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THE MEAN VALUES OF THE DOUBLE ZETA-FUNCTION

By

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Abstract. We investigate the mean square formulas of the Euler–Zagier type double zeta-function $\zeta_2(s_1, s_2)$ and provide the Ω results of the double zeta-function. We also calculate the double integral under certain conditions.

1. Introduction

Let $s_j = \sigma_j + it_j$ $(j = 1, 2)$ be complex variables with $\sigma_j, t_j \in \mathbf{R}$, and let $\zeta(s)$ be the Riemann zeta-function, which is defined as $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$ for $\mathrm{Re}\, s > 1$. The double zeta-function of Euler–Zagier type is defined by

$$\zeta_2(s_1, s_2) = \sum_{1 \le m < n} \frac{1}{m^{s_1} n^{s_2}},$$

which is absolutely convergent for $\sigma_2 > 1$ and $\sigma_1 + \sigma_2 > 2$. One can easily see that the reciprocity law

$$\zeta(s_1)\zeta(s_2) = \zeta(s_1 + s_2) + \zeta_2(s_1, s_2) + \zeta_2(s_2, s_1) \tag{1.1}$$

holds for $\sigma_1 > 1$ and $\sigma_2 > 1$. The function $\zeta_2(s_1, s_2)$ was applied to the proof of the mean value formula given by F. V. Atkinson [3] (see also A. Ivić [7]) in the theory of the Riemann zeta-function $\zeta(s)$. Atkinson gave the analytic continuation of $\zeta_2(s_1, s_2)$ to the region $\{(s_1, s_2) \in \mathbf{C}^2 \mid 0 < \sigma_1 < 1,\ 0 < \sigma_2 < 1\}$. The function $\zeta_2(s_1, s_2)$ is continued meromorphically to $\mathbf{C}^2$, which was studied by J. Q. Zhao [18] and S. Akiyama, S. Egami and Y. Tanigawa [1], independently. The double zeta-function $\zeta_2(s_1, s_2)$ has many applications to mathematical physics. In particular, some algebraic relations among the values of the double zeta-function

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$\zeta_2(s_1, s_2)$ at positive integers have been extensively studied [16]. Some analytic properties of $\zeta_2(s_1, s_2)$ have been obtained by Akiyama, Egami and Tanigawa, H. Ishikawa and K. Matsumoto [6], I. Kiuchi and Y. Tanigawa [11], I. Kiuchi, Y. Tanigawa and W. Zhai [12], K. Matsumoto [13], [14], and others.

1.1. Mean Square Formula—Concerning t_1. K. Matsumoto and H. Tsumura [15] were the first to study a new type of some mean value formulas of $\int_2^T |\zeta_2(s_1, s_2)|^2 \, dt_2$ with a fixed complex number s_1 and any large positive number $T > 2$. They derived two approximate formulas for $\zeta_2(s_1, s_2)$ and three mean value formulas for $\zeta_2(s_1, s_2)$ with respect to t_2, who particularly obtained

$$\int_2^T |\zeta_2(s_1, s_2)|^2 \, dt_2 = \left(\sum_{n=2}^{\infty} \left| \sum_{m=1}^{n-1} \frac{1}{m^{s_1}} \right|^2 \frac{1}{n^{2\sigma_2}} \right) T + O(T^{4-2\sigma_1-2\sigma_2} \log T) + O(T^{1/2})$$

for $\frac{1}{2} < \sigma_1 < 1$, $\frac{1}{2} < \sigma_2 < 1$ and $\frac{3}{2} < \sigma_1 + \sigma_2 \leq 2$, where the coefficient of the main term on the right-hand side of the above converges if $\sigma_2 > 1/2$ and $\sigma_1 + \sigma_2 > 3/2$. S. Ikeda, K. Matsuoka and Y. Nagata [5] studied the asymptotic behaviour of the integral $\int_2^T |\zeta_2(s_1, s_2)|^2 \, dt_2$ for $\sigma_2 \geq \frac{1}{2}$ and $\sigma_1 + \sigma_2 \geq \frac{3}{2}$, and obtained some asymptotic formulas. They also considered the mean value formula of $\int_2^T |\zeta_2(s_1, s_2)|^2 \, dt_1$ with a fixed complex number s_2 and any large positive number $T > 2$, who showed that

$$\int_2^T |\zeta_2(s_1, s_2)|^2 \, dt_1 = \left(\sum_{m=1}^{\infty} \frac{1}{m^{2\sigma_1}} \left| \zeta(s_2) - \sum_{n=1}^{m} \frac{1}{n^{s_2}} \right|^2 \right) T + \begin{cases} O(T^{4-2\sigma_1-2\sigma_2}) & \text{if } \frac{3}{2} < \sigma_1 + \sigma_2 < 2, \\ O(\log^2 T) & \text{if } \sigma_1 + \sigma_2 = 2. \end{cases} \tag{1.2}$$

Here, the coefficient of main term on the right-hand side of (1.2) converges if $\sigma_1 + \sigma_2 > \frac{3}{2}$. Furthermore, they deduced that the asymptotic formula

$$\int_2^T |\zeta_2(s_1, s_2)|^2 \, dt_1 = \frac{1}{|s_2 - 1|^2} T \log T + O(T) \tag{1.3}$$

holds on the line $\sigma_1 + \sigma_2 = \frac{3}{2}$. This result implied that the conjecture of Matsumoto and Tsumura on the line $\sigma_1 + \sigma_2 = \frac{3}{2}$ (see their conjecture (ii) in [15]) was true. Some mean value formulas given by I. Kiuchi and M. Minamide [10] were inspired by Matsumoto and Tsumura, and Ikeda, Matsuoka and Nagata. For the region $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ and $0 < \sigma_1 + \sigma_2 < \frac{3}{2}$, Kiuchi and Minamide recently considered five formulas of the integral $\int_2^T |\zeta_2(s_1, s_2)|^2 \, dt_1$ and showed that, for any sufficiently large positive number $T > 2$,

$$\int_2^T |\zeta_2(s_1, s_2)|^2 \, dt_1 = \frac{\zeta(2)}{4\pi |s_2 - 1|^2} T^2 + O(t_2^{-1/2} (\log t_2) T^{3/2}) \tag{1.4}$$

with $\sigma_1 + \sigma_2 = 1$ and $2 \le t_2 \le \frac{T^{1/3}}{\log T}$,

$$\int_2^T |\zeta_2(s_1, s_2)|^2 \, dt_1 = (2\pi)^{2\sigma_1 + 2\sigma_2 - 3} \frac{\zeta(4 - 2\sigma_1 - 2\sigma_2)}{(4 - 2\sigma_1 - 2\sigma_2)|s_2 - 1|^2} T^{4 - 2\sigma_1 - 2\sigma_2} + O(t_2^{-1/2} T^{5/2 - \sigma_1 - \sigma_2}) \tag{1.5}$$

with $1 < \sigma_1 + \sigma_2 < \frac{3}{2}$ and $2 \le t_2 \le T^{1 - (2/3)(\sigma_1 + \sigma_2)}$,

$$\int_2^T |\zeta_2(s_1, s_2)|^2 \, dt_1 = (2\pi)^{2\sigma_1 + 2\sigma_2 - 3} \frac{\zeta(4 - 2\sigma_1 - 2\sigma_2)}{(4 - 2\sigma_1 - 2\sigma_2)|s_2 - 1|^2} T^{4 - 2\sigma_1 - 2\sigma_2} + O(t_2^{1/2 - \sigma_1 - \sigma_2} T^{5/2 - \sigma_1 - \sigma_2}) \tag{1.6}$$

with $\frac{1}{2} < \sigma_1 + \sigma_2 < 1$ and $2 \le t_2 \le T^{(3 - 2\sigma_1 - 2\sigma_2)/(5 - 2\sigma_1 - 2\sigma_2)}$,

$$\int_2^T |\zeta_2(s_1, s_2)|^2 \, dt_1 = \frac{\zeta(3)}{12\pi^2 |s_2 - 1|^2} T^3 + \begin{cases} O(T^2) & \text{if } \sqrt{\log T} \le t_2 \le T^{1/2}, \\ O(t_2^{-1} T^2 \sqrt{\log T}) & \text{if } 2 \le t_2 \le \sqrt{\log T} \end{cases} \tag{1.7}$$

with $\sigma_1 + \sigma_2 = \frac{1}{2}$, and

$$\begin{aligned} &\int_2^T |\zeta_2(s_1, s_2)|^2 \, dt_1 \\ &= (2\pi)^{2\sigma_1 + 2\sigma_2 - 3} \frac{\zeta(4 - 2\sigma_1 - 2\sigma_2)}{(4 - 2\sigma_1 - 2\sigma_2)|s_2 - 1|^2} T^{4 - 2\sigma_1 - 2\sigma_2} \\ &\quad + \begin{cases} O(t_2^{1/2 - \sigma_1 - \sigma_2} T^{5/2 - \sigma_1 - \sigma_2}) & \text{if } T^{(1 - 2\sigma_1 - 2\sigma_2)/(3 - 2\sigma_1 - 2\sigma_2)} \le t_2 \\ & \qquad \le T^{(3 - 2\sigma_1 - 2\sigma_2)/(5 - 2\sigma_1 - 2\sigma_2)}, \\ O(t_2^{-1} T^{3 - 2\sigma_1 - 2\sigma_2}) & \text{if } 2 \le t_2 \le T^{(1 - 2\sigma_1 - 2\sigma_2)/(3 - 2\sigma_1 - 2\sigma_2)} \end{cases} \end{aligned} \tag{1.8}$$

with $0 < \sigma_1 + \sigma_2 < \frac{1}{2}$. Note that the O constants of the above formulas depend on σ_1 and σ_2. From (1.4)–(1.8), they also showed that

$$\zeta_2(\sigma_1 + it_1, \sigma_2 + it_2) = \Omega\left(\frac{t_1^{3/2 - \sigma_1 - \sigma_2}}{t_2}\right) \tag{1.9}$$

for $1 \leq \sigma_1 + \sigma_2 < \frac{3}{2}$, $2 \leq t_1 \leq T$ and $2 \leq t_2 \leq T^{1-(2/3)(\sigma_1+\sigma_2)-\varepsilon}$, and

$$\zeta_2(\sigma_1 + it_1, \sigma_2 + it_2) = \Omega\left(\frac{t_1^{3/2-\sigma_1-\sigma_2}}{t_2}\right) \tag{1.10}$$

for $0 < \sigma_1 + \sigma_2 < 1$, $2 \leq t_1 \leq T$ and $2 \leq t_2 \leq T^{(3-2\sigma_1-2\sigma_2)/(5-2\sigma_1-2\sigma_2)-\varepsilon}$, with ε being any small positive constant.

Ikeda, Matsuoka and Nagata made use of the mean value theorems for Dirichlet polynomials and suitable approximate formulas derived from the Euler–Maclaurin summation formula to obtain the formulas (1.2) and (1.3). However Kiuchi and Minamide used some mean value formulas of the Riemann zeta-function for the region $-1 < \sigma < \frac{3}{2}$ and a weak form of the approximate formula of Kiuchi, Tanigawa and Zhai for $\zeta_2(s_1, s_2)$ to obtain the above formulas (1.4)–(1.8). In (1.2) and (1.3), s_2 is a constant, but t_2 is not a constant in (1.4)–(1.8). This difference is important, because the analytic properties of $\zeta_2(s_1, s_2)$ depend on both s_1 and s_2.

1.2. Mean Square Formula—Concerning t_2. The main purpose of this paper is to prove the mean square formulas of the double zeta-function $\zeta_2(s_1, s_2)$ concerning the variable t_2 in place of the variable t_1 within the region $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ and $0 < \sigma_1 + \sigma_2 < \frac{3}{2}$. Some mean square formulas of the double zeta-function $\zeta_2(s_2, s_1)$ with respect to t_1 under the region $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ and $0 < \sigma_1 + \sigma_2 < \frac{3}{2}$ can be derived from the reciprocity law (1.1) and the formulas (1.4)–(1.8). Hence, it is sufficient to calculate the integral $\int_2^T |\zeta_2(s_2, s_1)|^2 \, dt_1$. Then we have the following eleven formulas.

THEOREM 1.1. *Suppose that $2 \leq t_1 \leq T$ and $2 \leq t_2 \leq \frac{T^{1/3}}{\log T}$. For any sufficiently large positive number $T > 2$, we have*

$$\int_2^T |\zeta_2(s_2, s_1)|^2 \, dt_1 = \frac{\zeta(2)}{4\pi|s_2 - 1|^2} T^2 + O(t_2^{-1/2}(\log t_2) T^{3/2}) \tag{1.11}$$

with $\frac{1}{2} < \sigma_1 < 1$, $0 < \sigma_2 < \frac{1}{2}$ and $\sigma_1 + \sigma_2 = 1$.

Taking $t_2 = \frac{T^{1/3}}{\log T}$ into the above, the right-hand side of (1.11) is estimated as $O(T^{4/3} \log^{3/2} T)$, but if we can take $t_2 = T$, then we can estimate that $O(T \log T)$.

Throughout our theorems, the O-constants of whose formulas depend on σ_1 and σ_2, and is independent of t_2. In short, t_2 is not a constant. This fact is one of important observations, because the analytic properties of $\zeta_2(s_1, s_2)$ depend on both variables s_1 and s_2.

Remark 1.1. For $2 \le t_1 \le T$ and $2 \le t_2 \le \frac{T^{1/3}}{\log T}$, we observe that the main terms on the right-hand side of (1.4) and (1.11) are the same one with $\frac{1}{2} < \sigma_1 < 1$, $0 < \sigma_2 < \frac{1}{2}$ and $\sigma_1 + \sigma_2 = 1$, but (1.11) does not coincide with (1.4), namely

$$\int_2^T |\zeta_2(s_1, s_2)|^2 \, dt_1 - \int_2^T |\zeta_2(s_2, s_1)|^2 \, dt_1 = O(t_2^{-1/2} (\log t_2) T^{3/2}). \tag{1.12}$$

Integrating (1.12) by the variable t_2, hence

$$\int_2^N \int_2^T \{|\zeta_2(s_2, s_1)|^2 - |\zeta_2(s_1, s_2)|^2\} \, dt_1 \, dt_2 = O(T^{3/2} N^{1/2} \log N)$$

for $2 < N \le \frac{T^{1/3}}{\log T}$. Then we deduce the double integral for $|\zeta_2(s_1, s_2)|^2$, namely

Corollary 1.1. *We have*

$$\frac{1}{TN} \int_2^N \int_2^T \{|\zeta_2(s_2, s_1)|^2 - |\zeta_2(s_1, s_2)|^2\} \, dt_1 \, dt_2 = O\left(\sqrt{\frac{T}{N}} \log N\right) \tag{1.13}$$

for $\frac{1}{2} < \sigma_1 < 1$, $0 < \sigma_2 < \frac{1}{2}$, $\sigma_1 + \sigma_2 = 1$ *and* $2 \le N \le \frac{T^{1/3}}{\log T}$. *In particular, taking* $N = \frac{T^{1/3}}{\log T}$ *into the above we have*

$$\int_2^N \int_2^T \{|\zeta_2(s_2, s_1)|^2 - |\zeta_2(s_1, s_2)|^2\} \, dt_1 \, dt_2 = O(T^{5/3} (\log T)^{1/2}).$$

Theorem 1.2. *Suppose that* $2 \le t_1 \le T$ *and* $\sigma_1 = \sigma_2 = \frac{1}{2}$. *For any sufficiently large positive number* $T > 2$, *we have*

$$\begin{aligned} \int_2^T |\zeta_2(s_2, s_1)|^2 \, dt_1 &= \frac{\zeta(2)}{4\pi |s_2 - 1|^2} T^2 \\ &\quad + \begin{cases} O(t_2^{-1/2} (\log t_2) T^{3/2}) & \text{if } (\log T)^{3/2} \le t_2 \le \dfrac{T^{1/3}}{\log T}, \\ O(t_2^{-5/6} (\log t_2) T^{3/2} (\log T)^{1/2}) & \text{if } 2 \le t_2 \le (\log T)^{3/2}. \end{cases} \end{aligned} \tag{1.14}$$

Theorem 1.3. *Suppose that* $2 \le t_1 \le T$ *and* $2 \le t_2 \le \frac{T^{1/3}}{\log T}$. *For any sufficiently large positive number* $T > 2$, *we have*

(1.15)
$$\int_2^T |\zeta_2(s_2, s_1)|^2 \, dt_1$$
$$= \frac{\zeta(2)}{4\pi|s_2 - 1|^2} T^2 + O(t_2^{-1/2}(\log t_2) T^{3/2})$$
$$+ O(t_2^{(2/3)\sigma_1}(\log t_2)^4 T^{2-2\sigma_1}) + O(t_2^{-1+(1/3)\sigma_1}(\log t_2)^2 T^{2-\sigma_1})$$

with $0 < \sigma_1 < \frac{1}{2}$, $\frac{1}{2} < \sigma_2 < 1$ *and* $\sigma_1 + \sigma_2 = 1$.

From Theorems 1.1 and 1.2, we immediately derive an alternative proof of the Ω-result in Kiuchi, Tanigawa and Zhai [12], namely

COROLLARY 1.2. *Let* $\sigma_1 + \sigma_2 = 1$, $\frac{1}{2} \le \sigma_1 < 1$, $0 < \sigma_2 \le \frac{1}{2}$ *and* $2 \le t_1 \le T$. *We have*

$$\zeta_2(\sigma_2 + it_2, \sigma_1 + it_1) = \Omega\left(\frac{t_1^{1/2}}{t_2}\right)$$

for $2 \le t_2 \le T^{1/3-\varepsilon}$, *where* ε *is any small positive constant.*

Under the condition $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ and $1 < \sigma_1 + \sigma_2 < \frac{3}{2}$, we shall consider the mean square formula for the double zeta-function $\zeta_2(s_2, s_1)$ with respect to the variable t_1.

THEOREM 1.4. *Suppose that* $2 \le t_1 \le T$, $2 \le t_2 \le T^{1-(2/3)(\sigma_1+\sigma_2)}$, $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ *and* $1 < \sigma_1 + \sigma_2 < \frac{3}{2}$. *For any sufficiently large positive number* $T > 2$, *we have*

(1.16)
$$\int_2^T |\zeta_2(s_2, s_1)|^2 \, dt_1 = (2\pi)^{2\sigma_1+2\sigma_2-3} \frac{\zeta(4 - 2\sigma_1 - 2\sigma_2)}{(4 - 2\sigma_1 - 2\sigma_2)|s_2 - 1|^2} T^{4-2\sigma_1-2\sigma_2}$$
$$+ O(t_2^{-1/2} T^{5/2-\sigma_1-\sigma_2})$$

with $\frac{1}{2} < \sigma_1 < 1$ *and* $0 < \sigma_2 < 1$,

(1.17)
$$\int_2^T |\zeta_2(s_2, s_1)|^2 \, dt_1 = (2\pi)^{2\sigma_2-2} \frac{\zeta(3 - 2\sigma_2)}{(3 - 2\sigma_2)|s_2 - 1|^2} T^{3-2\sigma_2} + O(t_2^{-1/2} T^{2-\sigma_2})$$
$$+ O(t_2^{-2/3-(1/3)\sigma_2}(\log t_2)^2 T^{2-\sigma_2}(\log T)^{1/2})$$

with $\sigma_1 = \frac{1}{2}$, $\frac{1}{2} < \sigma_2 < 1$ *and* $2 \le t_2 \le T^{2/3-(2/3)\sigma_2}$, *and*

$$(1.18)\qquad \int_2^T |\zeta_2(s_2,s_1)|^2\,dt_1$$
$$= (2\pi)^{2\sigma_1+2\sigma_2-3}\frac{\zeta(4-2\sigma_1-2\sigma_2)}{(4-2\sigma_1-2\sigma_2)|s_2-1|^2}T^{4-2\sigma_1-2\sigma_2}$$
$$+ O(t_2^{-1/2}T^{5/2-\sigma_1-\sigma_2}) + O(t_2^{-2/3-(1/3)\sigma_2}(\log t_2)^2T^{3-2\sigma_1-\sigma_2})$$
$$+ O(t_2^{(2/3)(1-\sigma_2)}(\log t_2)^4T^{2-2\sigma_1})$$

with $0<\sigma_1<\frac{1}{2}$ *and* $\frac{1}{2}<\sigma_2<1$.

As two applications of Theorem 1.4, we have the following.

Corollary 1.3. *Let* $1<\sigma_1+\sigma_2<\frac{3}{2}$ *and* $2\le t_1\le T$. *We have*

$$(1.19)\qquad \zeta_2(\sigma_2+it_2,\sigma_1+it_1)=\Omega\left(\frac{t_1^{3/2-\sigma_1-\sigma_2}}{t_2}\right)$$

with $\frac{1}{2}<\sigma_1<1$, $0<\sigma_2<1$, *and* $2\le t_2\le T^{1-(2/3)(\sigma_1+\sigma_2)-\varepsilon}$, *or* $\sigma_1=\frac{1}{2}$, $\frac{1}{2}<\sigma_2<1$ *and* $2\le t_2\le T^{2/3-(2/3)\sigma_2}$.

Remark 1.2. Comparing (1.19) with (1.9), the Ω-result of the right-hand side of (1.19) is same one on the right-hand side of (1.9), but the left-hand side of (1.19) does not coincide with the left-hand side of (1.9). It is observed that the variables s_2 and s_1 on the left-hand side of (1.19) are taken a change of the variables s_1 and s_2 on the left-hand side of (1.9), respectively. Therefore, the Ω-results of (1.19) and (1.9) depend the ratio of the order of t_1 to that of t_2; that is the inequalities $2\le t_1\le T$ and $2\le t_2\le T^{1-(2/3)(\sigma_1+\sigma_2)-\varepsilon}$.

This involves the result of Corollary 1.2. The formula (1.19) provides an improvement upon the Ω result in [12].

Corollary 1.4. *We have*

$$(1.20)\qquad \int_2^N\int_2^T\{|\zeta_2(s_2,s_1)|^2-|\zeta_2(s_1,s_2)|^2\}\,dt_1\,dt_2=O(T^{5/2-\sigma_1-\sigma_2}N^{1/2})$$

for $\frac{1}{2}<\sigma_1<1$, $0<\sigma_2<1$, $1<\sigma_1+\sigma_2<\frac{3}{2}$ *and* $2\le N\le T^{1-(2/3)(\sigma_1+\sigma_2)}$. *In particular, taking* $N=T^{1-(2/3)(\sigma_1+\sigma_2)}$ *and* $\sigma_1+\sigma_2=\frac{3}{2}-\varepsilon$ $(\varepsilon>0)$ *into* (1.20), *we have*

$$\int_2^N\int_2^T\{|\zeta_2(s_2,s_1)|^2-|\zeta_2(s_1,s_2)|^2\}dt_1\,dt_2=O(T^{1+\varepsilon}).$$

Furthermore, we shall consider the mean square formula for the double zeta-function $\zeta_2(s_2, s_1)$ with respect to the variable t_1 for the region $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ and $0 < \sigma_1 + \sigma_2 < 1$.

THEOREM 1.5. *Suppose that* $2 \le t_1 \le T$, $2 \le t_2 \le T^{(3-2\sigma_1-2\sigma_2)/(5-2\sigma_1-2\sigma_2)}$, $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ *and* $\frac{1}{2} < \sigma_1 + \sigma_2 < 1$. *For any sufficiently large positive number* $T > 2$, *we have*

$$\int_2^T |\zeta_2(s_2, s_1)|^2 \, dt_1 = (2\pi)^{2\sigma_1-2\sigma_2-3} \frac{\zeta(4-2\sigma_1-2\sigma_2)}{(4-2\sigma_1-2\sigma_2)|s_2-1|^2} T^{4-2\sigma_1-2\sigma_2} + O(t_2^{-1/2-(2/3)\sigma_2}(\log t_2) T^{5/2-\sigma_1-\sigma_2}) + O(t_2^{1-(4/3)\sigma_2}(\log t_2)^2 T) + O(t_2^{1/2-\sigma_1-\sigma_2} T^{5/2-\sigma_1-\sigma_2}) \tag{1.21}$$

with $\frac{1}{2} < \sigma_1 < 1$ *and* $0 < \sigma_2 < \frac{1}{2}$,

$$\int_2^T |\zeta_2(s_2, s_1)|^2 \, dt_1 = (2\pi)^{2\sigma_2-2} \frac{\zeta(3-2\sigma_2)}{(3-2\sigma_2)|s_2-1|^2} T^{3-2\sigma_2} + O(t_2^{-\sigma_2} T^{2-\sigma_2}) + O(t_2^{1-(4/3)\sigma_2}(\log t_2)^2 T \log T) + O(t_2^{-1/2-(2/3)\sigma_2}(\log t_2) T^{2-\sigma_2}(\log T)^{1/2}) \tag{1.22}$$

with $\sigma_1 = \frac{1}{2}$ *and* $0 < \sigma_2 < \frac{1}{2}$,

$$\int_2^T |\zeta_2(s_2, s_1)|^2 \, dt_1 = (2\pi)^{2\sigma_1+2\sigma_2-3} \frac{\zeta(4-2\sigma_1-2\sigma_2)}{(4-2\sigma_1-2\sigma_2)|s_2-1|^2} T^{4-2\sigma_1-2\sigma_2} + O(t_2^{-1/2-(2/3)\sigma_2}(\log t_2) T^{3-2\sigma_1-\sigma_2}) + O(t_2^{1-(4/3)\sigma_2}(\log t_2)^2 T^{2-2\sigma_1}) + O(t_2^{1/2-\sigma_1-\sigma_2} T^{5/2-\sigma_1-\sigma_2}) \tag{1.23}$$

with $0 < \sigma_1 < \frac{1}{2}$ *and* $0 < \sigma_2 < \frac{1}{2}$,

$$\int_2^T |\zeta_2(s_2, s_1)|^2 \, dt_1 = (2\pi)^{2\sigma_1-2} \frac{\zeta(3-2\sigma_1)}{(3-2\sigma_1)|s_2-1|^2} T^{3-2\sigma_1} + O(t_2^{-5/6}(\log t_2) T^{5/2-2\sigma_1}) + O(t_2^{1/3}(\log t_2)^2 T^{2-2\sigma_1}) + O(t_2^{-\sigma_1} T^{2-\sigma_1}) \tag{1.24}$$

with $0 < \sigma_1 < \frac{1}{2}$ *and* $\sigma_2 = \frac{1}{2}$, *and*

$$(1.25)\qquad \int_2^T |\zeta_2(s_2, s_1)|^2\, dt_1$$

$$= (2\pi)^{2\sigma_1+2\sigma_2-3} \frac{\zeta(4-2\sigma_1-2\sigma_2)}{(4-2\sigma_1-2\sigma_2)|s_2-1|^2} T^{4-2\sigma_1-2\sigma_2}$$

$$+ O(t_2^{-2/3-(1/3)\sigma_2}(\log t_2)^2 T^{3-2\sigma_1-\sigma_2}) + O(t_2^{(2/3)(1-\sigma_2)}(\log t_2)^4 T^{2-2\sigma_1})$$

$$+ O(t_2^{1/2-\sigma_1-\sigma_2} T^{5/2-\sigma_1-\sigma_2})$$

with $0 < \sigma_1 < \frac{1}{2}$ *and* $\frac{1}{2} < \sigma_2 < 1$.

Theorem 1.6. *Suppose that* $2 \le t_1 \le T$, $2 \le t_2 \le T^{1/2}$, $0 < \sigma_1 < \frac{1}{2}$, $0 < \sigma_2 < \frac{1}{2}$ *and* $\sigma_1 + \sigma_2 = \frac{1}{2}$. *For any sufficiently large positive number* $T > 2$, *we have*

$$(1.26)\qquad \int_2^T |\zeta_2(s_2, s_1)|^2\, dt_1$$

$$= \frac{\zeta(3)}{12\pi^2|s_2-1|^2} T^3 + O(t_2^{-1/2-(2/3)\sigma_2}(\log t_2) T^{5/2-\sigma_1})$$

$$+ O(t_2^{1-(4/3)\sigma_2}(\log t_2)^2 T^{2-2\sigma_1}) + O(T^2) + O(t_2^{-1} T^2 \sqrt{\log T}).$$

Theorem 1.7. *Suppose that* $2 \le t_1 \le T$, $2 \le t_2 \le T^{(3-2\sigma_1-2\sigma_2)/(5-2\sigma_1-2\sigma_2)}$, $0 < \sigma_1 < \frac{1}{2}$, $0 < \sigma_2 < \frac{1}{2}$ *and* $0 < \sigma_1 + \sigma_2 < \frac{1}{2}$. *For any sufficiently large positive number* $T > 2$, *we have*

$$(1.27)\qquad \int_2^T |\zeta_2(s_2, s_1)|^2\, dt_1$$

$$= (2\pi)^{2\sigma_1+2\sigma_2-3} \frac{\zeta(4-2\sigma_1-2\sigma_2)}{(4-2\sigma_1-4\sigma_2)|s_2-1|^2} T^{4-2\sigma_1-2\sigma_2}$$

$$+ O(t_2^{-1/2-(2/3)\sigma_2}(\log t_2) T^{3-2\sigma_1-\sigma_2}) + O(t_2^{1-(4/3)\sigma_2}(\log t_2)^2 T^{2-2\sigma_1})$$

$$+ O(t_2^{1/2-\sigma_1-\sigma_2} T^{5/2-\sigma_1-\sigma_2}) + O(t_2^{-1} T^{3-2\sigma_1-2\sigma_2}).$$

Notations. When $g(x)$ is a positive function of x for $x \ge x_0$, $f(x) = \Omega(g(x))$ means that $f(x) = o(g(x))$ does not hold as $x \to \infty$. In what follows, ε denotes any arbitrarily small positive number, not necessarily the same ones at each occurrence.

2. Other Mean Square Formula

For any large positive number $T > 2$, Ikeda, Matsuoka and Nagata [5] first studied a new type of mean values formula of $\int_2^T |\zeta_2(s_1, s_2)|^2 \, dt$, where $s_1 = \sigma_1 + it$ and $s_2 = \sigma_2 + it$. By using the approximate forms for the Dirichlet polynomials and technique of calculation for multiple sums, they showed that

$$\int_2^T |\zeta_2(s_1, s_2)|^2 \, dt = \left\{ \sum_{k=2}^{\infty} \left(\sum_{\substack{mn=k \\ m<n}} \frac{1}{m^{\sigma_1} n^{\sigma_2}} \right)^2 \right\} T + O(T^{4-2\sigma_1-2\sigma_2+\varepsilon}) + O(T^{1/2}) \tag{2.1}$$

for $\sigma_1 \le 1$ and $\frac{3}{2} < \sigma_1 + \sigma_2 < 2$, where the coefficient (the double sum) of the first term on the right-hand side of (2.1) converges for $\sigma_1 > \frac{1}{2}$ and $\sigma_1 + \sigma_2 > 1$. The method of their proof is standard version, but which is very complexity.

Secondly, we consider the mean value formulas for the double zeta-function $\zeta_2(s_1, s_2)$ with respect to the variable t for $s_1 = \sigma_1 + it$ and $s_2 = \sigma_2 + it$ in the region $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ and $0 < \sigma_1 + \sigma_2 < \frac{3}{2}$, whose proof makes use of the method of Kiuchi and Minamide [10] to obtain the mean square formulas for the double zeta-function $\zeta_2(s_1, s_2)$ under the region $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$, and $0 < \sigma_1 + \sigma_2 < \frac{3}{2}$. Then we derive the following formula.

THEOREM 2.1. *Suppose that* $0 < \sigma_1 < 1$ *and* $0 < \sigma_2 < 1$. *For any sufficiently large positive number* $T > 2$, *we have*

$$\int_2^T |\zeta_2(\sigma_1 + it, \sigma_2 + it)|^2 \, dt = \begin{cases} O(T^{4-2\sigma_1-2\sigma_2}) & \text{if } 0 < \sigma_1 + \sigma_2 < 1, \\ O(T^2 \log^2 T) & \text{if } \sigma_1 + \sigma_2 = 1, \\ O(T^2) & \text{if } 1 < \sigma_1 + \sigma_2 < \frac{3}{2}. \end{cases} \tag{2.2}$$

To improve this theorem, we must be obtained the sharper estimate for the function $E(s_1, s_2)$ in Lemma 3.1 below, but it is very difficult to find the sharper estimate of the function $E(s_1, s_2)$.

3. Some Lemmas

Kiuchi and Minamide used a weak form of the approximate formula of Kiuchi, Tanigawa and Zhai [12] to prove the formulas (1.4)–(1.8). We use the reciprocity relation (1.1), the formulas (1.4)–(1.8), the estimates (upper bound) for the Riemann zeta-function and the mean value formulas for the Riemann zeta-function to prove our theorems. The estimate of the function $E(s_1, s_2)$ mentioned in Lemma 3.1 was derived by Kiuchi and Minamide, who used the simpler estimation for the partial sum of the divisor function to get the estimate (3.2), whose lemma is only used for the proof of Theorem 2.1, and is not used for the proofs of Theorems 1.1–1.7. However, we use the reciprocity relation (1.1) to obtain the mean value formulas of the double zeta-function $\zeta_2(s_2, s_1)$ with respect to t_1 for $0 < \sigma_1 < 1$ and $0 < \sigma_2 < 1$.

Lemma 3.1. *Suppose that* $0 < \sigma_1 < 1$ *and* $0 < \sigma_2 < 1$. *Then we have*

$$\zeta_2(s_1, s_2) = \frac{\zeta(s_1 + s_2 - 1)}{s_2 - 1} - \frac{1}{2}\zeta(s_1 + s_2) + E(s_1, s_2) \tag{3.1}$$

where the error term $E(s_1, s_2)$ *is estimated as*

$$E(s_1, s_2) \ll \begin{cases} |t_2|^{3/2-\sigma_1-\sigma_2} & \text{if } 0 < \sigma_1 + \sigma_2 < 1, \\ |t_2|^{1/2} \log|t_2| & \text{if } \sigma_1 + \sigma_2 = 1, \\ |t_2|^{1/2} & \text{if } \sigma_1 + \sigma_2 > 1. \end{cases} \tag{3.2}$$

Note that this error term $E(s_1, s_2)$ is independent of t_1.

Lemma 3.2. *For* $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ *and any sufficiently large number* $T > 2$, *we have*

$$\int_2^T |\zeta_2(s_2, s_1)|^2 \, dt_1 = J_1 + J_2 + O(J_1^{1/2} J_2^{1/2}), \tag{3.3}$$

where

$$J_1 = \int_2^T |\zeta_2(s_1, s_2)|^2 \, dt_1, \tag{3.4}$$

and

$$J_2 = |\zeta(s_2)|^2 \int_2^T |\zeta(s_1)|^2 \, dt_1. \tag{3.5}$$

Proof. Using the reciprocity law (1.1), namely

$$\zeta_2(s_2, s_1) = \zeta(s_1)\zeta(s_2) - \zeta(s_1 + s_2) - \zeta_2(s_1, s_2)$$

and Schwarz's inequality, we deduce the following formula

$$\int_2^T |\zeta_2(s_2, s_1)|^2 \, dt_1 = J_1 + J_2 + J_3 + O(J_1^{1/2}J_2^{1/2} + J_2^{1/2}J_3^{1/2} + J_3^{1/2}J_1^{1/2})$$

within the region $0 < \sigma_1 < 1$ and $0 < \sigma_2 < 1$, and the integrals $J_1 = \int_2^T |\zeta_2(s_1, s_2)|^2 \, dt_1$, $J_2 = |\zeta(s_2)|^2 \int_2^T |\zeta(s_1)|^2 \, dt_1$ and $J_3 = \int_2^T |\zeta(s_1 + s_2)|^2 \, dt_1$. Now, using the analytic property of the mean value theorems for the Riemann zeta-function (see Lemma 3.3 below), there is a positive constant $c = c(\alpha, \beta)$ depending on α and β such that the inequality

$$\int_2^{T+\xi} |\zeta(\alpha + it)|^2 \, dt \le c \int_2^T |\zeta(\beta + it)|^2 \, dt \tag{3.6}$$

holds for $0 < \beta < \alpha < 2$ and $2 \le \xi \le T$. This constant c is independent of ξ. Thus, from (3.6) we obtain $J_3 = O(J_2)$, completing the proof of (3.3). □

We shall take the proof of the inequality (3.6) behind that of Lemma 3.3.

To deal with the integral (3.5), we need the following Lemma 3.3, which is the mean square formulas of the Riemann zeta-function for $-1 < \sigma < 2$.

LEMMA 3.3. *For any sufficiently large positive number* $T > 2$, *we have*

$$\int_2^T |\zeta(\sigma + it)|^2 \, dt = \zeta(2\sigma)T + O(1) \tag{3.7}$$

with $\sigma > 1$,

$$\int_2^T |\zeta(1 + it)|^2 \, dt = \zeta(2)T + O(\log T) \tag{3.8}$$

with $\sigma = 1$,

$$\int_2^T |\zeta(\sigma + it)|^2 \, dt = \zeta(2\sigma)T + (2\pi)^{2\sigma-1} \frac{\zeta(2-2\sigma)}{2-2\sigma} T^{2-2\sigma} + O(T^{(2/3)(1-\sigma)} \log^{2/9} T) \tag{3.9}$$

with $\frac{1}{2} < \sigma < 1$,

$$\int_2^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^2 dt = T \log \frac{T}{2\pi} + (2\gamma - 1)T + O(T^{1/2}) \tag{3.10}$$

with $\sigma = \frac{1}{2}$ *and the Euler constant* γ,

$$\int_2^T |\zeta(\sigma + it)|^2 \, dt = (2\pi)^{2\sigma-1} \frac{\zeta(2-2\sigma)}{2-2\sigma} T^{2-2\sigma} + \zeta(2\sigma)T + O(T^{1-\sigma}) \tag{3.11}$$

with $0 < \sigma < \frac{1}{2}$,

$$\int_2^T |\zeta(it)|^2 \, dt = \frac{\zeta(2)}{4\pi} T^2 + O(T \log T) \tag{3.12}$$

with $\sigma = 0$, *and*

$$\int_2^T |\zeta(\sigma + it)|^2 \, dt = (2\pi)^{2\sigma-1} \frac{\zeta(2-2\sigma)}{2-2\sigma} T^{2-2\sigma} + O(T^{1-2\sigma}) \tag{3.13}$$

with $-1 < \sigma < 0$.

Proof. The formulas (3.7) and (3.10) follow from the standard texts (see [7], [8] and [17]). From [4] and [9], we get the formulas (3.8) and (3.9), respectively. The formulas (3.11)–(3.13) derive from Lemma 2 in [10]. □

Proof of (3.6). We assume that $0 < \beta < \alpha < 2$ and $2 \le \xi \le T$. From Lemma 3.3, one can see that

$$\int_2^T |\zeta(\alpha + it)|^2 \, dt < c_1 \int_2^T |\zeta(\beta + it)|^2 \, dt$$

for any sufficiently large positive number $T > 2$ and some positive constant c_1. Thus, to prove (3.6) it is enough to show some positive constant c_2 such that

$$\int_T^{T+\xi} |\zeta(\alpha + it)|^2 \, dt < c_2 \int_2^T |\zeta(\beta + it)|^2 \, dt. \tag{3.14}$$

From Lemma 3.3, we have

$$\int_T^{T+\xi} |\zeta(\alpha + it)|^2 \, dt = \begin{cases} \zeta(2\alpha)\xi + O(1) & \text{if } \alpha > 1, \\ \zeta(2)\xi + O(\log T) & \text{if } \alpha = 1, \\ \zeta(2\alpha)\xi + O(T^{2-2\alpha}) & \text{if } \frac{1}{2} < \alpha < 1, \\ O(T \log T) & \text{if } \alpha = \frac{1}{2}, \\ O(T^{2-2\alpha}) & \text{if } 0 < \alpha < \frac{1}{2}. \end{cases}$$

Since $0 < \beta < \alpha < 2$ and $2 \le \xi \le T$, it follows from the above that we obtain the inequality (3.14). □

Furthermore, we need an upper bound for the Riemann zeta-function in the critical strip to prove Theorems 1.1–1.7, which is given by Lemma 2.5 in Kiuchi and Tanigawa [11] (see also Ivić [7], Titchmarsh [17]).

LEMMA 3.4. *For $t \ge t_0 > 1$ uniformly in σ, we have*

$$\zeta(\sigma + it) \ll \begin{cases} t^{(1/3)(1-\sigma)} \log^2 t & \text{if } \frac{1}{2} < \sigma \le 1, \\ t^{1/6} \log t & \text{if } \sigma = \frac{1}{2}, \\ t^{1/2-(2/3)\sigma} \log t & \text{if } 0 \le \sigma < \frac{1}{2}. \end{cases} \tag{3.15}$$

4. Proofs of Theorems

PROOFS OF THEOREMS 1.1–1.3. We shall evaluate the integral $\int_2^T |\zeta_2(s_2, s_1)|^2 \, dt_1$ under the condition $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ and $\sigma_1 + \sigma_2 = 1$. From (3.5), Lemmas 3.3 and 3.4, we have

$$\begin{aligned} J_2 &= |\zeta(s_2)|^2 \int_2^T |\zeta(s_1)|^2 \, dt_1 \\ &\ll \begin{cases} t_2^{1-(4/3)\sigma_2} (\log t_2)^2 T & \text{if } \frac{1}{2} < \sigma_1 < 1 \text{ and } 0 < \sigma_2 < \frac{1}{2}, \\ t_2^{1/3} (\log t_2)^2 T \log T & \text{if } \sigma_1 = \sigma_2 = \frac{1}{2}, \\ t_2^{(2/3)(1-\sigma_2)} (\log t_2)^4 T^{2-2\sigma_1} & \text{if } 0 < \sigma_1 < \frac{1}{2} \text{ and } \frac{1}{2} < \sigma_2 < 1. \end{cases} \end{aligned} \tag{4.1}$$

From (1.4) and (3.4), we have

$$J_1 = \frac{\zeta(2)}{4\pi |s_2 - 1|^2} T^2 + O(t_2^{-1/2} (\log t_2) T^{3/2}) \tag{4.2}$$

with $2 \le t_2 \le \frac{T^{1/3}}{\log T}$. Substituting (4.1) and (4.2) into (3.3), we observe that all error terms on the right-hand side of (3.3) are absorbed into $O(t_2^{-1/2} (\log t_2) T^{3/2})$ if $\frac{1}{2} < \sigma_1 < 1$, or into $O(t_2^{-1/2} (\log t_2) T^{3/2}) + O(t_2^{-5/6} (\log t_2) T^{3/2} (\log T)^{1/2})$ if $\sigma_1 = \frac{1}{2}$, or into $O(t_2^{-1/2} (\log t_2) T^{3/2}) + O(t_2^{(2/3)\sigma_1} (\log t_2)^4 T^{2-2\sigma_1}) + O(t_2^{-1+(1/3)\sigma_1} \cdot (\log t_2)^2 T^{2-\sigma_1})$ if $0 < \sigma_1 < \frac{1}{2}$. Hence, we obtain the formula (1.11), (1.14) and (1.15). □

PROOF OF THEOREM 1.4. In a similar manner to above, we shall evaluate the integral.

In a similar manner to above, we shall evaluate the integral $\int_2^T |\zeta_2(s_2,s_1)|^2\,dt_1$ under the condition $0<\sigma_1<1$, $0<\sigma_2<1$ and $1<\sigma_1+\sigma_2<\frac{3}{2}$. Then, by (3.5), Lemmas 3.3 and 3.4 we have

$$(4.3)\qquad J_2 \ll \begin{cases} t_2^{1-(4/3)\sigma_2}(\log t_2)^2 T & \text{if } \frac{1}{2}<\sigma_1<1 \text{ and } 0<\sigma_2<\frac{1}{2},\\ t_2^{1/3}(\log t_2)^2 T & \text{if } \frac{1}{2}<\sigma_1<1 \text{ and } \sigma_2=\frac{1}{2},\\ t_2^{(2/3)(1-\sigma_2)}(\log t_2)^4 T & \text{if } \frac{1}{2}<\sigma_1<1 \text{ and } \frac{1}{2}<\sigma_2<1,\\ t_2^{(2/3)(1-\sigma_2)}(\log t_2)^4 T\log T & \text{if } \sigma_1=\frac{1}{2} \text{ and } \frac{1}{2}<\sigma_2<1,\\ t_2^{(2/3)(1-\sigma_2)}(\log t_2)^4 T^{2-2\sigma_1} & \text{if } 0<\sigma_1<\frac{1}{2} \text{ and } \frac{1}{2}<\sigma_2<1. \end{cases}$$

From (1.5) and (3.4) we have

$$(4.4)\qquad J_1 = (2\pi)^{2\sigma_1+2\sigma_2-3}\frac{\zeta(4-2\sigma_1-2\sigma_2)}{(4-2\sigma_1-2\sigma_2)|s_2-1|^2}T^{4-2\sigma_1-2\sigma_2} + O(t_2^{-1/2}T^{5/2-\sigma_1-\sigma_2})$$

for $2\le t_2\le T^{1-(2/3)(\sigma_1+\sigma_2)}$. Substituting (4.3) and (4.4) into (3.3), we observe that all error terms on the right-hand side of (3.3) are absorbed into $O(t_2^{-1/2}T^{5/2-\sigma_1-\sigma_2})$ if $\frac{1}{2}<\sigma_1<1$ and $0<\sigma_2<1$, or into $O(t_2^{-1/2}T^{2-\sigma_2})+O(t_2^{-2/3-(1/3)\sigma_2}(\log t_2)^2T^{2-\sigma_2}(\log T)^{1/2})$ if $\sigma_1=\frac{1}{2}$ and $\frac{1}{2}<\sigma_2<1$, or into $O(t_2^{-1/2}T^{5/2-\sigma_1-\sigma_2})+O(t_2^{-2/3-(1/3)\sigma_2}(\log t_2)^2T^{3-2\sigma_1-\sigma_2})+O(t_2^{(2/3)(1-\sigma_2)}(\log t_2)^4T^{2-2\sigma_1})$ if $0<\sigma_1<\frac{1}{2}$ and $\frac{1}{2}<\sigma_2<1$. We derive the formulas (1.16), (1.17) and (1.18). □

Proof of Theorem 1.5. We shall evaluate the integral $\int_2^T |\zeta_2(s_2,s_1)|^2\,dt_1$ under the condition $0<\sigma_1<1$, $0<\sigma_2<1$, $\frac{1}{2}<\sigma_1+\sigma_2<1$ and $2\le t_2\le T^{(3-2\sigma_1-2\sigma_2)/(5-2\sigma_1-2\sigma_2)}$. From (3.5), Lemmas 3.3 and 3.4 we have

$$(4.5)\qquad J_2 \ll \begin{cases} t_2^{1-(4/3)\sigma_2}(\log t_2)^2 T & \text{if } \frac{1}{2}<\sigma_1<1 \text{ and } 0<\sigma_2<\frac{1}{2},\\ t_2^{1-(4/3)\sigma_2}(\log t_2)^2 T\log T & \text{if } \sigma_1=\frac{1}{2} \text{ and } 0<\sigma_2<\frac{1}{2},\\ t_2^{1-(4/3)\sigma_2}(\log t_2)^2 T^{2-2\sigma_1} & \text{if } 0<\sigma_1<\frac{1}{2} \text{ and } 0<\sigma_2<\frac{1}{2},\\ t_2^{1/3}(\log t_2)^2 T^{2-2\sigma_1} & \text{if } 0<\sigma_1<\frac{1}{2} \text{ and } \sigma_2=\frac{1}{2},\\ t_2^{(2/3)(1-\sigma_2)}(\log t_2)^4 T^{2-2\sigma_1} & \text{if } 0<\sigma_1<\frac{1}{2} \text{ and } \frac{1}{2}<\sigma_2<1. \end{cases}$$

From (1.6) and (3.4), we have

$$(4.6)\qquad J_1 = (2\pi)^{2\sigma_1+2\sigma_2-3}\frac{\zeta(4-2\sigma_1-2\sigma_2)}{(4-2\sigma_1-2\sigma_2)|s_2-1|^2}T^{4-2\sigma_1-2\sigma_2} + O(t_2^{1/2-\sigma_1-\sigma_2}T^{5/2-\sigma_1-\sigma_2}).$$

From (4.5) and (4.6) we have

(4.7) $J_1^{1/2}J_2^{1/2}$

$$\ll \begin{cases} t_2^{-1/2-(2/3)\sigma_2}(\log t_2)T^{5/2-\sigma_1-\sigma_2} & \text{if } \frac{1}{2}<\sigma_1<1 \text{ and } 0<\sigma_2<\frac{1}{2}, \\ t_2^{-1/2-(2/3)\sigma_2}(\log t_2)T^{2-\sigma_2}\sqrt{\log T} & \text{if } \sigma_1=\frac{1}{2} \text{ and } 0<\sigma_2<\frac{1}{2}, \\ t_2^{-1/2-(2/3)\sigma_2}(\log t_2)T^{3-2\sigma_1-\sigma_2} & \text{if } 0<\sigma_1<\frac{1}{2} \text{ and } 0<\sigma_2<\frac{1}{2}, \\ t_2^{-5/6}(\log t_2)T^{5/2-2\sigma_1} & \text{if } 0<\sigma_1<\frac{1}{2} \text{ and } \sigma_2=\frac{1}{2}, \\ t_2^{-2/3-(1/3)\sigma_2}(\log t_2)^2T^{3-2\sigma_1-\sigma_2} & \text{if } 0<\sigma_1<\frac{1}{2} \text{ and } \frac{1}{2}<\sigma_2<1. \end{cases}$$

Substituting (4.5), (4.6) and (4.7) into (3.3), we obtain the formulas (1.21), (1.22), (1.23), (1.24) and (1.25). □

Proof of Theorem 1.6. We shall evaluate the integral $\int_2^T |\zeta_2(s_2,s_1)|^2\,dt_1$ under the condition $0<\sigma_1<\frac{1}{2}$, $0<\sigma_2<\frac{1}{2}$ and $\sigma_1+\sigma_2=\frac{1}{2}$. Then, from (3.5), Lemmas 3.3 and 3.4 we have

$$J_2 \ll t_2^{1-(4/3)\sigma_2}(\log t_2)^2T^{2-2\sigma_1}. \tag{4.8}$$

From (1.7) and (3.4), we have

$$J_1 = \frac{\zeta(3)}{12\pi^2|s_2-1|^2}T^3 + \begin{cases} O(T^2) & \text{if } \sqrt{\log T}\le t_2\le T^{1/2}, \\ O(t_2^{-1}T^2\sqrt{\log T}) & \text{if } 2\le t_2\le\sqrt{\log T}. \end{cases} \tag{4.9}$$

We have, from (4.8) and (4.9)

$$J_1^{1/2}J_2^{1/2} \ll t_2^{-1/2-(2/3)\sigma_2}(\log t_2)T^{5/2-\sigma_1}. \tag{4.10}$$

Substituting (4.8), (4.9) and (4.10) into (3.3), we have the formula (1.26) within $2\le t_2\le T^{1/2}$. □

Proof of Theorem 1.7. We shall evaluate the integral $\int_2^T |\zeta_2(s_2,s_1)|^2\,dt_1$ under the condition $0<\sigma_1<\frac{1}{2}$, $0<\sigma_2<\frac{1}{2}$ and $0<\sigma_1+\sigma_2<\frac{1}{2}$. Then, from (3.5), Lemmas 3.3 and 3.4 we have

$$J_2 \ll t_2^{1-(4/3)\sigma_2}(\log t_2)^2T^{2-2\sigma_1}. \tag{4.11}$$

From (1.8) and (3.4), we have

$$(4.12)\qquad J_1 = (2\pi)^{2\sigma_1+2\sigma_2-3}\frac{\zeta(4-2\sigma_1-2\sigma_2)}{(4-2\sigma_1-2\sigma_2)|s_2-1|^2}T^{4-2\sigma_1-2\sigma_2}$$
$$+\begin{cases} O(t_2^{1/2-\sigma_1-\sigma_2}T^{5/2-\sigma_1-\sigma_2}) & \text{if } T^{(1-2\sigma_1-2\sigma_2)/(3-2\sigma_1-2\sigma_2)} \le t_2 \\ & \quad \le T^{(3-2\sigma_1-2\sigma_2)/(5-2\sigma_1-2\sigma_2)}, \\ O(t_2^{-1}T^{3-2\sigma_1-2\sigma_2}) & \text{if } 2 \le t_2 \le T^{(1-2\sigma_1-2\sigma_2)/(3-2\sigma_1-2\sigma_2)}. \end{cases}$$

From (4.11) and (4.12), we have

$$(4.13)\qquad J_1^{1/2}J_2^{1/2} \ll t_2^{-1/2-(2/3)\sigma_2}(\log t_2)T^{3-2\sigma_1-\sigma_2}.$$

Hence, we substitute (4.11), (4.12) and (4.13) into (3.3) to obtain the formula (1.27). □

Proof of Theorem 2.1. Assume that $s_1 = \sigma_1 + it$, $s_2 = \sigma_2 + it$, $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ and $0 < \sigma_1 + \sigma_2 < \frac{3}{2}$. From (3.1), we have

$$(4.14)\qquad \int_2^T |\zeta_2(s_1,s_2)|^2\,dt = K_1 + K_2 + K_3 + O(K_1^{1/2}K_2^{1/2} + K_2^{1/2}K_3^{1/2} + K_3^{1/2}K_1^{1/2}),$$

where $K_1 = \int_2^T \frac{|\zeta(s_1+s_2-1)|^2}{|s_2-1|^2}\,dt$, $K_2 = \frac{1}{4}\int_2^T |\zeta(s_1+s_2)|^2\,dt$ and $K_3 = \int_2^T |E(s_1,s_2)|^2\,dt$. From (3.2), we have

$$(4.15)\qquad K_3 = \begin{cases} O(T^{4-2\sigma_1-2\sigma_2}) & \text{if } 0 < \sigma_1+\sigma_2 < 1, \\ O(T^2\log^2 T) & \text{if } \sigma_1+\sigma_2 = 1, \\ O(T^2) & \text{if } 1 < \sigma_1+\sigma_2 < \frac{3}{2}. \end{cases}$$

From (3.7)–(3.12), we have

$$(4.16)\qquad K_2 = \begin{cases} O(T^{2-2\sigma_1-2\sigma_2}) & \text{if } 0 < \sigma_1+\sigma_2 < \frac{1}{2}, \\ O(T\log T) & \text{if } \sigma_1+\sigma_2 = \frac{1}{2}, \\ O(T) & \text{if } \frac{1}{2} < \sigma_1+\sigma_2 < \frac{3}{2}. \end{cases}$$

Integrating by parts and Lemma 3.3, it follows that

$$(4.17)\qquad K_1 = O\left(\max_{\substack{0<\sigma_1<1,\,0<\sigma_2<1\\ 0<\sigma_1+\sigma_2<3/2}}\{1, T^{2-2\sigma_1-2\sigma_2}\}\right).$$

Substituting (4.15), (4.16) and (4.17) into (4.14), we derive the formula (2.2). □

Using Lemma 3.3, we can easily calculate more precise formulas of K_1 and K_2, however, we can not obtain a more accurate formula for K_3 by Lemma 3.1.

Therefore we need the exact expression of the asymptotic formula of $E(s_1, s_2)$ for $0 < \sigma_1 < 1$ and $0 < \sigma_2 < 1$ to obtain more accurate asymptotic formulas for $\int_2^T |\zeta_2(s_1, s_2)|^2 \, dt$.

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