TSUKUBA J. MATH.
Vol. 41 No. 1 (2017), 21–41

ON THE GROUPS OF ISOMETRIES OF SIMPLE PARA-HERMITIAN SYMMETRIC SPACES

By

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Abstract. The main purpose in this paper is to completely determine the groups of isometries of simple para-Hermitian symmetric spaces. That enables us to also determine the groups of affine transformations of these spaces, with respect to the canonical affine connections.

1. Introduction and the Main Result in This Paper

This paper reports the following:

THEOREM 1. *Let G be a connected absolutely simple Lie group whose center is trivial, and let $(G/H, \sigma, I, g)$ be a para-Hermitian symmetric space of hyperbolic orbit type. Then,*

(1) *the metric g is the G-invariant extension of the Killing form of* $\mathrm{Lie}(G)$ *up to constant;*
(2) *when ∇^1 denotes the canonical affine connection on $(G/H, \sigma)$, the group $\mathrm{I}(G/H, g)$ of isometries coincides with the group $\mathrm{A}(G/H, \nabla^1)$ of affine transformations;*
(3) *the quotient group $\mathrm{I}(G/H, g)/\mathrm{I}(G/H, g)_0$ is determined as in Table* 1, *where $\mathrm{I}(G/H, g)_0$ is the identity component of $\mathrm{I}(G/H, g)$.*

Since the seminal pioneering work of K. Nomizu [No1], the theory of (affine) symmetric spaces has evolved. As a symmetric space $(G/H, \sigma)$ with a G-invariant para-complex structure I and with a G-invariant para-Hermitian metric g,

2010 *Mathematics Subject Classification*: Primary 53C35, Secondary 17B20.
Key words and phrases: para-Hermitian symmetric space, hyperbolic orbit type, absolutely simple Lie algebra, real form, group of isometries.
Received April 4, 2015.
Revised February 14, 2017.

Table 1. Structures of $\mathrm{I}(G/H, g)/\mathrm{I}(G/H, g)_0$

Type	$(\mathfrak{g}, \mathfrak{h})$	The root system	Z	Condition	$\mathrm{I}(G/H, g)/\mathrm{I}(G/H, g)_0$
AI	$(\mathfrak{sl}(n, \mathbf{R}), \mathfrak{sl}(k, \mathbf{R}) \oplus \mathfrak{sl}(n-k, \mathbf{R}) \oplus \mathbf{R})$	$\mathfrak{a}_{n-1}$	Z_k	$n \geq 3$, odd; $1 \leq k \leq n-1$	$\mathbf{Z}_2$
				$n \geq 4$, even; $1 \leq k \leq n-1, k \neq n/2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$
				$n \geq 4$, even; $k = n/2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
AII	$(\mathfrak{su}^*(2n), \mathfrak{su}^*(2k) \oplus \mathfrak{su}^*(2n-2k) \oplus \mathbf{R})$	$\mathfrak{a}_{n-1}$	Z_k	$n \geq 2$; $1 \leq k \leq n-1, k \neq n/2$	$\mathbf{Z}_2$
				$n \geq 2$, even; $k = n/2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$
AIII	$(\mathfrak{su}(n, n), \mathfrak{sl}(n, \mathbf{C}) \oplus \mathbf{R})$	$\mathfrak{c}_n$	Z_n	$n \geq 2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
AIV	$(\mathfrak{su}(1, 1), \mathbf{R})$	$\mathfrak{a}_1$	Z_1	—	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$
BI	$(\mathfrak{so}(p, q), \mathfrak{so}(p-1, q-1) \oplus \mathbf{R})$	$\mathfrak{b}_q$	Z_1	$p + q \geq 5$, odd; $2 \leq q < p$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$
BII	$(\mathfrak{so}(p, 1), \mathfrak{so}(p-1) \oplus \mathbf{R})$	$\mathfrak{a}_1$	Z_1	$p \geq 2$, even	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$
DI	$(\mathfrak{so}(p, q), \mathfrak{so}(p-1, q-1) \oplus \mathbf{R})$	$\mathfrak{b}_q$	Z_1	$p + q \geq 5$, even; $3 \leq q < p$, odd, respectively	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$
				$p + q \geq 6$, even; $2 \leq q < p$, even, respectively	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
	$(\mathfrak{so}(n, n), \mathfrak{so}(n-1, n-1) \oplus \mathbf{R})$	$\mathfrak{d}_n$	Z_1	$n \geq 5$, odd	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$
				$n \geq 4$, even	$\mathbf{Z}_2 \oplus ((\mathbf{Z}_2 \oplus \mathbf{Z}_2) \rtimes \mathbf{Z}_2)$
	$(\mathfrak{so}(n, n), \mathfrak{sl}(n, \mathbf{R}) \oplus \mathbf{R})$	$\mathfrak{d}_n$	Z_{n-1}, Z_n	$n \geq 5$, odd	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$
				$n \geq 6$, even	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
				$n = 4$	$\mathbf{Z}_2 \oplus ((\mathbf{Z}_2 \oplus \mathbf{Z}_2) \rtimes \mathbf{Z}_2)$
DII	$(\mathfrak{so}(p, 1), \mathfrak{so}(p-1) \oplus \mathbf{R})$	$\mathfrak{a}_1$	Z_1	$p \geq 3$, odd	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$

DIII	$(\mathfrak{so}^*(4n), \mathfrak{su}^*(2n) \oplus \mathbf{R})$	$\mathfrak{c}_n$	Z_n	$n \geq 3$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$
CI	$(\mathfrak{sp}(n, \mathbf{R}), \mathfrak{sl}(n, \mathbf{R}) \oplus \mathbf{R})$	$\mathfrak{c}_n$	Z_n	$n \geq 3$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$
CII	$(\mathfrak{sp}(n, n), \mathfrak{su}^*(2n) \oplus \mathbf{R})$	$\mathfrak{c}_n$	Z_n	$n \geq 2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$
EI	$(\mathfrak{e}_{6(6)}, \mathfrak{so}(5, 5) \oplus \mathbf{R})$	$\mathfrak{e}_6$	Z_1, Z_6	—	$\mathbf{Z}_2$
EIV	$(\mathfrak{e}_{6(-26)}, \mathfrak{so}(1, 9) \oplus \mathbf{R})$	$\mathfrak{a}_2$	Z_1, Z_2	—	$\mathbf{Z}_2$
EV	$(\mathfrak{e}_{7(7)}, \mathfrak{e}_{6(6)} \oplus \mathbf{R})$	$\mathfrak{e}_7$	Z_7	—	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$
EVII	$(\mathfrak{e}_{7(-25)}, \mathfrak{e}_{6(-26)} \oplus \mathbf{R})$	$\mathfrak{c}_3$	Z_3	—	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$

In this table, the notation of Lie algebras is that of in [He] and the symbols Z are the same as in Table 2.

S. Kaneyuki and M. Kozai have introduced the notion of para-Hermitian symmetric space $(G/H, \sigma, I, g)$ in [Ka-Koz]. A para-Hermitian symmetric space is a pseudo-Riemannian manifold. When we discuss a pseudo-Riemannian manifold (M, g), some natural problems arise:

(1) How many such metrics exist there on the manifold M?

(2) With what kind of structure is the group $\mathrm{I}(M, g)$ of isometries?

For almost effective semisimple para-Hermitian symmetric spaces, we well know the following fact contributed in [Ka-Koz]:

- *For an arbitrary almost effective semisimple para-Hermitian symmetric space* $(G/H, \sigma, I, g)$, *there exists a unique element* $Z \in \mathfrak{g} := \mathrm{Lie}(G)$ *such that* $\mathfrak{c}_{\mathfrak{g}}(Z) = \mathfrak{h}$ *and* $I_o = \mathrm{ad}_{\mathfrak{m}} Z$, *where o is the origin of* G/H *and* $\mathfrak{h}$ (*resp.* $\mathfrak{m}$) *is the* 1 (*resp.* -1) *eigenspace of* σ_* *in* $\mathfrak{g}$.
- *Furthermore, set* $\bar{I}$ (*resp.* $\bar{g}$) *as the G-invariant extension of* $\lambda_1 \mathrm{ad}_{\mathfrak{m}} Z$ (*resp. of* $\lambda_2 B_{\mathfrak{g}}|_{\mathfrak{m}\times\mathfrak{m}}$), *where* $\lambda_1 = \pm 1$ (*resp.* λ_2 *is any nonzero real number*). *Then the quadruplet* $(G/H, \sigma, \bar{I}, \bar{g})$ *is a para-Hermitian symmetric space, also*.

If there exists a nonzero real number λ such that the metric g is the G-invariant extension of $\lambda B_{\mathfrak{g}}|_{\mathfrak{m}\times\mathfrak{m}}$, then $\mathrm{I}(G/H, g)$ coincides with $\mathrm{I}(G/H, \bar{g})$. As a result, it is greatly important to determine the group $\mathrm{I}(G/H, \bar{g})$ of isometries in this case. Thereupon, under certain conditions, we determine the group of isometries for every para-Hermitian symmetric space.

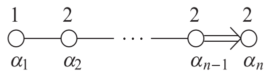
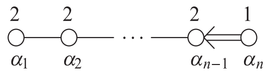
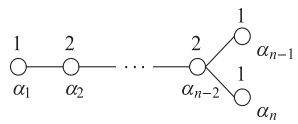
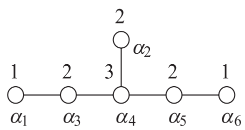
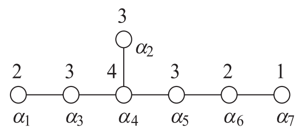
This paper is organized as follows:

§2 Preliminaries. In this section, we provide useful notation and recollect some definitions and facts on para-Hermitian symmetric spaces. An important result in this section is a theorem on para-Hermitian structures, and this is Proposition 2 and Theorem 1-(1).

§3 Relation between isometries and Lie algebra automorphisms. Next, under the same conditions as in Theorem 1, we assert Proposition 3. From this, we infer relation between isometries and Lie algebra automorphisms. Using this proposition, we briefly prove the main point in the section, namely Proposition 4. Those propositions imply Theorem 1-(2) and form a basis of Theorem 1-(3).

§4 A way to determine every quotient group $\mathrm{I}(G/H, g)/\mathrm{I}(G/H, g)_0$ and Examples. The final section presents a way to determine every group $\mathrm{I}(G/H, g)/\mathrm{I}(G/H, g)_0$ for $(G/H, \sigma, I, g)$ in Theorem 1. In addition, we consider a way to construct any space $(G/H, \sigma, I, g)$ in Theorem 1. With a similar way to this, we individually determine every group $\mathrm{I}(G/H, g)/\mathrm{I}(G/H, g)_0$ and obtain Tables 1 and 2.

Table 2. The unique elements corresponding to every system of nonzero roots

Type	Condition	Dynkin diagram with the coefficients of the maximal root	The element Z
$\mathfrak{a}_n$	$n \geq 1$		$Z_1, \dots, Z_n$
$\mathfrak{b}_n$	$n \geq 3$		Z_1
$\mathfrak{c}_n$	$n \geq 2$		Z_n
$\mathfrak{d}_n$	$n \geq 4$		Z_1, Z_{n-1}, Z_n
$\mathfrak{e}_6$	—		Z_1, Z_6
$\mathfrak{e}_7$	—		Z_7

Here $\{Z_i\}_i$ is the dual basis of $\{\alpha_i\}_i$.

2. Preliminaries

2.1. Notation. We use the following notation in this paper, where M is a manifold, G is a Lie group, and $\mathfrak{g}$ is a Lie algebra:

$\mathfrak{X}(M)$	the Lie algebra of vector fields on M,
$\mathrm{I}(M, g)$	the group of isometries of a pseudo-Riemannian manifold (M, g),
∇^1	the canonical affine connection on a symmetric space,
$\mathrm{A}(M, \nabla)$	the group of affine transformations of an affine manifold (M, ∇),
$\mathrm{Lie}(G)$	the Lie algebra of G,
$\mathrm{Aut}(G), \mathrm{Aut}(\mathfrak{g})$	the groups of automorphisms of G, $\mathfrak{g}$, respectively,

$\mathrm{Aut}(\mathfrak{g}, \phi) := \{\psi \in \mathrm{Aut}(\mathfrak{g}) \mid \phi \circ \psi = \psi \circ \phi\}$, for $\phi \in \mathrm{Aut}(\mathfrak{g})$,

$\mathrm{Int}(\mathfrak{g})$	the group of inner automorphisms of $\mathfrak{g}$,

$B_{\mathfrak{g}}$ the Killing form of $\mathfrak{g}$,
$C_G(Z) := \{x \in G \mid \mathrm{Ad}(x)Z = Z\}$, for $Z \in \mathrm{Lie}(G)$,
$\mathfrak{c}_{\mathfrak{g}}(Z) := \{X \in \mathfrak{g} \mid [Z, X] = 0\}$, for $Z \in \mathfrak{g}$,
$Z(G), \mathfrak{z}(\mathfrak{g})$ the centers of G, $\mathfrak{g}$, respectively,
G_0 the identity component of G,
G^σ the closed subgroup of G which consists of the fixed points of an involution σ of G,
A_x the inner automorphism of G by an element $x \in G$,
$\mathfrak{a}^*$ the dual space of a vector space $\mathfrak{a}$,
id_A the identity mapping of a set A.

2.2. Definitions and Well Known Propositions. Let us begin with a brief review of para-Hermitian symmetric space. First of all,

DEFINITION 1 (cf. [No1], p. 52, p. 53). (1) Set G as a connected Lie group, and H as a closed subgroup of G. The pair $(G/H, \sigma)$ of the homogeneous space G/H and an involution σ of G is said to be a *symmetric space*, if the following inclusion relation is satisfied:

$$(G^\sigma)_0 \subset H \subset G^\sigma;$$

(2) A symmetric space $(G/H, \sigma)$ is uniquely equipped with a G-invariant affine connection ∇^1 making an affine transformation of $\hat{\sigma}$, where $\hat{\sigma}(xH) := \sigma(x)H$ for $xH \in G/H$. We call the connection ∇^1 the *canonical affine connection* on $(G/H, \sigma)$.

REMARK 1. If a symmetric space $(G/H, \sigma)$ admits a G-invariant pseudo-Riemannian metric g, then the Levi-Civita connection induced by g coincides with the canonical affine connection ∇^1 (cf. [No1], p. 55). Additionally, on the compact-open topology, $\mathrm{I}(G/H, g)$ is a closed subgroup of $\mathrm{A}(G/H, \nabla^1)$ (cf. [No2], p. 823).

Second, we recollect the definition of para-Hermitian symmetric space:

DEFINITION 2 (cf. [Ka-Koz], p. 86–87). A *para-Hermitian symmetric space* is the name given to a quadruplet $(G/H, \sigma, I, g)$, where $(G/H, \sigma)$ is a symmetric space furnished with a G-invariant para-complex structure I and with a G-invariant para-Hermitian metric g.

Remark 2. Note that a 2-form ω, $\omega(X, Y) := g(X, IY)$ for $X, Y \in \mathfrak{X}(G/H)$, becomes symplectic. In other words, g is upgraded to a para-Kähler metric (cf. [Ka-Koz], p. 86).

Next, to characterize para-Hermitian symmetric spaces, we prepare a term on a Lie algebra:

Definition 3. A real Lie algebra $\mathfrak{g}$ is called *absolutely simple*, if its complexification $\mathfrak{g}_{\mathbf{C}}$ is simple. A Lie group G and a symmetric space $(G/H, \sigma)$ are equally called *absolutely simple*, if $\mathrm{Lie}(G)$ is absolutely simple.

Last, recall a well known proposition about para-Hermitian symmetric spaces:

Proposition 1 (cf. [Ka-Koz], p. 89–92, and [Koh], p. 306). *Let $(G/H, \sigma, I, g)$ be an absolutely simple para-Hermitian symmetric space. In addition, put $\mathfrak{g} :=$ $\mathrm{Lie}(G)$ and set $\mathfrak{h}$ (resp. $\mathfrak{m}$) as the 1 (resp. -1) eigenspace of σ_* in $\mathfrak{g}$. Moreover, set $\mathfrak{p}$ as the -1 eigenspace in $\mathfrak{g}$ of a Cartan involution θ of $\mathfrak{g}$ which commutes with σ_*. Then there exists a unique element $Z \in \mathfrak{z}(\mathfrak{h}) \cap \mathfrak{p}$ such that*

(1) $C_G(Z)_0 \subset H \subset C_G(Z)$,

(2) $\mathfrak{h} = \mathfrak{c}_{\mathfrak{g}}(Z) = \mathfrak{g}_0$, $\mathfrak{m} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$, *where $\mathfrak{g}_\lambda$ is the λ eigenspace in $\mathfrak{g}$ of* $\mathrm{ad}\, Z$,

(3) $I_o = \mathrm{ad}_{\mathfrak{m}}\, Z$, $\sigma_* = \exp \sqrt{-1}\pi\, \mathrm{ad}\, Z$,

(4) $\mathfrak{z}(\mathfrak{h}) = \mathbf{R}Z$.

Furthermore, a quadruplet $(G/\overline{H}, \sigma, \bar{I}, \bar{g})$ becomes a para-Hermitian symmetric space for an arbitrary open subgroup $\overline{H}$ of $C_G(Z)$, where $\bar{I}$ (resp. $\bar{g}$) is the G-invariant extension of $\lambda_1\, \mathrm{ad}_{\mathfrak{m}}\, Z$ (resp. of $\lambda_2 B_{\mathfrak{g}}|_{\mathfrak{m}\times\mathfrak{m}}$) and $\lambda_1 = \pm 1$ (resp. λ_2 is a nonzero real number).

2.3. An Invariant Para-Hermitian Metric. As the first step in this study, we uniformize the metrics on absolutely simple para-Hermitian symmetric spaces. Assume that $(G/H, \sigma, I, g)$, $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{m}$, and Z are the same symbols as in Proposition 1.

Proposition 2. *Any G-invariant para-Hermitian metric of G/H with respect to I is the G-invariant extension of $B_{\mathfrak{g}}|_{\mathfrak{m}\times\mathfrak{m}}$ up to constant. In particular, there exists a nonzero real number λ such that g is the G-invariant extension of $\lambda B_{\mathfrak{g}}|_{\mathfrak{m}\times\mathfrak{m}}$.*

Proof. Take any G-invariant para-Hermitian metric $\tilde{g}$ of G/H with respect to I. Set π as the projection from G onto G/H and set Ω as the pull back of $\tilde{\omega}$ with π: $\Omega := \pi^*\tilde{\omega}$, where $\tilde{\omega}(X, Y) := \tilde{g}(X, IY)$ for $X, Y \in \mathfrak{X}(G/H)$. Since $\tilde{\omega}$ is

G-invariant symplectic, Ω is a left-invariant closed 2-form on G. Let ∂ be the boundary operator of the trivial representation of $\mathfrak{g}$. Then, for $X, Y, W \in \mathfrak{g}$,

$$\begin{aligned}\partial\Omega(X, Y, W) &= -\Omega([X, Y], W) - \Omega([Y, W], X) - \Omega([W, X], Y) \\ &= W(\Omega(X, Y)) + X(\Omega(Y, W)) + Y(\Omega(W, X)) \\ &\quad - \Omega([X, Y], W) - \Omega([Y, W], X) - \Omega([W, X], Y) \\ &= d\Omega(X, Y, W) = 0;\end{aligned}$$

thus Ω is a 2-cochain. From the Whitehead lemma (e.g. [Va] Theorem 3.12.1, p. 220), the 1-cohomology space vanishes and the 2-cohomology space also dose. The latter implies that Ω is a 2-coboundary, and then there exists a nonzero element $\alpha \in \mathfrak{g}^*$ such that $\Omega = \partial\alpha$. This α is unique, because of the former. Moreover, by use of $\mathfrak{g} \cong \mathfrak{g}^*$ as a vector space with $X \leftrightarrow B_{\mathfrak{g}}(X, *)$, there uniquely exists an element $A \in \mathfrak{g}$ such that

$$\Omega(X, Y) = -B_{\mathfrak{g}}(A, [X, Y]) \quad \text{for all } X, Y \in \mathfrak{g}.$$

Here $\mathfrak{h} = \mathfrak{c}_{\mathfrak{g}}(A)$; indeed, $-B_{\mathfrak{g}}(A, [X, Y]) = \Omega(X, Y) = \tilde{\omega}(\pi_* X, \pi_* Y) \circ \pi$ for $X, Y \in \mathfrak{g}$ and both $B_{\mathfrak{g}}$ and $\tilde{\omega}$ are nondegenerate. In consequence, Proposition 1-(4) causes that there exists a nonzero unique real number $\tilde{\lambda}$ such that $A = \tilde{\lambda}Z$. Hence, for all $X, Y \in \mathfrak{m}$,

$$\begin{aligned}\tilde{g}_o(X, Y) &= \tilde{\omega}(X, I_o Y) = \Omega(X, I_o Y) = -B_{\mathfrak{g}}(A, [X, \operatorname{ad} Z(Y)]) \\ &= \tilde{\lambda}B_{\mathfrak{g}}(X, (\operatorname{ad} Z)^2(Y)) = \tilde{\lambda}B_{\mathfrak{g}}(X, Y).\end{aligned}$$

Recollecting that $\tilde{g}$ is G-invariant, we obtain this proposition. □

3. Relation between Isometries and Lie Algebra Automorphisms

3.1. An Isotropy Subgroup. In this section, $(G/H, \sigma, I, g)$, $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{m}$, and Z are the same symbols as in Proposition 1. Let us just consider the only case where G/H can be realized as a hyperbolic orbit with the adjoint representation Ad, namely hyperbolic orbit type. Suppose that G/H is of hyperbolic orbit type (that is, $H = C_G(Z)$) and $Z(G)$ is trivial. In addition, set $o \in G/H$ as the origin of G/H and let $\mathrm{I}(G/H, g, o)$ (resp. $\mathrm{A}(G/H, \nabla^1, o)$) be the isotropy subgroup at o of the isometric transformation group $\mathrm{I}(G/H, g)$ (resp. of the affine transformation group $\mathrm{A}(G/H, \nabla^1)$).

Here, we confirm the following proposition to achieve the main purpose in this paper:

PROPOSITION 3. $\mathrm{I}(G/H, g, o) = \mathrm{A}(G/H, \nabla^1, o) \cong \mathrm{Aut}(\mathfrak{g}, \sigma_*)$ *as a group. Moreover,* $\mathrm{I}(G/H, g) = \mathrm{A}(G/H, \nabla^1)$.

This proposition comes from four lemmas. At the beginning, $\mathrm{Aut}(\mathfrak{g}, \sigma_*)$ is distinguished by Z as follows:

LEMMA 1. *Put* $\mathrm{Aut}(\mathfrak{g}, Z)^{\pm} := \{\phi \in \mathrm{Aut}(\mathfrak{g}) \mid \phi(Z) = \pm Z\}$, *and as disjoint union*

$$\mathrm{Aut}(\mathfrak{g}, \sigma_*) = \mathrm{Aut}(\mathfrak{g}, Z)^+ \sqcup \mathrm{Aut}(\mathfrak{g}, Z)^-.$$

PROOF. Since Z is nonzero, this proof is completed if we prove $\mathrm{Aut}(\mathfrak{g}, \sigma_*) = \mathrm{Aut}(\mathfrak{g}, Z)^+ \cup \mathrm{Aut}(\mathfrak{g}, Z)^-$. Assume that $\phi \in \mathrm{Aut}(\mathfrak{g}, Z)^{\pm}$, and then $\phi \circ \sigma_* \circ \phi^{-1} = \exp \sqrt{-1}\pi \,\mathrm{ad}(\pm Z) = \sigma_*^{\pm 1} = \sigma_*$.

On the other hand, take any $\phi \in \mathrm{Aut}(\mathfrak{g}, \sigma_*)$. By use of $\phi(\mathfrak{z}(\mathfrak{h})) = \mathfrak{z}(\mathfrak{h})$ and of Proposition 1-(4), there exists a nonzero real number μ such that $\phi(Z) = \mu Z$. Applying $(\phi \circ I_o \circ \phi^{-1})^2 = \mathrm{id}_{\mathfrak{m}}$, we obtain $X = (\phi \circ I_o \circ \phi^{-1})^2(X) = (\phi \circ I_o \circ \phi^{-1}) \cdot (\mu \,\mathrm{ad}\, Z(X)) = \mu^2 X$ for all $X \in \mathfrak{m}$, and consequently have $\mu = \pm 1$. □

In the second place, we correlate $\mathrm{Aut}(\mathfrak{g}, \sigma_*)$ with $\mathrm{A}(G/H, \nabla^1, o)$:

LEMMA 2. *For an arbitrary* $\phi \in \mathrm{Aut}(\mathfrak{g}, \sigma_*)$, *there exists a unique* $\Phi \in \mathrm{Aut}(G)$ *such that* $\Phi(H) = H$ *and* $\Phi_* = \phi$. *Put* $\hat{\Phi}(xH) := \Phi(x)H$ *for* $xH \in G/H$, *and then* $\hat{\Phi} \in \mathrm{I}(G/H, g, o)$; *as a result,* $\hat{\Phi} \in \mathrm{A}(G/H, \nabla^1, o)$.

PROOF. G is connected, and there uniquely exists the universal covering group $(\tilde{G}, \rho)$ such that (1) $N := \mathrm{Ker}(\rho) \subset Z(\tilde{G})$ is a normal subgroup of $\tilde{G}$, (2) $\tilde{G}/N \cong G$ as a Lie group, and (3) $\mathrm{Lie}(\tilde{G}) \cong \mathfrak{g}$ as a Lie algebra. Since $Z(G)$ is trivial, we have $Z(\tilde{G}) = N$; and then $G \cong \tilde{G}/Z(\tilde{G})$ as a Lie group.

Take any $\phi \in \mathrm{Aut}(\mathfrak{g}, \sigma_*)$, and there exists a unique $\Psi \in \mathrm{Aut}(\tilde{G})$ such that $\Psi_* = \phi$. Since $\Psi(N) = N$, Ψ induces an automorphism of $\tilde{G}/N$: $\tilde{\Psi}(aN) := \Psi(a)N$. Hence there exists a unique $\Phi \in \mathrm{Aut}(G)$ such that $\Phi_* = \phi$. Here $\Phi(H) = H$; because, by Lemma 1, $\mathrm{Ad}\,\Phi(x)Z = \phi \circ \mathrm{Ad}\,x \circ \phi^{-1}(Z) = Z$ for all $x \in H$. Thus an automorphism $\hat{\Phi}$ of G/H is induced by $\hat{\Phi}(xH) := \Phi(x)H$, and then $\hat{\Phi}$ is an isometry. Indeed, $g_o(\hat{\Phi}_* X, \hat{\Phi}_* Y) = \lambda B_{\mathfrak{g}}(\phi(X), \phi(Y)) = g_o(X, Y)$ for all $X, Y \in \mathfrak{m}$, where λ is the real number associated with g and $B_{\mathfrak{g}}$ (cf. Proposition 2). □

Now set F_1, F_2, and F as the mappings provided in Lemma 2:

$$F_1 : \mathrm{Aut}(\mathfrak{g}, \sigma_*) \to \mathrm{Aut}(G, H), \quad \phi \mapsto \Phi,$$

$$F_2 : \mathrm{Aut}(G, H) \to \mathrm{A}(G/H, \nabla^1, o), \quad \Phi \mapsto \hat{\Phi},$$

$$F := F_2 \circ F_1 : \mathrm{Aut}(\mathfrak{g}, \sigma_*) \to \mathrm{A}(G/H, \nabla^1, o),$$

where $\mathrm{Aut}(G, H)$ is the subgroup of $\mathrm{Aut}(G)$ maintaining H. Note that F is a homomorphism.

To close, we prove that F_2 is a group isomorphism of $\mathrm{Aut}(G, H)$ onto $\mathrm{A}(G/H, \nabla^1, o)$, and accordingly heed the following:

LEMMA 3. *Put $\tau_x : aH \mapsto xaH$ for $aH \in G/H$ and $\tau : x \mapsto \tau_x$ for $x \in G$, and then the mapping $\mathrm{A}(G/H, \nabla^1, o) \to \mathrm{Aut}(G, H)$, $f \mapsto \tau^{-1} \circ \mathrm{A}_f \circ \tau$ is the inverse mapping of F_2.*

PROOF. First of all, let us prove that G/H is effective. Take any normal subgroup N of G contained in H. Since G is simple, N is discrete and then $N \subset Z(G) = \{e\}$; thus G/H is effective. This enables us to have $\tau_G = \mathrm{A}(G/H, \nabla^1)_0$ by proceeding in the similar way to that of Proposition 1.6 in [Ka], where $\tau_G = \{\tau_x \mid x \in G\}$.

Just take any $f \in \mathrm{A}(G/H, \nabla^1, o)$, and it consequently follows that $\mathrm{A}_f(\tau_x) \in \mathrm{A}(G/H, \nabla^1)_0$ for any $x \in G$. Thereupon put $\Phi_f := \tau^{-1} \circ \mathrm{A}_f \circ \tau$, and then $\Phi_f \in \mathrm{Aut}(G)$. Here $\Phi_f(H) = H$; indeed, $\tau_y(o) = f \circ \tau_h \circ f^{-1}(o) = f(\tau_h(o)) = f(o) = o$ for all $h \in H$, where $y := \Phi_f(h)$. As a result, an automorphism $\hat{\Phi}_f$ of G/H is induced as $\hat{\Phi}_f(xH) := \Phi_f(x)H$. This automorphism $\hat{\Phi}_f$ coincides with f; because $\hat{\Phi}_f(xH) = \Phi_f(x)H = zH = \tau_z(o) = f \circ \tau_x \circ f^{-1}(o) = f(xH)$ for all $x \in G$, where $z := \Phi_f(x)$. Therefore the mapping $f \mapsto \tau^{-1} \circ \mathrm{A}_f \circ \tau$ is F_2^{-1}. □

Applying Lemma 3, we obtain the following:

LEMMA 4. *There exists the inverse mapping of F, and this is $F' : \mathrm{A}(G/H, \nabla^1, o) \to \mathrm{Aut}(\mathfrak{g}, \sigma_*)$, $f \mapsto (\tau^{-1} \circ \mathrm{A}_f \circ \tau)_*$.*

PROOF. If $f \in \mathrm{A}(G/H, \nabla^1, o)$ and $\Phi_f := \tau^{-1} \circ \mathrm{A}_f \circ \tau$, then this mapping $\Phi_f \in \mathrm{Aut}(G, H)$ yields $(\Phi_f)_*|_{\mathfrak{h}} \in \mathrm{Aut}(\mathfrak{h})$. In consequence, $(\Phi_f)_*|_{\mathfrak{m}}$ is an isomorphism of $\mathfrak{m}$.

Here $(\Phi_f)_* \in \mathrm{Aut}(\mathfrak{g}, \sigma_*)$, because $\sigma_* \circ (\Phi_f)_* \circ \sigma_*(X) = \sigma_* \circ (\Phi_f)_*(X_1 - X_2) = (\Phi_f)_*(X_1) - (-(\Phi_f)_*(X_2)) = (\Phi_f)_*(X)$ for all $X = X_1 + X_2 \in \mathfrak{g}$ $(X_1 \in \mathfrak{h}, X_2 \in \mathfrak{m})$. Thus F' is well-defined, and we can easily confirm $F \circ F' = \mathrm{id}$ and $F' \circ F = \mathrm{id}$. This lemma is proved. □

Now, we are in a position to prove Proposition 3.

Proof of Proposition 3. For every $f \in \mathrm{A}(G/H, \nabla^1, o)$, f is an isometry of $(G/H, g)$. Indeed, put $\Phi_f := \tau^{-1} \circ \mathrm{A}_f \circ \tau$, and then $f_{*o} = (\Phi_f)_*|_{\mathfrak{m}}$. There exists a nonzero real number λ such that g is the G-invariant extension of $\lambda B_{\mathfrak{g}}|_{\mathfrak{m}\times\mathfrak{m}}$ (cf. Proposition 2). Hence $g_o(f_{*o}(X), f_{*o}(Y)) = \lambda B_{\mathfrak{g}}((\Phi_f)_*(X), (\Phi_f)_*(Y)) = \lambda B_{\mathfrak{g}}(X, Y) = g_o(X, Y)$ for all $X, Y \in \mathfrak{m}$; thus $f^*g = g$. This provides us with $\mathrm{I}(G/H, g, o) = \mathrm{A}(G/H, \nabla^1, o)$. From this relation and the lemmas stated above, $\mathrm{I}(G/H, g, o) = \mathrm{A}(G/H, \nabla^1, o) \cong \mathrm{Aut}(\mathfrak{g}, \sigma_*)$ comes. Let us show $\mathrm{I}(G/H, g) = \mathrm{A}(G/H, \nabla^1)$ from now on. Take any $f \in \mathrm{A}(G/H, \nabla^1)$, and there exists an element $x \in G$ such that $f \circ \tau_x(o) = o$. Since $\mathrm{I}(G/H, g, o) = \mathrm{A}(G/H, \nabla^1, o)$, the mapping $f = (f \circ \tau_x) \circ \tau_x^{-1} \in \mathrm{I}(G/H, g)$. In consequence, $\mathrm{I}(G/H, g) = \mathrm{A}(G/H, \nabla^1)$. □

3.2. The Connected Components of Transformation Groups. At last, we obtain the following:

Proposition 4. *Put* $\mathrm{Int}(\mathfrak{g}, Z)^+ := \mathrm{Aut}(\mathfrak{g}, Z)^+ \cap \mathrm{Int}(\mathfrak{g})$, *and then as a group*

$$\begin{aligned}\mathrm{I}(G/H, g)/\mathrm{I}(G/H, g)_0 &= \mathrm{A}(G/H, \nabla^1)/\mathrm{A}(G/H, \nabla^1)_0 \\ &\cong (\mathrm{Aut}(\mathfrak{g}, Z)^+ \sqcup \mathrm{Aut}(\mathfrak{g}, Z)^-)/\mathrm{Int}(\mathfrak{g}, Z)^+.\end{aligned}$$

Proof. The deduction in the proof of Proposition 3 connotes $\mathrm{I}(G/H, g)_0 = \tau_G$ and $\mathrm{I}(G/H, g, o) \cap \tau_G = \tau_H$, where $\tau_L = \{\tau_x \mid x \in L\}$ $(L = G$ or $H)$.

Moreover, the isomorphism F' in Lemma 4 correlates τ_h with $\mathrm{Ad}\, h$ through A_h for $h \in H$. Hence $\tau_H \cong \mathrm{Ad}\, H$ as a group.

Lastly, by use of $\mathrm{Ad}\, G = \mathrm{Int}(\mathfrak{g})$ and $H = C_G(Z)$, we have $\mathrm{Ad}\, H = \mathrm{Int}(\mathfrak{g}, Z)^+$.

Therefore, by Proposition 3 and Lemma 1,

$$\begin{aligned}\mathrm{I}(G/H, g)/\mathrm{I}(G/H, g)_0 &= \mathrm{A}(G/H, \nabla^1)/\mathrm{A}(G/H, \nabla^1)_0 = (\mathrm{I}(G/H, g, o) \circ \tau_G)/\tau_G \\ &\cong \mathrm{I}(G/H, g, o)/\tau_H \cong \mathrm{Aut}(\mathfrak{g}, \sigma_*)/\mathrm{Ad}\, H \\ &= (\mathrm{Aut}(\mathfrak{g}, Z)^+ \sqcup \mathrm{Aut}(\mathfrak{g}, Z)^-)/\mathrm{Int}(\mathfrak{g}, Z)^+.\end{aligned}$$

□

4. A Way to Determine Every Quotient Group $\mathrm{I}(G/H, g)/\mathrm{I}(G/H, g)_0$ and Examples

4.1. A Way to Determine Every Group $\mathrm{I}(G/H, g)/\mathrm{I}(G/H, g)_0$. In this subsection, we consider a way to investigate the structure of every group $\mathrm{I}(G/H, g)/\mathrm{I}(G/H, g)_0$ for $(G/H, \sigma, I, g)$ in Theorem 1 with Proposition 4.

Let G be a connected absolutely simple Lie group whose Lie algebra is $\mathfrak{g}$. Suppose that $Z(G)$ is trivial. Now, take any Cartan involution $\tilde{\theta}$ of $\mathfrak{g}$ and take a maximal abelian subspace $\tilde{\mathfrak{a}}$ in $\tilde{\mathfrak{p}}$, where $\tilde{\mathfrak{p}}$ is the -1 eigenspace of $\tilde{\theta}$ in $\mathfrak{g}$. Moreover, set $\tilde{\Delta}$ as the system of nonzero restricted roots of $(\mathfrak{g}, \tilde{\mathfrak{a}})$, $\tilde{\Delta}^+$ as a half system of $\tilde{\Delta}$, $\tilde{\gamma}$ as the maximal root of $\tilde{\Delta}^+$, and $\tilde{W} := \{\tilde{A} \in \tilde{\mathfrak{a}} \mid \tilde{\alpha}(\tilde{A}) \geq 0$ for all $\tilde{\alpha} \in \tilde{\Delta}^+\}$. Then the following is caused:

PROPOSITION 5. (1) *Take any para-Hermitian symmetric space* $(G/H, \sigma, I, g)$ *of hyperbolic orbit type. For any Cartan involution* θ *of* $\mathfrak{g}$ *which commutes with* σ_*, *there exists an inner automorphism* ϕ *of* $\mathfrak{g}$ *such that*

(a) $\tilde{Z} := \phi(Z) \in \tilde{W}$,

(b) $\tilde{\gamma}(\tilde{Z}) = 1$,

(c) $\mathrm{I}(G/H, g)/\mathrm{I}(G/H, g)_0 \cong (\mathrm{Aut}(\mathfrak{g}, \tilde{Z})^+ \sqcup \mathrm{Aut}(\mathfrak{g}, \tilde{Z})^-)/\mathrm{Int}(\mathfrak{g}, \tilde{Z})^+$,

where Z *is a unique element with Proposition* 1 *for the space* $(G/H, \sigma, I, g)$ *and the Cartan involution* θ.

(2) *For any* $\tilde{A} \in \tilde{W}$ *with* $\tilde{\gamma}(\tilde{A}) = 1$, *there exists a unique absolutely simple para-Hermitian symmetric space* $(G/\tilde{H}, \tilde{\sigma}, \tilde{I}, \tilde{g})$ *of hyperbolic type such that*

(a) *the element* $\tilde{A}$ *is a unique one satisfying the conditions* (1)–(4) *on Proposition* 1 *for* $(G/\tilde{H}, \tilde{\sigma}, \tilde{I}, \tilde{g})$ *and* $\tilde{\theta}$,

(b) $\mathrm{I}(G/\tilde{H}, \tilde{g})/\mathrm{I}(G/\tilde{H}, \tilde{g})_0 \cong (\mathrm{Aut}(\mathfrak{g}, \tilde{A})^+ \sqcup \mathrm{Aut}(\mathfrak{g}, \tilde{A})^-)/\mathrm{Int}(\mathfrak{g}, \tilde{A})^+$,

where the uniqueness of metric is up to constant.

PROOF. (1) The Lie algebra $\mathfrak{g}$ is (semi)simple, and then there exists an inner automorphism ϕ_1 of $\mathfrak{g}$ such that $\tilde{\theta} = \phi_1 \circ \theta \circ \phi_1^{-1}$. Hence $\tilde{\theta}(\phi_1(Z)) = \phi_1(\theta(Z)) = -\phi_1(Z)$; in brief $Z_1 := \phi_1(Z) \in \tilde{\mathfrak{p}}$. Take a maximal abelian subspace $\tilde{\mathfrak{a}}_1$ in $\tilde{\mathfrak{p}}$ containing Z_1, and then the subspaces $\tilde{\mathfrak{a}}$ and $\tilde{\mathfrak{a}}_1$ are conjugate under the action of K, where K is a maximal compact subgroup of G whose Lie algebra is the 1 eigenspace of $\tilde{\theta}$ in $\mathfrak{g}$. In consequence, there exists an inner automorphism ϕ_2 of $\mathfrak{g}$ such that $\phi_2(\tilde{\mathfrak{a}}_1) = \tilde{\mathfrak{a}}$; thus $Z_2 := \phi_2(Z_1) \in \tilde{\mathfrak{a}}$. Let M be the subgroup of G with Lie algebra $\tilde{\mathfrak{a}}$. Then the Weyl group of $\tilde{\Delta}$ coincides with the Weyl group $N_K(\tilde{\mathfrak{a}})/Z_K(\tilde{\mathfrak{a}}) = N_G(\tilde{\mathfrak{a}})/Z_G(\tilde{\mathfrak{a}})$ of (G, M), where $N_L(\tilde{\mathfrak{a}})$ (resp. $Z_L(\tilde{\mathfrak{a}})$) is the normalizer (resp. the centralizer) of $\tilde{\mathfrak{a}}$ in $L = G$ or K. This causes that there exists a mapping ϕ_3 in the Weyl group of (G, M) such that $\phi_3(Z_2) \in \tilde{W}$. As a result, we obtain $\phi := \phi_3 \circ \phi_2 \circ \phi_1 \in \mathrm{Int}(\mathfrak{g})$ with $\tilde{Z} := \phi(Z) \in \tilde{W}$.

Notice that $\tilde{\gamma}(\tilde{Z}) \geq 0$ is an eigenvalue of $\mathrm{ad}\,\tilde{Z}$ in $\mathfrak{g}$ and is consequently one of $\mathrm{ad}\,Z$ in $\mathfrak{g}$, and then $\tilde{\gamma}(\tilde{Z}) = 0$ or 1. Since the element Z is nonzero and $\tilde{Z}$ is also, the map $\mathrm{ad}\,\tilde{Z}$ must generate a nonzero eigenvalue in $\mathfrak{g}$. Hence there exists a root $\tilde{\beta} \in \tilde{\Delta}^+$ such that $\tilde{\beta}(\tilde{Z}) > 0$. Owing to $\tilde{\gamma}(\tilde{Z}) \geq \tilde{\beta}(\tilde{Z})$, the value $\tilde{\gamma}(\tilde{Z})$ is 1.

Moreover, By Proposition 4, $\mathrm{I}(G/H, g)/\mathrm{I}(G/H, g)_0 \cong (\mathrm{Aut}(\mathfrak{g}, Z)^+ \sqcup \mathrm{Aut}(\mathfrak{g}, Z)^-)/\mathrm{Int}(\mathfrak{g}, Z)^+ \cong (\mathrm{Aut}(\mathfrak{g}, \tilde{Z})^+ \sqcup \mathrm{Aut}(\mathfrak{g}, \tilde{Z})^-)/\mathrm{Int}(\mathfrak{g}, \tilde{Z})^+$.

(2) The condition $\tilde{A} \in \tilde{W}$ with $\tilde{\gamma}(\tilde{A}) = 1$ causes the decomposition $\mathfrak{g} = \tilde{\mathfrak{g}}_{-1} \oplus \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1$, where $\tilde{\mathfrak{g}}_\lambda$ is the λ eigenspace of $\operatorname{ad} \tilde{A}$ in $\mathfrak{g}$. Thereupon, put $\tilde{H} := C_G(\tilde{A})$, $\tilde{\sigma} := \mathrm{A}_{\exp \sqrt{-1}\pi \tilde{A}}$, and $\tilde{\mathfrak{m}} := \tilde{\mathfrak{g}}_{-1} \oplus \tilde{\mathfrak{g}}_1$. Additionally, set $\tilde{I}$ (resp. $\tilde{g}$) as the G-invariant extension of $\operatorname{ad}_{\tilde{\mathfrak{m}}} \tilde{A}$ (resp. of $B_{\mathfrak{g}}|_{\tilde{\mathfrak{m}} \times \tilde{\mathfrak{m}}}$). Then the quadruplet $(G/\tilde{H}, \tilde{\sigma}, \tilde{I}, \tilde{g})$ becomes an absolutely simple para-Hermitian symmetric space of hyperbolic type.

Note in passing that $\tilde{\theta} \circ \tilde{\sigma}_* = \tilde{\sigma}_* \circ \tilde{\theta}$, and there exists a unique element $\tilde{A}_0$ such that (1)–(4) on Proposition 1 for $(G/\tilde{H}, \tilde{\sigma}, \tilde{I}, \tilde{g})$ and $\tilde{\theta}$. This element $\tilde{A}_0$ is unique and the one $\tilde{A}$ satisfies the same conditions for $(G/\tilde{H}, \tilde{\sigma}, \tilde{I}, \tilde{g})$ and $\tilde{\theta}$; and consequently $\tilde{A}_0$ is just $\tilde{A}$. Therefore Proposition 4 induces $\mathrm{I}(G/\tilde{H}, \tilde{g})/\mathrm{I}(G/\tilde{H}, \tilde{g})_0 \cong (\mathrm{Aut}(\mathfrak{g}, \tilde{A})^+ \sqcup \mathrm{Aut}(\mathfrak{g}, \tilde{A})^-)/\mathrm{Int}(\mathfrak{g}, \tilde{A})^+$.

Assume that a quadruplet $(G/\overline{H}, \bar{\sigma}, \bar{I}, \bar{g})$, also, satisfies the conditions (2)-(a) and (2)-(b) on Proposition 5. Then there exists a unique element $\bar{A}_0$ such that (1)–(4) on Proposition 1 for $(G/\overline{H}, \bar{\sigma}, \bar{I}, \bar{g})$ and $\tilde{\theta}$. This assures $(G/\overline{H}, \bar{\sigma}, \bar{I}) = (G/\tilde{H}, \tilde{\sigma}, \tilde{I})$. In addition, Proposition 2 causes that $\bar{g}$ is the G-invariant extension of $B_{\mathfrak{g}}|_{\tilde{\mathfrak{m}} \times \tilde{\mathfrak{m}}}$ up to constant. Hence $(G/\overline{H}, \bar{\sigma}, \bar{I}, \bar{g})$ coincides with $(G/\tilde{H}, \tilde{\sigma}, \tilde{I}, \tilde{g})$, where the coincidence of metrics is up to constant. □

REMARK 3. Take an arbitrary space $(G/H, \sigma, I, g)$ in Theorem 1, namely an absolutely simple para-Hermitian symmetric space of hyperbolic orbit type under the condition which the center of G is trivial. Assume that this Lie group G is the above G fixed first in this subsection. Then, for any Cartan involution θ of $\mathfrak{g}$ which commutes with σ_*, there exists a unique element Z such that (1)–(4) on Proposition 1 for $(G/H, \sigma, I, g)$ and θ. Additionally, by Proposition 5-(1), there exists an inner automorphism ϕ of $\mathfrak{g}$ such that $\tilde{Z} := \phi(Z) \in \tilde{W}$, $\tilde{\gamma}(\tilde{Z}) = 1$, and $\mathrm{I}(G/H, g)/\mathrm{I}(G/H, g)_0 \cong (\mathrm{Aut}(\mathfrak{g}, \tilde{Z})^+ \sqcup \mathrm{Aut}(\mathfrak{g}, \tilde{Z})^-)/\mathrm{Int}(\mathfrak{g}, \tilde{Z})^+$. Simultaneously, by Proposition 5-(2), there exists the para-Hermitian symmetric space $(G/\tilde{H}, \tilde{\sigma}, \tilde{I}, \tilde{g})$ of hyperbolic orbit type for the element $\tilde{Z}$. This space $(G/\tilde{H}, \tilde{\sigma}, \tilde{I}, \tilde{g})$ coincides just with $(G/H, \sigma, I, g)$ because $\tilde{Z} = \phi(Z)$ and $\phi \in \mathrm{Int}(\mathfrak{g})$, where the coincidence of metrics is up to constant.

Therefore, an arbitrary para-Hermitian symmetric space in Theorem 1 can be generated from an element $\tilde{A} \in \tilde{W}$ with $\tilde{\gamma}(\tilde{A}) = 1$ in an absolutely simple Lie algebra, where $\tilde{\gamma}$ and $\tilde{W}$ are the same constructed with the above way for this Lie algebra. Hence, it is sufficient to determine $\mathrm{I}(G/H, g)/\mathrm{I}(G/H, g)_0$ for an arbitrary space $(G/H, \sigma, I, g)$ in Theorem 1 that we determine $(\mathrm{Aut}(\mathfrak{g}, \tilde{A})^+ \sqcup \mathrm{Aut}(\mathfrak{g}, \tilde{A})^-)/\mathrm{Int}(\mathfrak{g}, \tilde{A})^+$ for all $\tilde{A} \in \tilde{W}$ with $\tilde{\gamma}(\tilde{A}) = 1$ in every absolutely simple Lie algebra $\mathfrak{g}$, where $\tilde{\gamma}$ and $\tilde{W}$ are the same constructed with the above way for $\mathfrak{g}$. With the paper [Ta], we have already known the group $\mathrm{Aut}(\mathfrak{g})/\mathrm{Int}(\mathfrak{g})$. Thus, we can individually determine $(\mathrm{Aut}(\mathfrak{g}, \tilde{A})^+ \sqcup \mathrm{Aut}(\mathfrak{g}, \tilde{A})^-)/\mathrm{Int}(\mathfrak{g}, \tilde{A})^+$ for all $\tilde{A} \in \tilde{W}$ with $\tilde{\gamma}(\tilde{A}) = 1$ in $\mathfrak{g}$.

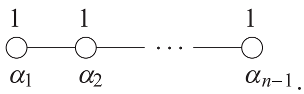
4.2. Examples of the Classical Type. Parenthetically, we note that the notation of the Lie groups and Lie algebras in this paper is that of in [He].

Our aim in this subsection is to present our procedure for determining $(\mathrm{Aut}(\mathfrak{g}, \tilde{A})^+ \sqcup \mathrm{Aut}(\mathfrak{g}, \tilde{A})^-)/\mathrm{Int}(\mathfrak{g}, \tilde{A})^+$ for $\mathfrak{g}$ of the classical type. Let us consider the two types $\mathfrak{sl}(n, \mathbf{R})$ and $\mathfrak{so}(n, n)$.

Type $\mathfrak{sl}(n, \mathbf{R})$. Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{sl}(n, \mathbf{R})$ and let $\tilde{\theta} : X \mapsto -{}^tX$ for $X \in \mathfrak{g}$, where $n \geq 3$. Then the 1 eigenspace $\tilde{\mathfrak{k}}$ of $\tilde{\theta}$ in $\mathfrak{g}$ is $\mathfrak{so}(n)$ and the -1 eigenspace $\tilde{\mathfrak{p}}$ is $\{X \in \mathfrak{sl}(n, \mathbf{R}) \mid {}^tX = X\}$. Now we choose

$$\tilde{\mathfrak{a}} := \left\{ \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix} \middle| \sum_{i=1}^n a_i = 0 \right\} \subset \tilde{\mathfrak{p}},$$

$\tilde{\Delta} := \{\pm(e_i - e_j) \mid 1 \leq i < j \leq n\}$, and $\tilde{\Delta}^+ := \{e_i - e_j \mid 1 \leq i < j \leq n\}$, and have the fundamental system $\tilde{\Pi} \subset \tilde{\Delta}^+$ as $\{\alpha_k := e_k - e_{k+1} \mid 1 \leq k \leq n-1\}$ and $\tilde{\gamma} = \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}$:



Set $\{Z_1, \ldots, Z_{n-1}\}$ as the dual basis of $\tilde{\Pi}$, namely

$$Z_i = \frac{1}{n} \begin{pmatrix} (n-i)E_i & 0 \\ 0 & -iE_{n-i} \end{pmatrix},$$

where E_l denotes the unit matrix of order l. Every element $\tilde{A} \in \tilde{W}$ with $\tilde{\gamma}(\tilde{A}) = 1$ in the absolutely simple Lie algebra $\mathfrak{g}$ is any in $\{Z_1, \ldots, Z_{n-1}\}$. In this connection, the eigenspaces $\mathfrak{g}_{-1}$, $\mathfrak{g}_1$, and $\mathfrak{g}_0$ of $\mathrm{ad}\, Z_i$ in $\mathfrak{g}$ are evaluated as

$$\mathfrak{g}_{-1} = \left\{ \begin{pmatrix} O & O \\ X_{n-i,i} & O \end{pmatrix} \middle| X_{n-i,i} : (n-i) \times i \text{ real matrix} \right\},$$

$$\mathfrak{g}_1 = \left\{ \begin{pmatrix} O & X_{i,n-i} \\ O & O \end{pmatrix} \middle| X_{i,n-i} : k \times (n-i) \text{ real matrix} \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} X_i & O \\ O & X_{n-i} \end{pmatrix} \middle| \begin{array}{c} \mathrm{Tr}(X_i + X_{n-i}) = 0, \\ X_i : i \times i, X_{n-i} : (n-i) \times (n-i) \text{ real matrix, respectively} \end{array} \right\}$$

$$\cong \mathfrak{sl}(i, \mathbf{R}) \oplus \mathfrak{sl}(n-i, \mathbf{R}) \oplus \mathbf{R}.$$

Let us assume that $1 \leq k \leq n-1$ and let us be just about to determine $(\mathrm{Aut}(\mathfrak{g}, Z_k)^+ \sqcup \mathrm{Aut}(\mathfrak{g}, Z_k)^-)/\mathrm{Int}(\mathfrak{g}, Z_k)^+$.

(*Case* I: *n is odd*). With the paper [Ta], we see that $\mathrm{Aut}(\mathfrak{g})/\mathrm{Int}(\mathfrak{g}) \cong \mathbf{Z}_2$ is generated by $\tilde{\theta}$. Here,

Lemma 5. $\mathrm{Int}(\mathfrak{g}, Z_k)^- := \{\phi \in \mathrm{Int}(\mathfrak{g}) \mid \phi(Z_k) = -Z_k\}$ *is empty.*

Proof. If there existed a real matrix

$$x = \begin{pmatrix} A_k & B_{k,n-k} \\ C_{n-k,k} & D_{n-k} \end{pmatrix}$$

such that $\mathrm{Ad}\, x(Z_k) = -Z_k$, then the following conditions would be satisfied:

$$(n-k)A = -(n-k)A, \quad -kB = -(n-k)B, \quad (n-k)C = kC, \quad -kD = kD.$$

Hence we would have $A = O$ and $D = O$, and would obtain $B = O$ and $C = O$ because $n \neq 2k$; as a result, $x = O$ and $\mathrm{Ad}\, x(Z_k) = O$. □

Since $\mathrm{Int}(\mathfrak{g}, Z_k)^- = \varnothing$ and $\tilde{\theta}(Z_k) = -Z_k$, the following holds:

$$(\mathrm{Aut}(\mathfrak{g}, Z_k)^+ \sqcup \mathrm{Aut}(\mathfrak{g}, Z_k)^-)/\mathrm{Int}(\mathfrak{g}, Z_k)^+ = \{[\mathrm{id}_{\mathfrak{g}}], [\tilde{\theta}]\} \cong \mathbf{Z}_2.$$

(*Case* II: *n is even*). The paper [Ta] reports that $\mathrm{Aut}(\mathfrak{g})/\mathrm{Int}(\mathfrak{g}) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$ is generated by $\tilde{\theta}$ and $\mathrm{Ad}\, I_{1,n-1}$, where

$$I_{1,n-1} = \begin{pmatrix} -1 & O \\ O & E_{n-1} \end{pmatrix}.$$

By a similar way to Lemma 5, $\mathrm{Int}(\mathfrak{g}, Z_k)^-$ is empty in the condition $n \neq 4k$. If k is just n over 4, for instance $\mathrm{Ad}\, a$ is an inner automorphism of $\mathfrak{g}$ moving Z_k to $-Z_k$. Here

$$a = \begin{pmatrix} O_k & B_k \\ C_k & O_k \end{pmatrix}; \quad (B_k, C_k) = (E_k, -E_k) \text{ if } k \text{ is odd}, \ B_k = C_k = E_k \text{ if } k \text{ is even}.$$

Notice that $\mathrm{Int}(\mathfrak{g}, Z_k)^- = \mathrm{Ad}\, a \circ \mathrm{Int}(\mathfrak{g}, Z_k)^+$ and $\tilde{\theta}$ is commutable with $\mathrm{Ad}\, a$ if $n = 4k$: thus

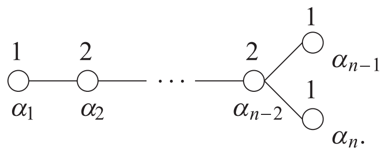
$$(\mathrm{Aut}(\mathfrak{g}, Z_k)^+ \sqcup \mathrm{Aut}(\mathfrak{g}, Z_k)^-)/\mathrm{Int}(\mathfrak{g}, Z_k)^+$$

$$\cong \begin{cases} \mathbf{Z}_2 \oplus \mathbf{Z}_2 & \text{by } \tilde{\theta} \text{ and } \mathrm{Ad}\, I_{1,n-1} \quad (n \neq 4k), \\ \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 & \text{by } \tilde{\theta},\ \mathrm{Ad}\, I_{1,n-1}, \text{ and } \mathrm{Ad}\, a \quad (n = 4k). \end{cases}$$

Type $\mathfrak{so}(n,n)$. Put $\mathfrak{g} := \mathfrak{so}(n,n)$ and $\tilde{\theta} := \mathrm{Ad}\, I_{n,n}$, and then $\tilde{\mathfrak{k}} \cong \mathfrak{so}(n) \oplus \mathfrak{so}(n)$ and

$$\tilde{\mathfrak{p}} = \left\{ \begin{pmatrix} O & X_n \\ {}^tX_n & O \end{pmatrix} \middle| X_n : n \text{ real matrix} \right\},$$

where $n \geq 5$. Choose the abelian subspace $\tilde{\mathfrak{a}}$ in $\tilde{\mathfrak{p}}$ as the one constructed by all diagonal matrix of order n and take $\tilde{\Delta}^+ := \{e_i \pm e_j \mid 1 \leq i < j \leq n\}$. These induce that the fundamental system $\tilde{\Pi} \subset \tilde{\Delta}^+$ is $\{\alpha_k := e_k - e_{k+1}, \alpha_n := e_{n-1} + e_n \mid 1 \leq k \leq n-1\}$ and that $\tilde{\gamma} = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$:



Denote the dual basis of $\tilde{\Pi}$ by $\{Z_1, \ldots, Z_n\}$. Then an element $\tilde{A} \in \tilde{W}$ with $\tilde{\gamma}(\tilde{A}) = 1$ in the absolutely simple Lie algebra $\mathfrak{g}$ is only Z_1, Z_{n-1}, or Z_n; concretely

$$Z_k = \begin{pmatrix} O & X(k) \\ {}^tX(k) & O \end{pmatrix}, \quad X(1) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$X(n-1) = -\frac{1}{2} I_{n-1,1}, \quad X(n) = \frac{1}{2} E_n.$$

Here the element Z_{n-1} can be mapped to Z_n by the involution $\mathrm{Ad}\, I_{n-1,1}$ of $\mathfrak{g}$. Accordingly, it is sufficient that we consider the two cases with Z_1 or Z_n. Furthermore, we obtain $\mathfrak{c}_{\mathfrak{g}}(Z_1) \cong \mathfrak{so}(n-1, n-1) \oplus \mathbf{R}$ and $\mathfrak{c}_{\mathfrak{g}}(Z_n) \cong \mathfrak{sl}(n, \mathbf{R}) \oplus \mathbf{R}$.

(*Case* I: *n is odd*). An result of [Ta] is that $\mathrm{Aut}(\mathfrak{g})/\mathrm{Int}(\mathfrak{g}) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$ is generated by $\tilde{\theta}$ and $\mathrm{Ad}\, J_n$, where

$$J_n = \begin{pmatrix} O & E_n \\ -E_n & O \end{pmatrix}.$$

On both of the cases with Z_1 and Z_n: the set $\mathrm{Int}(\mathfrak{g}, Z_k)^-$ is empty by a similar way to Lemma 5 and the maps $\tilde{\theta}$ and $\mathrm{Ad}\, J_n$ move Z_k to $-Z_k$. Hence $(\mathrm{Aut}(\mathfrak{g}, Z_k)^+ \sqcup \mathrm{Aut}(\mathfrak{g}, Z_k)^-)/\mathrm{Int}(\mathfrak{g}, Z_k)^+ \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$ by $\tilde{\theta}$ and $\mathrm{Ad}\, J_n$, where $k = 1$ or n.

(*Case* II: *n is even*). Applying the paper [Ta] in the case, we see that $\mathrm{Aut}(\mathfrak{g})/\mathrm{Int}(\mathfrak{g}) \cong (\mathbf{Z}_2 \oplus \mathbf{Z}_2) \rtimes \mathbf{Z}_2$ is generated by $\mathrm{Ad}\, a$, $\mathrm{Ad}\, J_n$, and $\mathrm{Ad}\, I_{1,n-1}$. We obtain $\mathrm{Ad}\, a \in \mathrm{Aut}(\mathfrak{g}, Z_k)^+$ and $\mathrm{Ad}\, J_n\, \mathrm{Aut}(\mathfrak{g}, Z_k)^-$. However, we just note that $\mathrm{Ad}\, I_{1,n-1} \notin \mathrm{Aut}(\mathfrak{g}, Z_k)^+ \sqcup \mathrm{Aut}(\mathfrak{g}, Z_k)^-$. Here $k = 1$ or n and

$$a = I_{1,n-1} \times I_{1,n-1} = \begin{pmatrix} I_{1,n-1} & O \\ O & I_{1,n-1} \end{pmatrix}.$$

On the one case with Z_1: the inner automorphism $\mathrm{Ad}\, b$ of $\mathfrak{g}$ carries Z_1 to $-Z_1$, where $b = I_{1,n-1} \times -I_{1,n-1}$. Moreover, by this inner automorphism, $[\mathrm{Ad}\, I_{1,n-1}] \in (\mathrm{Aut}(\mathfrak{g}, Z_1)^+ \sqcup \mathrm{Aut}(\mathfrak{g}, Z_1)^-)/\mathrm{Int}(\mathfrak{g}, Z_1)^+$; particularly $\mathrm{Ad}\, I_{1,n-1} \circ \mathrm{Ad}\, b \in \mathrm{Aut}(\mathfrak{g}, Z_k)^+$. By regarding the algebraic relation among these, the following holds:

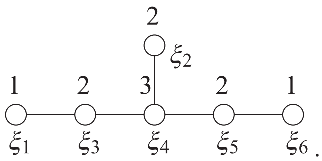
$$\begin{aligned}&(\mathrm{Aut}(\mathfrak{g}, Z_k)^+ \sqcup \mathrm{Aut}(\mathfrak{g}, Z_k)^-)/\mathrm{Int}(\mathfrak{g}, Z_k)^+ \\ &\qquad \cong \mathbf{Z}_2 \oplus ((\mathbf{Z}_2 \oplus \mathbf{Z}_2) \rtimes \mathbf{Z}_2) \quad \text{by } \mathrm{Ad}\, b, \mathrm{Ad}\, a, \mathrm{Ad}\, J_n, \text{ and } \mathrm{Ad}\, I_{1,n-1}.\end{aligned}$$

On the other case (with Z_n): there exists an inner automorphism of $\mathfrak{g}$ which can transfer Z_n to $-Z_n$, for instance the Cartan involution $\tilde{\theta} = \mathrm{Ad}\, I_{n,n}$. Nevertheless, $[\mathrm{Ad}\, I_{1,n-1}] \notin (\mathrm{Aut}(\mathfrak{g}, Z_1)^+ \sqcup \mathrm{Aut}(\mathfrak{g}, Z_1)^-)/\mathrm{Int}(\mathfrak{g}, Z_1)^+$ by any inner automorphism of $\mathfrak{g}$. This is realized by a similar way to Lemma 5. As a result,

$$\begin{aligned}&(\mathrm{Aut}(\mathfrak{g}, Z_1)^+ \sqcup \mathrm{Aut}(\mathfrak{g}, Z_1)^-)/\mathrm{Int}(\mathfrak{g}, Z_1)^+ \\ &\qquad \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \quad \text{by } \tilde{\theta}, \ \mathrm{Ad}\, a, \text{ and } \mathrm{Ad}\, J_n.\end{aligned}$$

4.3. Examples of the Exceptional Type. In the last place, we aim to present our proceeding of the exceptional type. Let us consider the two types $\mathfrak{e}_{6(6)}$ and $\mathfrak{e}_{7(-25)}$.

Type $\mathfrak{e}_{6(6)}$. Let $\mathfrak{g}_{\mathbf{C}}$ be the complex Lie algebra $(\mathfrak{e}_6)_{\mathbf{C}}$ and let $\mathfrak{h}_{\mathbf{C}}$ be a Cartan subalgebra of $\mathfrak{g}_{\mathbf{C}}$. Then we have the system of nonzero roots $\Delta_{\mathbf{C}}$ of $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ and a fundamental system $\Pi_{\mathbf{C}} := \{\xi_1, \dots, \xi_6\}$ of $\Delta_{\mathbf{C}}$ with the coefficients of the maximal root as follows:



Let $\mathfrak{g}_{\mathbf{C}} = \mathfrak{h}_{\mathbf{C}} \oplus \bigoplus_{\xi \in \Delta_{\mathbf{C}}} \mathfrak{g}_\xi$ be the root space decomposition, where $\mathfrak{g}_\xi$ is the root space of $\xi \in \Delta_{\mathbf{C}}$. Then there exist a Weyl basis $\{X_\xi\}_{\xi \in \Delta_{\mathbf{C}}}$ of $\mathfrak{g}_{\mathbf{C}} \bmod \mathfrak{h}_{\mathbf{C}}$, a set $\{H_\xi\}_{\xi \in \Delta_{\mathbf{C}}}$, and a set $\{N_{\xi,\eta}\}_{(\xi,\eta) \in \Delta_{\mathbf{C}} \times \Delta_{\mathbf{C}}}$ of real numbers such that

$$X_\xi \in \mathfrak{g}_\xi, \quad [X_\xi, X_{-\xi}] = H_\xi \in \mathfrak{h}_{\mathbf{C}}, \quad B_{\mathfrak{g}_{\mathbf{C}}}(H_\xi, H) = \xi(H) \text{ (for all } H \in \mathfrak{h}_{\mathbf{C}}) \quad \text{for } \xi \in \Delta_{\mathbf{C}},$$

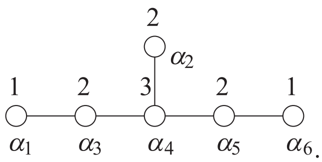
and

$$N_{\xi,\eta} = -N_{-\xi,-\eta}, \quad [X_\xi, X_\eta] = \begin{cases} N_{\xi,\eta} X_{\xi+\eta} & (\xi + \eta \in \Delta_{\mathbf{C}}) \\ 0 & (\xi + \eta \neq 0 \text{ and } \notin \Delta_{\mathbf{C}}) \end{cases}$$

for each pair $(\xi, \eta) \in \Delta_{\mathbf{C}} \times \Delta_{\mathbf{C}}$. In addition, the real Lie algebra $\mathfrak{g}_u := \sum_{\xi \in \Delta_{\mathbf{C}}} \mathbf{R}(\sqrt{-1} H_\xi) + \sum_{\xi \in \Delta_{\mathbf{C}}} \mathbf{R}(X_\xi - X_{-\xi}) + \sum_{\xi \in \Delta_{\mathbf{C}}} \mathbf{R}(\sqrt{-1}(X_\xi + X_{-\xi}))$ is a compact real form of $\mathfrak{g}_{\mathbf{C}}$.

Now, define the involution $\rho_{\mathbf{C}}$ of $\mathfrak{g}_{\mathbf{C}}$ by $\xi_1 \leftrightarrow \xi_6$, $\xi_2 \leftrightarrow \xi_2$, $\xi_3 \leftrightarrow \xi_5$, and $\xi_4 \leftrightarrow \xi_4$. Moreover, put $\tilde{\theta} := \rho_{\mathbf{C}} \circ \exp \sqrt{-1}\pi \operatorname{ad} T_2$, where $\{T_1, \dots, T_6\}$ is the dual basis of $\Pi_{\mathbf{C}}$. Then $\tilde{\theta}$ is an involution of $\mathfrak{g}_{\mathbf{C}}$ with $\tilde{\theta}(\mathfrak{g}_u) = \mathfrak{g}_u$. When $\mathfrak{k}$ (resp. $\sqrt{-1}\mathfrak{p}$) is the 1 (resp. -1) eigenspace of $\tilde{\theta}|_{\mathfrak{g}_u}$ in $\mathfrak{g}_u$, the subalgebra $\mathfrak{k}$ of $\mathfrak{g}_u$ is $\mathfrak{sp}(4)$ (cf. the list on p. 305 in [Mu]). Thus the real form $\mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}_{\mathbf{C}}$ is the Lie algebra $\mathfrak{e}_{6(6)}$, and we accordingly settle $\mathfrak{g}$ as $\mathfrak{k} \oplus \mathfrak{p}$.

Assume $\tilde{\theta}$ to be $\tilde{\theta}|_{\mathfrak{g}}$, and then $\tilde{\theta}$ is a Cartan involution of $\mathfrak{g}$. Take a maximal abelian subspace $\tilde{\mathfrak{a}}$ in $\mathfrak{p}$, and set $\tilde{\Delta}$ as the system of nonzero restricted roots of $(\mathfrak{g}, \tilde{\mathfrak{a}})$ and $\tilde{\Delta}^+$ as a half system of $\tilde{\Delta}$. Then the fundamental system $\tilde{\Pi} \subset \tilde{\Delta}^+$ is the $\{\alpha_1, \dots, \alpha_6\}$ with the coefficients of the maximal root as follows:



Let $\tilde{\gamma}$ be the maximal root of $\tilde{\Delta}^+$ and let $\{Z_1, \dots, Z_6\}$ be the dual basis of $\tilde{\Pi}$. It is only Z_1 or Z_6 that an element $\tilde{A} \in \tilde{W} = \{\tilde{A} \in \tilde{\mathfrak{a}} \mid \tilde{\alpha}(\tilde{A}) \geq 0 \text{ for all } \tilde{\alpha} \in \tilde{\Delta}^+\}$ with $\tilde{\gamma}(\tilde{A}) = 1$ in the absolutely simple Lie algebra $\mathfrak{g}$. This element Z_6 can be translated to Z_1 by the involution of $\mathfrak{g}$ defined by $\alpha_1 \leftrightarrow \alpha_6$, $\alpha_2 \leftrightarrow \alpha_2$, $\alpha_3 \leftrightarrow \alpha_5$, and $\alpha_4 \leftrightarrow \alpha_4$. Accordingly, let us consider the case with the element Z_1.

Here the algebra $\mathfrak{c}_{\mathfrak{g}}(Z_1)$ is $\mathfrak{so}(5,5) \oplus \mathbf{R}$ (cf. the list on p. 97 in [Ka-Koz]). In addition, the paper [Ta] informs us of what $\mathrm{Aut}(\mathfrak{g})/\mathrm{Int}(\mathfrak{g}) \cong \mathbf{Z}_2$ is generated by $\tilde{\theta}$. In addition,

Lemma 6. $\mathrm{Int}(\mathfrak{g}, Z_1)^- = \emptyset$.

Proof. Put $G := E_{6(6)}$ and $\sigma := \mathrm{A}_{\exp \sqrt{-1}\pi Z_1}$, and then we see $(G^\sigma)_0 \subset C_G(Z_1) \subset G^\sigma$. By Theorem 3.6.8-(2) on p. 219 in [Yo], the fixed points G^σ has two connected components and $G^\sigma = \mathbf{R}^* \times \mathit{Spin}(5,5)$, where $\mathbf{R}^* = \mathbf{R} \backslash \{0\}$.

Setting $\mathbf{R}^+$ as the set of positive numbers, we have $\mathbf{R}^+ \times \mathit{Spin}(5,5) = (G^\sigma)_0 \subset C_G(Z_1)$. In particular,

$$\{1\} \times \mathit{Spin}(5,5) \subset C_G(Z_1).$$

Since $\mathbf{R}^* \times \{e\} \subset Z(G^\sigma)$,

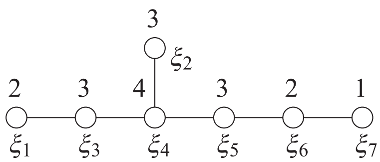
$$\mathbf{R}^* \times \{e\} \subset C_G(Z_1).$$

Owing to these conditions, $G^\sigma = \mathbf{R}^* \times \mathit{Spin}(5,5) \subset (\mathbf{R}^* \times \{e\})(\{1\} \times \mathit{Spin}(5,5)) \subset C_G(Z_1)C_G(Z_1) \subset C_G(Z_1)$, and then this assures $C_G(Z_1) = G^\sigma$ and $\mathrm{Int}(\mathfrak{g}, Z_1)^-$ is empty. □

Lemma 6 and what the involution $\tilde{\theta}$ transfers Z_1 to $-Z_1$ report a result:

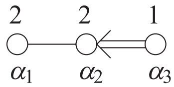
$$(\mathrm{Aut}(\mathfrak{g}, Z_1)^+ \sqcup \mathrm{Aut}(\mathfrak{g}, Z_1)^-)/\mathrm{Int}(\mathfrak{g}, Z_1)^+ = \{[\mathrm{id}_{\mathfrak{g}}], [\tilde{\theta}]\} \cong \mathbf{Z}_2.$$

Type $\mathfrak{e}_{7(-25)}$. Let $\mathfrak{g}_{\mathbf{C}}$ be the complex Lie algebra $(\mathfrak{e}_7)_{\mathbf{C}}$. Similarly to the type $\mathfrak{e}_{6(6)}$, we settle a Weyl basis, the compact real form $\mathfrak{g}_u$, and the dual basis $\{T_1, \dots, T_7\}$ of a fundamental system $\{\xi_1, \dots, \xi_7\}$ with the coefficients of the maximal root as follows:



Here, the involution $\tilde{\theta} := \exp \sqrt{-1}\pi \,\mathrm{ad}\, T_7$ of $\mathfrak{g}_{\mathbf{C}}$ leaves $\mathfrak{g}_u$ invariant. Hence we have the decomposition $\mathfrak{g}_u = \mathfrak{k} \oplus \sqrt{-1}\mathfrak{p}$, where $\mathfrak{k} = \mathfrak{e}_6 \oplus \sqrt{-1}\mathbf{R}$ (resp. $\sqrt{-1}\mathfrak{p}$) is the 1 (resp. -1) eigenspace of $\tilde{\theta}|_{\mathfrak{g}_u}$ in $\mathfrak{g}_u$ (cf. [Mu]). Thus the real form $\mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}_{\mathbf{C}}$ is the Lie algebra $\mathfrak{e}_{7(-25)}$, and accordingly put $\mathfrak{g} := \mathfrak{k} \oplus \mathfrak{p}$.

Think of $\tilde{\theta}$ as $\tilde{\theta}|_{\mathfrak{g}}$, and then $\tilde{\theta}$ is a Cartan involution of $\mathfrak{g}$. Take a maximal abelian subspace $\tilde{\mathfrak{a}}$ in $\mathfrak{p}$. With due order, we have a fundamental system $\tilde{\Pi} = \{\alpha_1, \alpha_2, \alpha_3\}$ with the coefficients of the maximal root as follows:



Set $\tilde{\gamma}$ as the maximal root of $\tilde{\Pi}$ and set $\{Z_1, Z_2, Z_3\}$ as the dual basis of $\tilde{\Pi}$, and then an element $\tilde{A} \in \tilde{W}$ with $\tilde{\gamma}(\tilde{A}) = 1$ in the absolutely simple Lie algebra $\mathfrak{g}$ is only Z_3.

Here, the algebra $\mathfrak{c}_{\mathfrak{g}}(Z_3)$ is $\mathfrak{e}_{6(-26)} \oplus \mathbf{R}$ (cf. [Ka-Koz]) and the group $\mathrm{Aut}(\mathfrak{g})/\mathrm{Int}(\mathfrak{g}) \cong \mathbf{Z}_2$ is generated by $\tilde{\theta}$ (cf. [Ta]). Additionally, the mapping ψ in the Weyl group of $G := E_{7(-25)}$ related to the following reflection is an inner automorphism of $\mathfrak{g}$ carrying Z_3 to $-Z_3$:

$$S_{\alpha_1+\alpha_2+\alpha_3} \circ S_{\alpha_1+\alpha_2} \circ S_{\alpha_1+\alpha_2+\alpha_3} \circ S_{\alpha_1+\alpha_2} \circ S_{\alpha_1+\alpha_2+\alpha_3} \circ S_{\alpha_3} \circ S_{\alpha_1+\alpha_2}$$
$$\circ S_{\alpha_2} \circ S_{\alpha_2+\alpha_3} \circ S_{\alpha_1+\alpha_2} \circ S_{\alpha_1+\alpha_2+\alpha_3},$$

where S_α is the reflection along a root α of $\mathfrak{a}$.

Since the involution $\tilde{\theta}$ transfers Z_3 to $-Z_3$ and commutes with ψ,

$$(\mathrm{Aut}(\mathfrak{g}, Z_1)^+ \sqcup \mathrm{Aut}(\mathfrak{g}, Z_1)^-)/\mathrm{Int}(\mathfrak{g}, Z_1)^+ = \{[\mathrm{id}_{\mathfrak{g}}], [\tilde{\theta}], [\psi], [\tilde{\theta} \circ \psi]\} \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2.$$

With a similar way to the above examples, we individually determine $(\mathrm{Aut}(\mathfrak{g}, \tilde{A})^+ \sqcup \mathrm{Aut}(\mathfrak{g}, \tilde{A})^-)/\mathrm{Int}(\mathfrak{g}, \tilde{A})^+$ and accordingly obtain Tables 1 and 2.

Acknowledgment

The authors would like to express their profound gratitude to Professor Nobutaka Boumuki. Without his guidance and persistence assistance, this paper would not have been materialized. In addition, what the authors would wish to do is to appreciate the referee providing specific comments and fruitful suggestions on earlier version of this paper.

References

[He] Helgason, S., Differential Geometry, Lie groups, and Symmetric Spaces, Amer. Math. Soc., Providence, RI, 2001.

[Ka] Kaneyuki, S., On classification of para-Hermitian symmetric spaces, Tokyo J. Math. **8** (1985), no. 2, 473–482.

[Ka-Koz] Kaneyuki, S. and Kozai, M., Paracomplex structures and affine symmetric spaces, Tokyo J. Math. **8** (1985), no. 1, 81–98.
[Koh] Koh, S. S., On affine symmetric spaces, Amer. Math. Soc. **119** (1965), 291–309.
[Mu] Murakami, S., Sur la classification des algèbres de Lie réelles et simples, Osaka J. Math. **2** (1965), 291–307.
[No1] Nomizu, K., Invariant affine connections on homogeneous spaces, Amer. J. Math. **76** (1954), 33–65.
[No2] Nomizu, K., On the group of affine transformations of an affinely connected manifold, Proc. Amer. Math. Soc. **4** (1955), 816–823.
[Ta] Takeuchi, M., On the fundamental group and the group of isometries of a symmetric space, J. Fac. Sci. Univ. Tokyo Sect. I **10** (1964), 88–123.
[Va] Varadarajan, V. S., Lie Groups, Lie Algebras, and Their Representations, Springer, New York, 1984.
[Yo] Yokota, I., Realizations of involutive automorphisms σ and G^σ of exceptional linear Lie groups G, part I, $G = G_2$, F_4, and E_6, Tsukuba J. Math. **14** (1990), no. 1, 185–223.

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