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Cockett, J. R. B. [Cockett, J. Robin B.] (3-CALG-C);
Cruttwell, G. S. H. (3-MTAU);
Lemay, J. -S. P. [Lemay, Jean-Simon Pacaud] (4-OX-DCS)

Differential equations in a tangent category I: Complete vector fields, flows, and exponentials. (English summary)

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While many ideas from differential geometry have been generalized to tangent categories [J. R. B. Cockett and G. S. H. Cruttwell, Theory Appl. Categ. **32** (2017), Paper No. 26, 835–888; MR3684725; Cah. Topol. Géom. Différ. Catég. **59** (2018), no. 1, 10–92; MR3792842; G. S. H. Cruttwell and R. B. B. Lucyshyn-Wright, J. Homotopy Relat. Struct. **13** (2018), no. 4, 867–925; MR3870775], one cannot turn vector fields into flows in an arbitrary tangent category. Nonetheless, it is possible to characterize the structure which enables such solutions, which this paper aims to do.

The central idea is that of a *curve object* allowing unique solutions of ordinary differential equations. Its definition (Definition 4.1) resembles that of a (parametrized) natural number object and possesses a similarly powerful universal property leading to a rich theory. Just as a natural number object acquires a commutative monoid structure, so too does a curve object (Theorem 4.19). However, there is a crucial difference between curve objects and natural number objects. On the one hand, natural number objects are initial so that every system has a unique solution. On the other hand, curve objects are preinitial so that solutions are unique but need not exist.

A synopsis of the paper, consisting of six sections, goes as follows.

- §2 focuses on vector fields and dynamical systems in a tangent category. It is shown (Proposition 2.18) that a vector field in the tangent category of vector fields is a pair of commuting vector fields.
- §3 defines what it means to give a solution in a dynamical system, developing some initial results regarding such solutions.
- §4 defines the notion of a curve object, establishing many of the key results. When a tangent category has a curve object, one can show that many standard results from differential geometry hold. There is a bijection between complete vector fields and flows (Theorem 4.24). There is a correspondence between commuting vector fields and commuting flows (Theorem 4.33), whose proof is structural, in sharp contrast with the calculated proof in differential geometry. The sum of commuting complete vector fields is complete (Proposition 4.37).
- §5 investigates the consequences of combining differential and curve object structure in the presence of linear completeness. It is established that a differential curve object is a differential exponential rig which has a bilinear action on every differential object and differential bundle in the tangent category.
- §1 is an introduction, and §6 is concerned with conclusions and future results.

Hirokazu Nishimura

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