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Toroidal prefactorization algebras associated to holomorphic fibrations and a relationship to vertex algebras. (English) Zbl 07367636

Adv. Math. 386, Article ID 107799, 42 p. (2021)

Given a finite-dimensional complex Lie algebra with invariant form $(\mathfrak{g}, \langle , \rangle)$, the corresponding affine Lie algebra $\mathfrak{g}$ is a central extension of the loop algebra

$$\mathfrak{g}\left[z, z^{-1}\right] = \mathfrak{g} \otimes \mathbb{C}\left[z, z^{-1}\right]$$

by a one-dimensional center $\mathbb{C}\mathbf{k}$. It is familiar [*E. Frenkel* and *D. Ben-Zvi*, Vertex algebras and algebraic curves. Providence, RI: American Mathematical Society (AMS) (2004; Zbl 1106.17035)] that, for each $K \in \mathbb{C}$, the vacuum module

$$V_K(\widehat{\mathfrak{g}}) = \mathrm{Ind}_{\widehat{\mathfrak{g}}^+}^{\widehat{\mathfrak{g}}} \mathbb{C}_K$$

where

$$\widehat{\mathfrak{g}}^+ = \mathfrak{g}[z] \oplus \mathbb{C}\mathbf{k} \subset \widehat{\mathfrak{g}}$$

and $\mathbb{C}_K$ denotes its one-dimensional representation on which the first summand acts trivially and $\mathbf{k}$ acts by K, is of the structure of a vertex algebra and can be realized geometrically on a smooth complex curve X [*A. Beilinson* and *V. Drinfeld*, Chiral algebras. Providence, RI: American Mathematical Society (2004; Zbl 1138.17300)], and tied closely to the geometry of the moduli space $Bun_G(X)$ of principal G-bundles on X.

More generally, for a commutative $\mathbb{C}$-algebra R, one can construct the Lie algebra $\mathfrak{g}_R = \mathfrak{g} \otimes_{\mathbb{C}} R$ and its universal central extension $\widehat{\mathfrak{g}}_R$. The case $R = \mathbb{C}\left[z, z^{-1}\right]$ corresponds to the affine Kac-Moody algebra, while $\widehat{\mathfrak{g}}_R$ is known as the $(n+1)$-toroidal algebra in case of $R = \mathbb{C}\left[z^{\pm 1}, z_1^{\pm 1}, \dots, z_n^{\pm 1}\right]$. The principal objective in this paper is to show that, for $R = A\left[z, z^{-1}\right]$ with a commutative $\mathbb{C}$-algebra A, one can associate to $\widehat{\mathfrak{g}}_R$ a vertex algebra analogous to $V(\widehat{\mathfrak{g}}_R)$ which has a geometric realization. Connections between toroidal algebras and vertex algebras have been explored in [*S. Berman* et al., Contemp. Math. 297, 1–26 (2002; Zbl 1018.17017); *S. Eswara Rao* and *R. V. Moody*, Commun. Math. Phys. 159, No. 2, 239–264 (1994; Zbl 0808.17018); *H. Li* et al., J. Algebra 365, 50–82 (2012; Zbl 1345.17019); *R. V. Moody* et al., Geom. Dedicata 35, No. 1–3, 283–307 (1990; Zbl 0704.17011)].

It is observed in this paper that Theorem (Theorem 2.9 and Proposition 2.10). When $R = A[z, z^{-1}]$, $V(\widehat{\mathfrak{g}}_R)$ has the structure of a vertex algebra. Furthermore, this structure is functorial in A.

This paper is largely devoted to giving a geometric realization of $V(\widehat{\mathfrak{g}}_R)$ in the language of factorization algebras [*K. Costello* and *O. Gwilliam*, Factorization algebras in quantum field theory. Volume 1. Cambridge: Cambridge University Press (2016; Zbl 1377.81004); Volume 2. Cambridge: Cambridge University Press (2021; Zbl 07376333)], where it was shown that there is a close relationship between a certain class of prefactorization algebras on $\mathbb{C}$ and vertex algebras. The following theorem of Gostello and Gwilliam plays a central role in this paper.

Theorem (Zbl 1377.81004, Theorem 5.3.3). Let $\mathcal{F}$ be a unital S^1-equivariant holomorphically translation invariant prefactorization algebra on $\mathbb{C}$ abiding by certain natural conditions. Then the vector space

$$V(\mathcal{F}) = \bigoplus_{t \in \mathbb{Z}} H^*(\mathcal{F}^{(l)}(\mathbb{C}))$$

has the structure of a vertex algebra, where $\mathcal{F}^{(l)}(\mathbb{C})$ denotes the l-th eigenspace of in $\mathcal{F}(\mathbb{C})$.

The authors begin with two pieces of data to construct prefactorization algebras.

- A locally trivial holomorphic fibration $\pi : E \to X$ of complex manifolds with fiber F.
- A Lie algebra with invariant bilinear form $(\mathfrak{g}, \langle , \rangle)$.

When F is a smooth affine complex variety with a trivial fibration $E = X \times F$, we obtain a chain of inclusions of factorization enveloping algebras

$$\mathcal{G}_{\mathfrak{g},\pi}^{alg} \subset \mathcal{G}_{\mathfrak{g},\pi} \subset \mathcal{F}_{\mathfrak{g},\pi}$$

corresponding to the inclusion of sheaves of dg Lie algebras

$$(\mathfrak{g} \otimes H^0(F, \mathcal{O}_F^{alg}) \otimes \Omega_X^{0,*}, \overline{\partial}) \subset (\mathfrak{g} \otimes \Gamma(F, \Omega_F^{0,*}) \otimes \Omega_X^{0,*}, \overline{\partial}) \subset \mathfrak{g}_\pi$$

which extends to the central extensions, where denotes the sheaf of algebraic functions.

The principal result in this paper is

Theorem (Theorem 5.2). Let F be a smooth affine variety, and $\pi : \mathbb{C} \times F \to \mathbb{C}$ the trivial fibration with fiber F. Then

1. The toroidal prefactorization algebra $\mathcal{G}_{\mathfrak{g},\pi}^{alg}$ abides by the hypothesis of the above theorem of Gostello and Gwilliam.
2. The vertex algebra $V(\mathcal{G}_{\mathfrak{g},\pi}^{alg})$ is isomorphic to the toroidal vertex algebra $V(\widehat{\mathfrak{g}}_R)$, with

$$R = H^0(F, \mathcal{O}_F^{alg}) \left[t, t^{-1}\right]$$

The proof of the above theorem breaks down into two major steps.

1. It is verified that the various technical hypothesis of the above theorem of Gostello and Gwilliam are satisfied.
2. The second step is a somewhat lengthy direct calculation after the approach taken in [*B. Williams*, Lett. Math. Phys. 107, No. 12, 2189–2237 (2017; Zbl 06814928)] for the Virasoro factorization algebra and in [Zbl 1377.81004, §5.5.5] for the affine factorization algebra.

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

17B69 Vertex operators; vertex operator algebras and related structures
81R10 Infinite-dimensional groups and algebras motivated by physics, including Virasoro, Kac-Moody, W-algebras and other current algebras and their representations
32Q99 Complex manifolds

Keywords:

factorization algebras; toroidal algebras; vertex algebras

Full Text: DOI

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