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Cosheaves. (English) Zbl 07420165
Theory Appl. Categ. 37, 1080-1148 (2021)

While the theory of sheaves is well developed, the theory of cosheaves is still in its infancy. This is because *cofiltered limits* are not exact in the usual categories like sets, abelian groups, rings or modules. On the contrary, *filtered colimits* are exact in the above categories.

The author [Theory Appl. Categ. 31, 1134–1175 (2016; Zbl 1373.18012)] has shown that, for precosheaves with values in a locally presentable category (or the opposite of such a category), the inclusion of cosheaves is in fact a coreflection. The principal objective in this paper is to lay a foundation for *homology* theory of cosheaves. The principal results are concerned with the most important properties of precosheaves (Theorem 3.1.1) and cosheaves (Theorem 3.3.1), as well as construction of homology theory for precosheves (Theorem 3.2.1) and cosheaves (Theorem 3.4.1).

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MSC:

18F10 Grothendieck topologies and Grothendieck topoi
18F20 Presheaves and sheaves, stacks, descent conditions (category-theoretic aspects)
18G05 Projectives and injectives (category-theoretic aspects)
18G10 Resolutions; derived functors (category-theoretic aspects)
55P55 Shape theory
55Q07 Shape groups
14F20 Étale and other Grothendieck topologies and (co)homologies

Keywords:

cosheaves; precosheves; cosheafification; pro-category; cosheaf homology; precosheaf homology; Čech homology; shape theory; pro-homology; pro-homotopy; locally presentable categories

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References:

[1] B.3. Pro-homotopy and pro-homology. Let Top be the category of topological spaces and continuous mappings. The following categories are closely related to Top: the category H (Top) of homotopy types, the category Pro (H (Top)) of pro-homotopy types, and the category H (Pro (Top)) of homotopy types of pro-spaces. The latter category is used in strong shape theory. It is finer than the former which is used in shape theory. The pointed versions Pro (H (Top *)) and H (Pro (Top *)) are defined similarly. One of the most important tools in strong shape theory is a strong expansion (see [Mardešić, 2000], conditions (S1) and (S2) on p. 129). In this paper, it is sufficient to use a weaker notion: an H (Top)-expansion ([Mardešić and Segal, 1982, §I.4.1], conditions (E1) and (E2)). Those two conditions are equivalent to the following

[2] B.3.1. Definition. Let X be a topological space. A morphism X → (Y j) j⊠I in Pro (H (Top)) is called an H (Top)-expansion (or simply expansion) if for any polyhedron P the fol-lowing mapping lim − → j [Y j , P] = lim − → j Hom H(Top) (Y j , P) −→ Hom H(Top) (X, P) = [X, P]

[3] is bijective where [Z, P] is the set of homotopy classes of continuous mappings from Z to P . An expansion is called polyhedral (or an H (Pol)-expansion) if all Y j are polyhedra.

[4] The pointed version of this notion (an H (Pol *)-expansion) is defined similarly. 2. For any (pointed) topological space X there exists an H (Pol)-expansion (an H (Pol *)-expansion), see [Mardešić and Segal, 1982, Theorem I.4.7 and I.4.10].

[5] Any two H (Pol)-expansions (H (Pol *)-expansions) of a (pointed) topological space X are isomorphic in the category Pro (H (Pol)) (Pro (H (Pol *))), see [Mardešić and Segal, 1982, Theorem I.2.6].

[6] B.3.3. Remark. Theorem 8 from [Mardešić and Segal, 1982, App.1, §3.2], shows that an H (Pol)-or an H (Pol *)-expansion for X can be constructed using nerves of normal (see Definition B.1.11) open coverings of X. Pro-homotopy is defined in [Mardešić and Segal, 1982, p. 121]:

[7] B.3.4. Definition. For a (pointed) topological space X, define its pro-homotopy pro-sets pro-π n (X) := (π n (Y j)) j⊠J

[8] where X → (Y j) j⊠J is an H (Pol)-expansion if n = 0, and an H (Pol *)-expansion if n ⊠ 1. Similar to the "usual" algebraic topology, pro-π 0 is a pro-set (an object of Pro (Set)), pro-π 1 is a pro-group (an object of Pro (Gr)), and pro-π n are abelian

pro-groups (objects of Pro (Ab)) for n ⊠ 2.

[9] Pro-homology groups are defined in [Mardešić and Segal, 1982, §II.3.2], as follows:

[10] B.3.5. Definition. For a topological space X, and an abelian group G, define its pro-homology groups as pro-H n (X, G) := (H n (Y j , G)) j⊠J

[11] Adámek, J. and Rosický, J. (1994). Locally presentable and accessible categories, volume 189 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge. · Zbl 0795.18007

[12] Artin, M. (1962). Grothendieck topologies. Notes on a Seminar held in Spring 1962. Harvard University, Cambridge, MA.

[13] Artin, M., Grothendieck, A., and Verdier, J.-L. (1972a). Théorie des topos et cohomolo-gieétale des schémas. Tome 1: Théorie des topos. Lecture Notes in Mathematics, Vol. 269. Springer-Verlag, Berlin. Séminaire de Géométrie Algébrique du Bois-Marie 1963-1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.

[14] Artin, M., Grothendieck, A., and Verdier, J.-L. (1972b). Théorie des topos et cohomologié etale des schémas. Tome 2. Lecture Notes in Mathematics, Vol. 270. Springer-Verlag, Berlin. Séminaire de Géométrie Algébrique du Bois-Marie 1963-1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.

[15] Artin, M. and Mazur, B. (1986). Etale homotopy, volume 100 of Lecture Notes in Math-ematics. Springer-Verlag, Berlin. Reprint of the 1969 original. · Zbl 0182.26001

[16] Bredon, G. E. (1997). Sheaf theory, volume 170 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition. · Zbl 0874.55001

[17] Bucur, I. and Deleanu, A. (1968). Introduction to the theory of categories and functors. With the collaboration of Peter J. Hilton and Nicolae Popescu. Pure and Applied Mathematics, Vol. XIX. Interscience Publication John Wiley \& Sons, Ltd., London-New York-Sydney. · Zbl 0197.29205

[18] Eckmann, B. and Hilton, P. J. (1966). Exact couples in an abelian category. J. Algebra, 3:38-87. · Zbl 0178.34306

[19] Gelfand, S. I. and Manin, Y. I. (2003). Methods of homological algebra. Springer Mono-graphs in Mathematics. Springer-Verlag, Berlin, second edition. · Zbl 1006.18001

[20] Grothendieck, A. (1957). Sur quelques points d'algèbre homologique. Tôhoku Math. J. (2), 9:119-221. · Zbl 0118.26104

[21] Kashiwara, M. and Schapira, P. (2006). Categories and sheaves, volume 332 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Math-ematical Sciences]. · Zbl 1118.18001

[22] Springer-Verlag, Berlin.

[23] Mac Lane, S. (1998). Categories for the working mathematician, volume 5 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition. · Zbl 0906.18001

[24] Mardešić, S. (2000). Strong shape and homology. Springer Monographs in Mathematics. Springer-Verlag, Berlin. · Zbl 0939.55007

[25] Mardešić, S. and Segal, J. (1982). Shape theory, volume 26 of North-Holland Mathematical Library. North-Holland Publishing Co., Amsterdam. · Zbl 0495.55001

[26] McCord, M. C. (1966). Singular homology groups and homotopy groups of finite topo-logical spaces. Duke Math. J., 33:465-474. · Zbl 0142.21503

[27] Noebeling, G. (1962).Über die Derivierten des Inversen und des direkten Limes einer Modulfamilie. Topology, 1:47-61.

[28] Prasolov, A. V. (2013). On the universal coefficients formula for shape homology. Topology Appl., 160(14):1918-1956. · Zbl 1287.55003

[29] Prasolov, A. V. (2016). Cosheafification. Theory Appl. Categ., 31:1134-1175. · Zbl 1373.18012

[30] Roos, J.-E. (1961). Sur les foncteurs dérivés de lim ← − . Applications. C. R. Acad. Sci. Paris, 252:3702-3704. · Zbl 0102.02501

[31] Schneiders, J.-P. (1987). Cosheaves homology. Bull. Soc. Math. Belg. Sér. B, 39(1):1-31. · Zbl 0663.55004

[32] Sugiki, Y. (2001). The category of cosheaves and Laplace transforms, volume 2001-33 of UTMS Preprint Series. University of Tokyo, Tokyo.

[33] Tamme, G. (1994). Introduction toétale cohomology. Universitext. Springer-Verlag, Berlin. Translated from the German by Manfred Kolster.

[34] Weibel, C. A. (1994). An introduction to homological algebra, volume 38 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge. Institute of Mathematics and Statistics The University of Tromsø -The Arctic University of Norway N-9037 Tromsø, Norway Email: andrei.prasolov@uit.no · Zbl 0797.18001

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