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AN INVERSE SPECTRAL UNIQUENESS IN EXTERIOR TRANSMISSION PROBLEM

By

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Abstract. We consider an inverse spectral theory in a domain with the cavity in a penetrable inhomogeneous medium. An ODE system is constructed piecewise through the ODE eigenfunctions inside and outside the cavity. Then the ODE system is connected to the PDE system via the analytic continuation property of the Helmholtz equation. For each scattered angle, we describe its eigenvalue density in the complex plane, and prove an inverse uniqueness on the inhomogeneity by the measurements in the far-fields. We take advantage of the symmetry near infinity.

1. Introduction and Preliminaries

In this paper, we apply the Sturm-Liouville theory to the inverse eigenvalue problem in the following scattering problem.

$$\begin{cases} \Delta u(x) + k^2 n(x) u(x) = 0, & x \in \mathbf{R}^3; \\ u(x) = u^i(x) + u^s(x), & x \in \mathbf{R}^3 \backslash D; \\ \lim_{|x| \to \infty} |x| \left\{ \frac{\partial u^s(x)}{\partial |x|} - iku^s(x) \right\} = 0, \end{cases} \tag{1.1}$$

where

$u(x)$ is the total wave;

$u^s(x)$ is the scattered wave;

$u^i(x) := e^{ikx \cdot d}$, $\quad k \in \mathbf{C}$, $x \in \mathbf{R}^3$, $d \in \mathbf{S}^2$, which is the incident wave.

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The problem occurs when the plane waves are perturbed by the inhomogeneity specified by the index of refraction $n(x)$. The inverse problem is to recover the information on the index of refraction $n(x)$ by the measurements of the scattered wave-fields in the far-fields. The problem is common in many disciplines of science and technology such as sonar and radar, geophysical sciences, astrophysics, and non-destructive testing in instrument manufacturing.

Out of the numerical motivation in their research in inverse scattering theory, Kirsch [17], and Colton and Monk [11] reduce the problem (1.1) into the following class of inverse spectral problem.

$$\begin{cases} \Delta w + k^2 n(x) w = 0 & \text{in } D'; \\ \Delta v + k^2 v = 0 & \text{in } D'; \\ w = v & \text{on } \partial D'; \\ \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} & \text{on } \partial D', \end{cases} \tag{1.2}$$

where ν is the unit outer normal. In this paper, we assume that D is a starlike domain in $\mathbf{R}^3$ containing the origin with the boundary ∂D, and that $\operatorname{supp}(1-n)$ is **outside** D, simple, and contained in some bounded domain D'. The inhomogeneity $n \in \mathscr{C}^2(\mathbf{R}^3)$, $n(x) > 0$ for all $x \in \mathbf{R}^3$, and the Laplacian in this paper is given by

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin\varphi}\frac{\partial}{\partial\varphi}\sin\varphi\frac{\partial}{\partial\varphi} + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2}{\partial\theta^2}. \tag{1.3}$$

Let us assume the boundary ∂D is defined by

$$R = R(\hat{x}) \in \mathscr{C}^1(\mathbf{S}^2; \mathbf{R}^+), \tag{1.4}$$

where $\mathbf{S}^2$ is the unit sphere, $\hat{x} := (\theta, \varphi)$ is the spherical coordinate, and $r := |x|$. The equation (1.2) is called the homogeneous exterior transmission eigenvalue problem [12, 13, 14]. We say k is an exterior transmission eigenvalue if and only if it parametrizes a non-trivial eigenfunction pair of (1.2).

The exterior transmission problem happens naturally when the plane waves are perturbed in the exterior of the cavity D surrounded by certain inhomogeneity. The free wave fields are generated in the cavity, and propagate through the inhomogeneity defined by the index of refraction to the far-fields. The inverse problem is to find the index of refraction by the measurements in the far-fields. We refer the scattering and inverse scattering theory of this problem to [1, 4, 12, 13, 14, 21]. To ensure the well-posedness of the scattered wave fields, we impose the Sommerfeld radiation conditions to (1.2).

$$\lim_{r\to\infty} r\left\{\frac{\partial w}{\partial r} - ikw\right\} = 0; \tag{1.5}$$

$$\lim_{r\to\infty} r\left\{\frac{\partial v}{\partial r} - ikv\right\} = 0, \tag{1.6}$$

which is typical in scattering theory [12, 16].

Let us expand the solution (w, v) of (1.2) in two series of spherical harmonics by Rellich theory [12, p. 32, p. 227]. This is a classic result holds for the Helmholtz equation outside a sphere. Here we choose the sphere large enough such that it contains the perturbation n. Then the following asymptotic identities hold.

$$\begin{cases} v(x;k) = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} a_{l,m} j_l(kr) Y_l^m(\hat{x}); \\ w(x;k) = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} b_{l,m} \frac{y_l(r)}{r} Y_l^m(\hat{x}), \end{cases} \tag{1.7}$$

where $r := |x|$, $R_0 \le r < \infty$; $\hat{x} = (\theta, \varphi) \in \mathbf{S}^2$; j_l is the spherical Bessel function of first kind of order l. The summations converge uniformly and absolutely on the compact subsets of $|x| = r \ge R_0 \gg 0$, with a sufficiently large R_0 containing D'.

The spherical harmonics

$$Y_l^m(\theta, \varphi) := \sqrt{\frac{2l+1}{4\pi} \frac{(l-|m|)!}{(l+|m|)!}} P_l^{|m|}(\cos\theta) e^{im\varphi}, \quad m = -l, \dots, l;\ l = 0, 1, 2, \dots \tag{1.8}$$

is a complete orthonormal system in $L^2(\mathbf{S}^2)$, and

$$P_l^m(t) := (1-t^2)^{m/2} \frac{d^m P_l(t)}{dt^m}, \quad m = 0, 1, \dots, l,$$

where the Legendre polynomials P_l, $l = 0, 1, \dots,$ give a complete orthogonal system in $L^2[-1, 1]$. We refer the details on the spherical harmonics and its applications to integral geometry to Groemer's book [15].

According to the orthogonality of the spherical harmonics [12, 15], the functions

$$\begin{cases} v_{l,m}(x;k) := a_{l,m} j_l(kr) Y_l^m(\hat{x}); \\ w_{l,m}(x;k) := \frac{b_{l,m} y_l(r)}{r} Y_l^m(\hat{x}) \end{cases} \tag{1.9}$$

satisfy the first two equations of (1.2) independently on the compact subsets in $|x| \ge R_0 \gg 0$.

Given one **fixed** incident $\hat{x} \in \mathbf{S}^2$, we can rotate the geometry and the perturbation on the $\hat{x}$ around the origin. Accordingly, we can extend uniquely the summands $\{v_{l,m}(x;k)\}$ and $\{w_{l,m}(x;k)\}$ into

$$\{x \in \mathbf{R}^3 \mid |x| \leq R_0\}$$

along that fixed incident $\hat{x} \in \mathbf{S}^2$ by applying the Laplacian (1.3). For each $w_{l,m}(x;k)$ solving the Helmholtz equation on the angle $\hat{x}$, the Fourier coefficient $y_l(r;k)$ is equivalent to satisfy the following ODE:

$$\begin{cases} y_l'' + \left(k^2 n(r\hat{x}) - \frac{l(l+1)}{r^2}\right) y_l = 0, \quad 0 < r < \infty; \\ \lim_{r \to 0^+} \left\{\frac{y_l(r)}{r} - j_l(kr)\right\} = 0. \end{cases} \tag{1.10}$$

The behavior of the Bessel function $j_l(kr)$ near $r = 0$ is found in [2, p. 437]. The coefficients $a_l(r;k)$ and $b_l(r;k)$ are renormalized by the initial condition in (1.10). We note that

$$j_l(z) = \frac{z^l}{2^{l+1} l!} \int_0^{\pi} \cos(z \cos \theta) \sin^{2l+1} \theta \, d\theta, \quad z \in \mathbf{C}, \tag{1.11}$$

which we refer to [2, p. 438]. Hence, we deduce that

$$\lim_{r \to 0} \frac{y_l(r;k)}{r^{l+1}} = \frac{k^l}{2^{l+1} l!} \lim_{r \to 0} \int_0^{\pi} \cos(kr \cos \theta) \sin^{2l+1} \theta \, d\theta < \infty. \tag{1.12}$$

Independently, we refer the initial condition in (1.10) to the work of [23, p. 354], and [25, (2.19), (2.20)].

Surely, $y_l(r;k)$ **depends** on the incident $\hat{x}$ in $|x| \leq R_0$. We denote the solution of (1.10) as $\hat{y}_l(r;k)$. We will demonstrate the correspondence between the spectrum of (1.10) and (1.2) in Lemma 3.9, Lemma 3.10, and Lemma 3.11. The eigenfunction of (1.10) is analytically connected to the eigenfunction of (1.2).

From the assumption of (1.2), the support of $\mathrm{supp}(1-n)$ is simple **outside** D. Thus the boundary condition/transmission condition is valid for $|x| \geq R_0 \gg 0$. This is the analytic continuation property of the generalized Helmholtz equation. Let k be an eigenvalue of (1.2). Hence, we have

$$\begin{cases} a_{l,m} j_l(kr)|_{r=R_0} = \frac{b_{l,m}\hat{y}_l(r)}{r}\Big|_{r=R_0}; \\ a_{l,m} \partial_r j_l(kr)|_{r=R_0} = \partial_r \frac{b_{l,m}\hat{y}_l(r)}{r}\Big|_{r=R_0}, \end{cases} \tag{1.13}$$

where $R_0 \gg 0$, and the system is independently of m. Now we apply the Sommerfield radiation condition (1.5) and (1.6) to $w_{l,m}(x;k)$ and $v_{l,m}(x;k)$ respectively, and deduce from the uniqueness that

$$a_{l,m} = 1;$$

$$b_{l,m} = 1,$$

and then we say there is no redundant multiples of the eigenfunctions. Now we are looking for any $k \in \mathbf{C}$ such that $\frac{y_l(r;k)}{r} = j_l(kr)$, $r \gg R_0$. We define

$$\hat{n}(r) := n(r\hat{x}). \tag{1.14}$$

For $-l \le m \le l$, $l = 0, 1, 2, \dots,$ the existence of the non-zero constants in (1.13) is reduced to finding the zeros of

$$\hat{D}_l(k; r = R_0) := \det \begin{pmatrix} j_l(kr)|_{r=R_0} & -\frac{\hat{y}_l(r)}{r}\Big|_{r=R_0} \\ \{j_l(kr)\}'|_{r=R_0} & -\left\{\frac{\hat{y}_l(r)}{r}\right\}'\Big|_{r=R_0} \end{pmatrix}. \tag{1.15}$$

If $\hat{y}_l(r;k_0)$ solves (1.10) and (1.13), then $\hat{y}_l(r;k_0)$ solves (1.10) and $\hat{D}_l(k_0) = 0$, which is an algebraic constraint. In this paper, we study the zero set of (1.13). The theory on the zeros of the entire function theory plays a role.

We state the following inverse spectral theorem of (1.2).

Theorem 1.1. *Let n^j be an unknown inhomogeneity to the background index of refraction* 1 *in* (1.2), $j = 1, 2$. *If n^1 and n^2 have the same set of eigenvalues of* (1.2) *in* $\mathbf{C}$, *then $n^1 \equiv n^2$.*

We may compare the result with [8, 9, 13, 14].

2. Asymptotic Solutions of ODE

Let us consider the ODE with the Liouville transformation [5, 6, 12, 22, 24] for some fixed $\hat{x}$:

$$z_l(\xi) := [n(r\hat{x})]^{1/4} y_l(r;k),$$

where

$$\xi(r) = \int_0^r [n(\rho\hat{x})]^{1/2}\, d\rho. \tag{2.1}$$

Therefore,

$$z_l'' + \left[k^2 - q(\xi) - \frac{l(l+1)}{\xi^2}\right] z_l = 0, \tag{2.2}$$

in which

$$q(\xi) := \frac{n''(r\hat{x})}{4[n(r\hat{x})]^2} - \frac{5}{16}\frac{[n'(r\hat{x})]^2}{[n(r\hat{x})]^3} + \frac{l(l+1)}{r^2 n(r\hat{x})} - \frac{l(l+1)}{\xi^2}. \tag{2.3}$$

Let us drop the superscript on $\hat{x}$ for notation simplicity if the context is clear. The general solution of (2.2) has two independent fundamental solutions. Let us apply the results from [5, Lemma 3.3], and consider $z_l(\xi; k)$ solving the following ODE.

$$\begin{cases} -z_l''(\xi) + \frac{l(l+1)z_l(\xi)}{\xi^2} + q(\xi)z_l(\xi) = k^2 z_l(\xi); \\ z_l(R;k) = -b; \quad z_l'(R;k) = a, \quad a, b \in \mathbf{R}, \end{cases} \tag{2.4}$$

where the function $q(\xi)$ is assumed to be real-valued and square-integrable and $l \geq -1/2$. The following estimate holds for $0 \leq \xi \leq R$.

$$\begin{aligned} &\left| z_l(\xi;k) + b\cos k(R-\xi) + a\frac{\sin k(R-\xi)}{k} \right| \\ &\qquad \leq \frac{K(\xi)}{|k|}\exp\{|\Im k|(R-\xi)\}, \quad |k| \geq 1, \end{aligned} \tag{2.5}$$

where

$$K(\xi) \leq \exp\left\{\int_\xi^R \frac{|l(l+1)|}{t^2} + |q(t)|\,dt\right\}.$$

We note here that the ODE (2.4) starts at $\xi = R$ and moves to the origin while [5, Lemma 3.3] starts at 1, and then moves toward the origin. We make it a **two-way** construction of solutions, which is the most important ingredient of this paper. For the ODE starting at $\xi = R$, that is, if and only $r = R$, and then moving to the infinity, we have

$$\begin{aligned} &\left| z_l(\xi;k) + b\cos k(\xi-R) - a\frac{\sin k(\xi-R)}{k} \right| \\ &\qquad \leq \frac{\tilde{K}(\xi)}{|k|}\exp\{|\Im k|(\xi-R)\}, \quad |k| \geq 1, \end{aligned} \tag{2.6}$$

where

$$\tilde{K}(\xi) \leq \exp\left\{\int_R^\xi \frac{|l(l+1)|}{t^2} + |q(t)|\,dt\right\}. \tag{2.7}$$

For the application in this paper, we combine the estimates of the solution of (2.4) by considering the initial condition $\hat{D}_l(R;k) = 0$ for $r \geq R$, $R = R(\hat{x})$, which is equivalent to the following algebraic system due to (1.13).

$$\hat{y}_l(R;k) = Rj_l(R;k); \tag{2.8}$$

$$\hat{y}_l'(R;k) = j_l(Rk) + Rkj_l'(Rk), \quad k \in \mathbf{C}, \tag{2.9}$$

which is the transmission condition of (1.2) on ∂D and radiation condition (1.5) and (1.6). Most importantly, the general solution of (2.4) for $r \geq R$ is spanned by two of its fundamental solutions as in the following lemma.

LEMMA 2.1. *For k near real axis, the following asymptotics holds.*

$$\hat{y}_l(r;k) = [j_l(Rk) + Rkj_l'(Rk)]\frac{\sin\{k[\xi(r) - R]\}}{k} + Rj_l(Rk)\cos\{k[\xi(r) - R]\} + O\left(\frac{1}{k}\right), \quad r \geq R. \tag{2.10}$$

Particularly, $\hat{y}_l(r;k)$ is bounded in $0i + \mathbf{R}$.

PROOF. (2.10) follows from the general theory of ODE and (2.6) if we are required by the initial condition (2.8) and (2.9). All functions in (2.10) are bounded in $0i + \mathbf{R}$, because of (2.6). □

The analysis is reversible into the domain D by considering (2.4) with initial condition (2.8) and (2.9).

3. Polyá-Cartwright-Levinson Theory

We collect a few facts from entire function theory [7, 18, 19, 20].

DEFINITION 3.1. Let $f(z)$ be an integral function of order ρ, and let $N(f, \alpha, \beta, r)$ denote the number of the zeros of $f(z)$ inside the angle $[\alpha, \beta]$ and $|z| \leq r$. We define the density function as

$$\Delta_f(\alpha, \beta) := \lim_{r\to\infty} \frac{N(f, \alpha, \beta, r)}{r^\rho}, \tag{3.1}$$

and

$$\Delta_f(\beta) := \Delta_f(\alpha_0, \beta), \tag{3.2}$$

with some fixed $\alpha_0 \notin E$ such that E is at most a countable set [3, 7, 18, 19, 20].

Definition 3.2. Let $f(z)$ be an integral function of finite order ρ in the angle $[\theta_1, \theta_2]$. We call the following quantity as the indicator function of the function $f(z)$.

$$h_f(\theta) := \lim_{r\to\infty} \frac{\ln|f(re^{i\theta})|}{r^\rho}, \quad \theta_1 \le \theta \le \theta_2. \tag{3.3}$$

Lemma 3.3. *Let f, g be two entire functions. Then the following two inequalities hold.*

$$h_{fg}(\theta) \le h_f(\theta) + h_g(\theta), \quad \textit{if one limit exists}; \tag{3.4}$$

$$h_{f+g}(\theta) \le \max_\theta\{h_f(\theta), h_g(\theta)\}, \tag{3.5}$$

where the equality in (3.4) *holds if one of the functions is of completely regular growth, and secondly the equality* (3.5) *holds if the indicator of the two summands are not equal at some θ_0.*

Proof. We can find the details in [19]. □

Definition 3.4. The following quantity is called the width of the indicator diagram of entire function f:

$$d = h_f\left(\frac{\pi}{2}\right) + h_f\left(-\frac{\pi}{2}\right). \tag{3.6}$$

The distribution on the zeros of entire function of exponential type is described precisely in the following Cartwright's theorem [7, 19, 20]. The following statements are from Levin [19, Ch. 5, Sec. 4].

Theorem 3.5 (Cartwright). *Let f be an entire function of exponential type with zero set $\{a_k\}$. We assume f satisfies one of the following conditions*:

$$\textit{the integral } \int_{-\infty}^{\infty} \frac{\ln^+|f(x)|}{1+x^2}\,dx \textit{ exists.}$$

$$|f(x)| \textit{ is bounded on the real axis.}$$

Then

1. *$f(z)$ is of class A and of completely regular growth, and its indicator diagram is an interval on the imaginary axis*;

2. *all of the zeros of the function* $f(z)$, *except possibly those of a set of zero density, lie inside arbitrarily small angles* $|\arg z| < \varepsilon$ *and* $|\arg z - \pi| < \varepsilon$, *where the density*

$$\Delta_f(-\varepsilon, \varepsilon) = \Delta_f(\pi - \varepsilon, \pi + \varepsilon) = \lim_{r\to\infty} \frac{N(f, -\varepsilon, \varepsilon, r)}{r}$$
$$= \lim_{r\to\infty} \frac{N(f, \pi - \varepsilon, \pi + \varepsilon, r)}{r}, \tag{3.7}$$

is equal to $\frac{d}{2\pi}$, *where* d *is the width of the indicator diagram in* (3.6). *Furthermore, the limit* $\delta = \lim_{r\to\infty} \delta(r)$ *exists, where*

$$\delta(r) := \sum_{\{|a_k|<r\}} \frac{1}{a_k};$$

3. *moreover,*

$$\Delta_f(\varepsilon, \pi - \varepsilon) = \Delta_f(\pi + \varepsilon, -\varepsilon) = 0; \tag{17}$$

4. *the function* $f(z)$ *can be represented in the form*

$$f(z) = cz^m e^{i\kappa z} \lim_{r\to\infty} \prod_{\{|a_k|<r\}} \left(1 - \frac{z}{a_k}\right),$$

where c, m, κ *are constants and* κ *is real*;

5. *the indicator function of* f *is of the form*

$$h_f(\theta) = \sigma|\sin\theta|. \tag{3.8}$$

We refer the last statement to Levin [20, p. 126].

Lemma 3.6. *We have the following indicator functions.*

$$h_{j_l'(kR_0)}(\theta) = h_{j_l(kR_0)}(\theta) = |R_0 \sin\theta|, \quad \theta \in [0, 2\pi].$$

Proof. The spherical Bessel functions $j_l(kR_0)$ and $j_l'(kR_0)$ behave asymptotically like $\frac{\sin R_0 k}{k}$ and $\cos R_0 k$ respectively by considering the analysis in (2.4). The analysis on the Bessel function is classic [2]. We refer the computation on their indicator functions to Cartwright theory [3, 7, 18, 19, 20]. We have applied the technique in inverse problems [8, 9, 10, 13]. □

Lemma 3.7. *The following asymptotic identity holds.*

$$h_{\hat{D}_l(k;R_0)}(\theta) = h_{j_l'(kR_0)}(\theta) + h_{\hat{y}_l(R_0;k)}(\theta), \quad \theta \in [0, 2\pi]. \tag{3.9}$$

Proof. We begin with (1.15).

$$\hat{D}_l(k;R_0) = -j_l(kR_0)\frac{\hat{y}_l'(R_0;k)}{R_0} + j_l(kR_0)\frac{\hat{y}_l(R_0;k)}{R_0^2} + kj_l'(kR_0)\frac{\hat{y}_l(R_0;k)}{R_0} \tag{3.10}$$

$$= \frac{kj_l'(kR_0)\hat{y}_l(R_0;k)}{R_0}\left\{1 - \frac{1}{k}\frac{j_l(kR_0)}{j_l'(kR_0)}\frac{\hat{y}_l'(R_0;k)}{\hat{y}_l(R_0;k)} + \frac{1}{kR_0}\frac{j_l(kR_0)}{j_l'(kR_0)}\right\}$$

$$= \frac{kj_l'(kR_0)\hat{y}_l(r;k)}{R_0}\left\{\hat{\alpha}_l(k) + O\left(\frac{1}{k}\right)\right\}, \tag{3.11}$$

in which

$$\hat{\alpha}_l(k) := 1 - \frac{1}{k}\frac{j_l(kR_0)}{j_l'(kR_0)}\frac{\hat{y}_l'(R_0;k)}{\hat{y}_l(R_0;k)}, \tag{3.12}$$

where we see that non-zero

$$\frac{j_l(kR_0)}{j_l'(kR_0)} = O(1)$$

and non-zero

$$\frac{\hat{y}_l'(R_0;k)}{\hat{y}_l(R_0;k)} = O(k)$$

away from its poles. Moreover, Lemma 2.1 implies that $\hat{y}_l'(R_0;k)$ and $k\hat{y}_l(R_0;k)$ are asymptotically periodic functions. They are bounded when suitably away from the real axis. Thus, (3.3) shows the Lindelöf's indicator function $h_{\hat{\alpha}_l}(\theta) = 0$. We refer the step-by-step computation to [3, 8, 9, 10, 19, 20]. However, Lindelöf's indicator function for (3.11) is

$$h_{\hat{D}_l(k;R_0)}(\theta) = h_{j_l'(kR_0)}(\theta) + h_{\hat{y}_l(R_0;k)}(\theta), \quad \theta \neq 0. \tag{3.13}$$

Here we use (3.13).

If $\hat{\alpha}_l(k) \equiv 0$, then we have the non-zero second term in (3.10). The indicator function is calculated similarly, and thus (3.9) is proven again. □

Lemma 3.8. *We have the following indicator functions for $\hat{y}_l(r;k)$ and $\hat{y}_l'(r;k)$ for $r \geq R$.*

$$h_{\hat{y}_l'(r;k)}(\theta) = h_{\hat{y}_l(r;k)}(\theta) = |\xi(r)|\,|\sin\theta|, \quad \theta \in [0, 2\pi].$$

Proof. We apply (3.13) to Lemma 2.1. The Liouville transform of (2.2) is

$$\xi(r) = \int_0^r \sqrt{n(\rho\hat{x})}\, d\rho, \quad r \geq R,$$

in which $n = 1$ outside $D' \backslash D$ by assumption. $\square$

We look for a zero set of certain entire function that contains the eigenvalues of (1.2). The sequence $\{\hat{D}_l(k; r = R_0)\}_{\hat{x}}$ plays a role in this paper.

Lemma 3.9. *k is an eigenvalue of* (1.2) *if and only if k is zero of $\hat{D}_l(k; r = R_0)$ for some l and some $\hat{x} \in \mathbf{S}^2$, where R_0 is sufficiently large and given in* (1.7).

Proof. Let $k \in \mathbf{C}$ be an eigenvalue of (1.2). By Rellich theory, the expansion (1.7) holds uniquely. Hence, $\hat{D}_l(k; r = R_0) = 0$ holds for all l, $\hat{x}$, and in particular for some l. We choose some incident $\hat{x}$ and extend the $\hat{y}_l(r)$ into $|x| \leq R_0$ along the $\hat{x}$ according to the construction (1.10) and (2.5), and then (1.13) applies.

For the sufficient condition, we let k_0 solve $\tilde{D}_l(k; R_0) = 0$ for some l and along some $\tilde{x} \in \mathbf{S}^2$. That is, we deduce from (1.13) and (1.10) that

$$\begin{cases} \tilde{y}_l'' + \left(k_0^2 n(r\tilde{x}) - \frac{l(l+1)}{r^2}\right)\tilde{y}_l = 0, \quad 0 < r < R_0; \\ \frac{\tilde{y}_l(r;k_0)}{r}\Big|_{r=R_0} = j_l(k_0 r)|_{r=R_0}; \\ \partial_r \frac{\tilde{y}_l(r;k_0)}{r}\Big|_{r=R_0} = \partial_r j_l(k_0 r)|_{r=R_0}; \\ \lim_{r\to 0^+}\left\{\frac{\tilde{y}_l(r;k_0)}{r} - j_l(kr)\right\} = 0, \end{cases} \tag{3.14}$$

in which the function $j_l(k_0 r)$ is defined in $|x| \gg 0$ and independent of $\hat{x}$. Then we use (1.13) as an initial condition that works for all $\hat{x} \in \mathbf{S}^2$, where $\hat{x} \neq \tilde{x}$. That is, we consider the uniqueness and existence of ODE

$$\begin{cases} \hat{y}_l'' + \left(k_0^2 n(r\hat{x}) - \frac{l(l+1)}{r^2}\right)\hat{y}_l = 0, \quad 0 < r < R_0; \\ \frac{\hat{y}_l(r;k_0)}{r}\Big|_{r=R_0} = j_l(k_0 r)|_{r=R_0}; \\ \partial_r \frac{\hat{y}_l(r;k_0)}{r}\Big|_{r=R_0} = \partial_r j_l(k_0 r)|_{r=R_0}; \\ \frac{\hat{y}_l(r;k_0)}{r}\Big|_{r=R} = j_l(k_0 r)|_{r=R}; \\ \partial_r \frac{\hat{y}_l(r;k_0)}{r}\Big|_{r=R} = \partial_r j_l(k_0 r)|_{r=R}; \\ \lim_{r\to 0^+}\left\{\frac{\hat{y}_l(r;k_0)}{r} - j_l(kr)\right\} = 0, \quad \hat{x} \neq \tilde{x}, \end{cases}$$

in which the given k_0 defines some coefficients $\hat{y}_l(r;k_0)$ in $\mathbf{R}^3$ constructed as in (2.5), (2.6), and (2.10). Moreover,

$$w_{l,m}(x;k) = \frac{b_{l,m} y_l(r)}{r} Y_l^m(\hat{x})$$

is independent of $\hat{x}$ for $|x| \gg R_0$. Hence, we require $\hat{y}_l(r;k_0) = \tilde{y}_l(r;k_0)$ for large $r = |x| \gg R_0$.

By assumption, $\mathrm{supp}\{1-n\}$ is simple, the boundary condition on $|x| = R$ and $|x| = R_0$ holds simultaneously due to the analytic continuation property of the Helmholtz equation. Therefore, there exists eigenfunction pair $\{w(x;k_0), v(x;k_0)\}$ in $R \leq |x| \leq R_0$. The extension routes of the Fourier coefficients $\hat{y}_l(r;k_0)$ up to $r = 0$ are illustrated in Figure 1.

To finish the extension inside D, we consider the ODE with the given k_0,

$$\begin{cases} \hat{y}_l'' + \left(k_0^2 n(r\hat{x}) - \frac{l(l+1)}{r^2}\right)\hat{y}_l = 0, \quad 0 < r < R; \\ \frac{\hat{y}_l(r;k_0)}{r}\Big|_{r=R} = j_l(k_0 r)|_{r=R}; \\ \partial_r \frac{\hat{y}_l(r;k_0)}{r}\Big|_{r=R} = \partial_r j_l(k_0 r)|_{r=R}; \\ \lim_{r\to 0^+}\left\{\frac{\hat{y}_l(r;k_0)}{r} - j_l(kr)\right\} = 0. \end{cases} \tag{3.15}$$

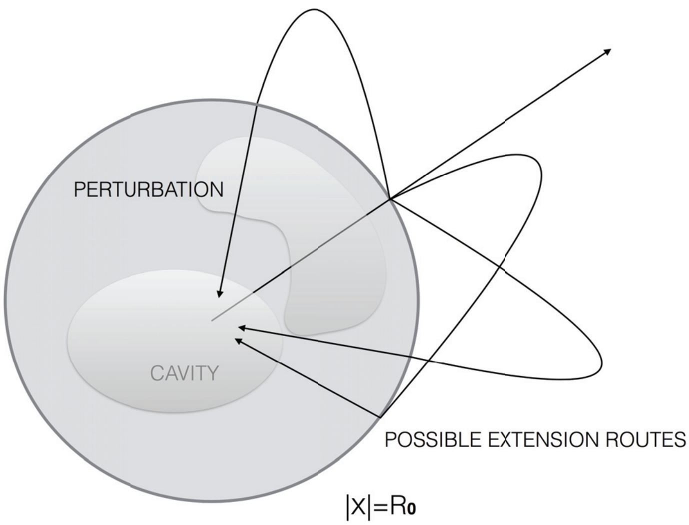

Figure 1: Rays of Extension Routes of ODE Eigenfunction

The index $n \equiv 1$ inside D. The uniqueness of the ODE implies that $\frac{\hat{y}_l(r;k_0)}{r} = j_l(k_0 r)$. The ODE holds for all other incident angle $\hat{x} \neq \tilde{x} \in \mathbf{S}^2$, so we deduce that $w(x;k_0) = v(x;k_0)$ inside D.

Taking directional derivatives near D,

$$\frac{\partial w(x;k_0)}{\partial \nu} = \frac{\partial v(x;k_0)}{\partial \nu} \quad \text{on } \partial D.$$

That makes $\{w, v\}$ a pair of eigenfunctions of (1.2). □

The effective support of $\{1-n\}$ may not be minimal as shown by the following lemma.

Lemma 3.10. *Given a $\hat{x} \in \mathbf{S}^2$ and a fixed k, $\hat{D}_l(k;r)$ is locally constant near $r = R_0$ whenever $x \notin \operatorname{supp}\{1-n\}$.*

Proof. Let us add the initial condition (1.13) to

$$\hat{y}_l''(r) + \left(k^2 n(r\hat{x}) - \frac{l(l+1)}{r^2}\right)\hat{y}_l(r) = 0.$$

The function $j_l(kr)$ and $\frac{\hat{y}_l(r)}{r}$ satisfy the same ODE outside the perturbation. The lemma is proven by the uniqueness of ODE. □

Lemma 3.11. *If k is an eigenvalue of* (1.2), *then $\hat{D}_l(k;r) = 0$ for $x \in D$.*

Proof. We have $w = v$ inside D. The uniqueness of Rellich's lemma (1.7) and the uniqueness of ODE imply that $\frac{\hat{y}_l(r;k)}{r} = j_l(kr)$ for $r \leq R$. This proves the lemma. □

4. A Proof of Theorem 1.1

Proof. Let n^1 and n^2 be two indices of refraction with solutions $y_l^1(r;k)$ and $y_l^2(r;k)$ respectively with the set of exterior eigenvalues $\mathscr{E}$. The density of the set $\mathscr{E}$ is given by the indicator function in Lemma 3.7 and (3.7) in Theorem 3.5.

By applying Lemma 3.9 and (1.13), we have for each fixed $\hat{x} \in \mathbf{S}^2$, the zeroth coefficient

$$\hat{y}_0^1(r;k) = \hat{y}_0^2(r;k); \tag{4.1}$$

$$\partial_r \hat{y}_0^1(r;k) = \partial_r \hat{y}_0^2(r;k), \quad k \in \mathscr{E},\ r = R_0. \tag{4.2}$$

Let

$$F(k) := \hat{y}_0^1(R_0;k) - \hat{y}_0^2(R_0;k).$$

According to Lemma 3.3, we know that the indicator function

$$h_F(\theta) = \max\{h_{\hat{y}_0^1(R_0;k)}(\theta), h_{\hat{y}_0^2(R_0;k)}(\theta)\},$$

in which, by Lemma 3.8, we have

$$h_{\hat{y}_0^j(R_0;k)}(\theta) = \left| R + \int_R^{R_0} \sqrt{n^j(\rho\hat{x})}\, d\rho \right| |\sin\theta|, \quad j = 1,2.$$

We apply Lemma 3.7, and Lemma 3.8 to deduce that

$$h_{\hat{D}_0^j(k;\,R_0)}(\theta) > h_{\hat{y}_0^j(R_0;k)}(\theta), \quad j = 1,2,$$

and thus the exterior spectrum $\mathscr{E}$ renders greater angle-wise density than the solution set of (4.1). This contradicts the maximal density of the zero set in Cartwright Theorem as stated in (3.7). Hence,

$$\hat{y}_0^1(R_0;k) \equiv \hat{y}_0^2(R_0;k); \tag{4.3}$$

$$\partial_r \hat{y}_0^1(R_0;k) \equiv \partial_r \hat{y}_0^2(R_0;k). \tag{4.4}$$

Hence, n^1 and n^2 have the same set of norming constants and two independent spectra, Dirichlet and Neumann, to the following equation.

$$\begin{cases} \hat{y}_l'' + \left(k^2 n(r\hat{x}) - \frac{l(l+1)}{r^2}\right)\hat{y}_l = 0, \quad 0 < r < R_0; \\ \lim\limits_{r\to 0^+} \left\{\frac{y_l(r)}{r} - j_l(kr)\right\} = 0. \end{cases} \tag{4.5}$$

By the inverse uniqueness of the Bessel operator [5, Theorem 1.2, Theorem 1.3], we have $n^1(r\hat{x}) \equiv n^2(r\hat{x})$ in $0 \le r \le R_0$. The argument can be carried to all $\hat{x} \in \mathbf{S}^2$. This proves Theorem 1.1. □

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