagram of algebras", Preprint, arXiv:2002.00886, Def. 4.6].

§4 incorporates the twist c by adding a formal element with certain relations, resulting in the bounded powerseries operad $\mathrm{Quilt_b}[[c]]$, while the above morphism R only involves the multiplication m and the functors f of the data of a prestack $(\mathcal{A}, m, f, c)$. The morphism R is extended to a morphism (Theorem 4.17)

$$R_c : \mathrm{Quilt_b}[[c]] \to \boldsymbol{End}(s\boldsymbol{C}_{GS}(\mathcal{A}))$$

The operad $\mathrm{Quilt_b}[[c]]$ is the counterpart of the operad mQuilt from [*E. Hawkins*, "Operations on the Hochschild bicomplex of a diagram of algebras", Preprint, arXiv:2002.00886, Def. 5.2]. Just as Hawkins constructed a morphism

$$L_\infty \to \mathrm{Quilt}$$

the authors construct a more involved morphism (Theorem 4.10)

$$L_\infty \to \mathrm{Quilt_b}[[c]]$$

by extending to an infinite series of higher components incorporating the element c. Putting Theorem 4.10 and 4.17 together, $s\boldsymbol{C}_{GS}(\mathcal{A})$ is endowed with an L_∞-structure. In the case of presheaves, this coincides on reduced and normalized cochains with the L_∞-structure from [*E. Hawkins*, "Operations on the Hochschild bicomplex of a diagram of algebras", Preprint, arXiv:2002.00886, Theorem 7.13].

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

18F20 Presheaves and sheaves, stacks, descent conditions (category-theoretic aspects)
18M60 Operads (general)

Keywords:

prestack; Hochschild cohomology; Gerstenhaber-Schack complex; operad; brace operations; L_∞-algebra

Full Text: DOI

References:

[1] Barmeier, S.; Frégier, Y., Deformation-obstruction theory for diagrams of algebras and applications to geometry, J. Noncommut. Geom., 14, 3, 1019-1047 (2020) · Zbl 1479.16028 · doi:10.4171/JNCG/385

[2] Dinh Van, H.; Lowen, W., The Gerstenhaber-Schack complex for prestacks, Adv. Math., 330, 173-228 (2018) · Zbl 1408.18038 · doi:10.1016/j.aim.2018.02.023

[3] Frégier, Y.; Markl, M.; Yau, D., The \(L_\infty \)-deformation complex of diagrams of algebras, N. Y. J. Math., 15, 353-392 (2009) · Zbl 1183.14004

[4] Gerstenhaber, M.; Schack, SD, On the deformation of algebra morphisms and diagrams, Trans. Am. Math. Soc., 279, 1, 1-50 (1983) · Zbl 0544.18005 · doi:10.1090/S0002-9947-1983-0704600-5

[5] Gerstenhaber, M.; Schack, SD; Hazewinkel, M.; Gerstenhaber, M., Algebraic cohomology and deformation theory, Deformation Theory of Algebras and Structures and Applications (Il Ciocco, 1986). NATO ASI Series (Series C: Mathematical and Physical Sciences), 11-264 (1988), Dordrecht: Kluwer Academic Publishers, Dordrecht

[6] Gerstenhaber, M.; Voronov, AA, Homotopy \(G\)-algebras and moduli space operad, Int. Math. Res. Not., 1995, 3, 141-153 (1995) · Zbl 0827.18004 · doi:10.1155/S1073792895000110

[7] Getzler, E., Jones, J.D.S.: Operads, homotopy algebra and iterated integrals for double loop spaces. Preprint, arxiv: hep-th/9403055

[8] Hawkins, E.: Operations on the Hochschild bicomplex of a diagram of algebras. arXiv:2002.00886

[9] Kontsevich, M., Deformation quantization of Poisson manifolds, Lett. Math. Phys., 66, 3, 157-216 (2003) · Zbl 1058.53065 · doi:10.1023/B:MATH.0000027508.00421.bf

[10] Kontsevich, M., Soibelman, Y.: Deformations of Algebras Over Operads and the Deligne Conjecture. Conférence Moshé Flato 1999, Volume I (Dijon). Mathematical Physics Studies, vol. 21, pp. 255-307. Kluwer Academic Publishers, Dordrecht (2000) · Zbl 0972.18005

[11] Lowen, W.: Algebroid prestacks and deformations of ringed spaces. Trans. Am. Math. Soc. 360(9), 1631-1660 (2008). Preprint, arxiv: math.AG/0511197v2 · Zbl 1135.13008

[12] Lowen, W.; van den Bergh, M., A Hochschild cohomology comparison theorem for prestacks, Trans. Am. Math. Soc., 363, 2, 969-986 (2011) · Zbl 1268.16009 · doi:10.1090/S0002-9947-2010-05288-2

[13] McClure, J.E., Smith, J.H.: A solution of Deligne's Hochschild cohomology conjecture. Recent Progress in Homotopy Theory (Baltimore, MD, 2000). Contemporary Mathematics, vol. 293, pp. 153-193. American Mathematical Society, Providence, RI (2002) · Zbl 1009.18009