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Unitary transformations of fibre functors. (English) Zbl 07481856

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By Tannaka duality, a compact group G is equivalent to its category of finite-dimensional representations $\mathrm{Rep}(G)$, with canonical unitary monoidal *fiber functor*

$$F : \mathrm{Rep}(G) \to \mathrm{Hilb}$$

where Hilb is the category of finite-dimensional Hilbert spaces and linear maps. One can generalize this notion of representation to C^**-tensor categories with conjugates*, which, as well as their fiber functors, are to be described by the representation theory of compact quantum groups and their associated Hopf-Galois objects. The author ["Unitary pseudonatural transformations", Preprint, arXiv:2004.12760] introduced a notion of *unitary pseudonatural transformation* relating two monoidal functors, or more generally two pseudofunctors, demonstrating that the 2-category $\mathrm{Fun}(\mathcal{C}, \mathrm{Hilb})$ of unitary fiber functors, unitary pseudonatural transformations and modifications is of certain nice properties. The principal objective in this paper is to study unitary pseudonatural transformations of C^*-tensor categories with conjugates, or equivalently, provided that a fiber functor exists, representation categories of compact quantum groups.

A synopsis of the paper goes as follows.

§2 provides necessary background material for this paper.

§3 addresses the relationship between unitary pseudonatural transformations and Hopf-Galois theory. Let $\mathcal{C}$ be a C^*-tensor category with conjugates. Whenever a fiber functor $F : \mathcal{C} \to \mathrm{Hilb}$ exists, one can construct a monoidal equivalence

$$\mathcal{C} \simeq \mathrm{Rep}(G)$$

for a compactquantum group G, which means that the category $\mathcal{C}$ is to be understood in terms of the compact quantum group G, or rather its dual Hopf $*$-algebra A_G. Let $F_1, F_2 : \mathcal{C} \to \mathrm{Hilb}$ be fiber functors corresponding to compact quantum groups G_1, G_2. Then one can construct an A_{G_2}-A_{G_1}-bi-Hopf-Galois object Z linking the two fiber functors. As a generalization of the known fact [*T. Banica*, "Symmetries of a generic coaction", Preprint, arXiv:math/9811060, Theorem 4.4.1] claiming that the 1-dimensional $*$-representations of an A_{G_2}-A_{G_1}-bi-Hopf-Galois object correspond to unitary monoidal natural transformations $F_1 \to F_2$, it is shown (Theorem 3.14) that there is an isomorphism of categories between

- The category $\mathrm{Rep}(Z)$ of finite-dimensional $*$-representations of and intertwining linear maps.
- The category $\mathrm{Hom}(F_1, F_2)$ of unitary pseudonatural transformations $F_1 \to F_2$ and modifications.

§4 discusses the Morita classification/construction of accessible unitary pseudonatural transformations and fiber functors. Given a fiber functor $F : \mathcal{C} \to \mathrm{Hilb}$, the author exploits Morita theory to classify

- unitary fiber functors accessible from F by a unitary pseudonatural transformation in terms of Morita equivalence classes of simple Frobenius monoids in $\mathrm{Rep}(A_G)$, and
- unitary pseudonatural transformations from F in terms of $*$-isomorphism classes of simple Frobenius monoids in $\mathrm{Rep}(A_G)$.

§5 shows that finite-dimensional quantum graph isomorphisms are unitary pseudonatural transformations. The author establishes an equivalence between the following 2-categories (Theorem 5.20):

- The 2-category QGraph_X of quantum graphs quantum isomorphic to X as objects, quantum isomorphisms as 1-morphisms, and intertwiners as 2-morphisms.
- The 2-category $\mathrm{Fun}(\mathrm{Rep}(G_X), \mathrm{Hilb})_X$ of fiber functors accessible by a unitary pseudonatural transformation from the canonical fiber functor on $\mathrm{Rep}(G_X)$ as objects, unitary pseudonatural transformations as 1-morphisms, and modifications as 2-morphisms.

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MSC:

18M40 Dagger categories, categorical quantum mechanics
81R50 Quantum groups and related algebraic methods applied to problems in quantum theory
18M30 String diagrams and graphical calculi
20G42 Quantum groups (quantized function algebras) and their representations

Full Text: DOI

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