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Homotopy coherent theorems of Dold-Kan type. (English) Zbl 07483896
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Given an abelian category A, the classical Dold-Kan correspondence [*D. M. Kan*, Trans. Am. Math. Soc. 87, 330–346 (1958; Zbl 0090.39001); *A. Dold*, Ann. Math. (2) 68, 54–80 (1958; Zbl 0082.37701)] gives an equivalence of categories

$$\mathrm{Fun}(\Delta^{\mathrm{op}}, A) \simeq \mathrm{Ch}_{\geq 0}(A)$$

between simplicail objects in A and connective chain complexes in A. By replacing the simplex category Δ by other categories of similar combinatorial nature, many related equivalences have been constructed [*T. Church* et al., Duke Math. J. 164, No. 9, 1833–1910 (2015; Zbl 1339.55004); *R. Helmstutler*, J. Pure Appl. Algebra 218, No. 7, 1302–1323 (2014; Zbl 1286.55008); *S. Lack* and *R. Street*, J. Pure Appl. Algebra 219, No. 10, 4343–4367 (2015; Zbl 1317.18016); *S. Lack* and *R. Street*, J. Pure Appl. Algebra 224, No. 3, 1364–1366 (2020; Zbl 1423.18032); *T. Pirashvili*, Math. Ann. 318, No. 2, 277–298 (2000; Zbl 0963.18006); *J. Słomińska*, J. Algebra 274, No. 1, 118–137 (2004; Zbl 1042.18010); *J. Słomińska*, Bull. Pol. Acad. Sci., Math. 59, No. 1, 33–40 (2011; Zbl 1225.18001)].

The principal objective in this paper is to simultaneously generalize these equivalences in the homotopy coherent context of ∞-categories. To this end, categories B endowed with the structure

$$\mathbb{B} = (B, E, E^{\vee})$$

of a so-called *DK-triple* (Definition 3.1.1) are studied, while a pointed category $N_0 = N_0(\mathbb{B})$ is associated to each DK-triple $\mathbb{B}$. The princpal result is the following theorem.

Theorem. For each weakly idempotent complete additive ∞-category $\mathcal{A}$, the DK-triple $\mathbb{B}$ induces a natural equivalence

$$\mathrm{Fun}(B, \mathcal{A}) \simeq \mathrm{Fun}^0(N_0, \mathcal{A})$$

between the ∞-categories of diagrams $B \to \mathcal{A}$ and of pointed diagrams $N_0 \to \mathcal{A}$.

A synopsis of the paper goes as follows.

- §2 gives preliminaries concerning ∞-categorical notation and tools (§2.1), pointed ∞-categories (§2.2), quotient categories and coherent chain complexes (§2.3), additive and semiadditive ∞-categories (§2.4) and weakly idempotent complete ∞-categories (§2.5).
- §3 states the main theorem (§3.3) after DK-triples (§3.1) and key constructions (§3.2).
- §4 is concerned with examples of the main theorem. §4.1 explains how to equip the simplex category $B = \Delta^{\mathrm{op}}$ with the structure of a DK-triple, a similar discussion can be seen in [*S. Lack* and *R. Street*, J. Pure Appl. Algebra 219, No. 10, 4343–4367 (2015; Zbl 1317.18016), Example 3.2]. §4.2 addresses an important class of examples of diagonalizable DK-triples out of consideration for partial maps with respect to certain factorization systems after [loc. cit., Example 3.1].
- §5 gives the proof of the main theorem (§5.3) after cofinality lemmas (§5.1) and inductive construction in the reduced case (§5.2).
- §6 deals with comparison with other works. §6.1 argues that when $\mathcal{A}$ is an idempotent complete additive ordinary category, the main theorem recovers the general Dold-Kan type equivalence of Lack and Street [loc. cit., Theorem 6.8]. §6.2 compares the author's approach with Lurie's stable Dold-Kan correspondence [`https://people.math.harvard.edu/~lurie/papers/HA.pdf`, Theorem 1.2.4.1] claiming that

  $$\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{D}) \simeq \mathrm{Fun}(\mathbb{N}, \mathcal{D})$$

  where $\mathcal{D}$ is a stable ∞-category, and we have

  $$\mathrm{Fun}(\mathbb{N}, \mathcal{D}) \simeq \mathrm{Ch}_{\geq 0}(D)$$

as was shown by *S. Ariotta* ["Coherent cochain complexes and Beilinson t-structures, with an appendix by Achim Krause", Preprint, arXiv:2109.01017].

§7 addresses further tools and formulas.

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MSC:

18Gxx Homological algebra in category theory, derived categories and functors
18Exx Categorical algebra
18Axx General theory of categories and functors

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