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研究課題名（和文）Beyond Generalized Moonshine

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研究成果の概要（和文）：本研究の主要結果は以下の通り:1)モンスター対称性を持つムーンシャイン頂点作用素代数の自己双対整数形式を構成した。これにより、1994年のRybaによるモジュラームーンシャイン予想を解決したことになる。２)未知定数の集合が１のべき根であることを証明したことにより、申請者の先行研究である一般ムーンシャインの定理を拡張させた。３)モンスター群とLeech格子の自己同型写像との対応に対するorbifold双対予想（1993年Tuite）を解決した。４)157個のモンストラスでない完全複製可能関数のうち154個に対して、それらが正則頂点作用素代数の自己同型のトレース関数として現れないことを証明した。

研究成果の学術的意義や社会的意義
Monstrous moonshine is a mathematical phenomenon that was discovered when someone noticed some large numbers from calculations in two different fields of mathematics were very similar. My work in this project has answered some old questions about the nature of this connection and similar phenomena.

研究成果の概要（英文）：In this project, I have four main results:1) I have constructed a self-dual integral form of the Moonshine vertex operator algebra, with monster symmetry. This resolves Ryba's 1994 Modular moonshine conjecture. 2) I strengthened my earlier proof of the Generalized Moonshine conjecture, by showing that a collection of ambiguous constants are roots of unity. 3) I proved Tuite's 1993 orbifold duality conjecture, connecting non-Fricke elements of the monster to certain fixed-point free automorphisms of the Leech lattice. 4) Together with T. Komuro and S. Urano, I showed that 154 of the 157 non-monstrous completely replicable functions cannot arise as traces of automorphisms of holomorphic vertex operator algebras.

研究分野：数学

キーワード：代数　ムーンシャイン　頂点作用素代数　モジュラー形式

１．研究開始当初の背景

(1) Monstrous moonshine arose in the 1970s, when McKay noticed that the q^1-coefficient of the modular j-function is 196884, while the smallest faithful complex representation of the largest sporadic simple group, called the monster, has dimension 196883. Computation with more coefficients and representations suggested that this was not a coincidence, but instead strongly suggested the existence of an infinite dimensional graded representation of the monster, whose graded dimension gives the q-expansion of the j-function. Further investigation by Conway and Norton suggested that in addition to the graded dimension, the graded characters of elements of the monster also yield interesting modular functions. In particular, Conway and Norton formulated the Monstrous Moonshine conjecture, asserting that the characters are Hauptmoduln, which are modular functions with strong symmetry properties. The representation in question, called $V^\natural$, was constructed in 1988 by Frenkel, Lepowsky, and Meurman using methods inspired by string theory, and Borcherds used it in 1992 to prove the Monstrous Moonshine conjecture.

Additional computations by Conway, Queen, and Norton yielded numerical evidence of more moonshine-like phenomena, connecting representations of subgroups of the monster with modular functions. This eventually led to two separate conjectures generalizing the Monstrous Moonshine conjecture: Norton's Generalized Moonshine conjecture (1987), and Ryba's Modular moonshine conjecture (1994). Furthermore, Tuite (1993) proposed a conjecture connecting $V^\natural$ with the Leech lattice vertex operator algebra by way of many cyclic orbifold constructions, and showed that the Hauptmodul property of moonshine functions would follow from suitable uniqueness statements, such as Frenkel, Lepowsky, and Meurman's uniqueness conjecture for $V^\natural$ (1988).

Norton's conjecture has been reinterpreted as a statement in orbifold conformal field theory about twisted modules of $V^\natural$. I solved Norton's conjecture in 2016 with a key step being my work with M. Miyamoto on rationality of solvable orbifolds. However, stronger forms of this conjecture have been proposed in light of numerical evidence and analogies from Mathieu Moonshine. In particular there is a collection of undetermined constants relating the modular functions in Generalized Moonshine, and Gaberdiel, Persson, Ronellenfitsch, and Volpato proposed that these constants are controlled by a specific 3-cocycle in the cohomology of the monster, by way of some explicit formulas.

Ryba's Modular moonshine conjecture was reinterpreted by Borcherds and Ryba as a statement about Tate cohomology with coefficients in a self-dual integral form of $V^\natural$. This refined conjecture was mostly solved by Borcherds and Ryba in the late 1990s, but their proof assumed the existence of a self-dual integral form of $V^\natural$.

Earlier progress on the problem of defining monster-symmetric forms of $V^\natural$ over rings smaller than $\mathbb{C}$ includes Frenkel-Lepowsky-Meurman's construction of a $\mathbb{Q}$-form in 1988, Borcherds-Ryba's construction of a self-dual $\mathbb{Z}[1/2]$-form in 1996, and Dong-Griess's construction of a $\mathbb{Z}$-form that is not self-dual in 2012.

２．研究の目的

(1) The original main goal of this project was to prove a strong form of Generalized Moonshine conjecture, and consider questions that connect moonshine with other mathematical and physical questions. In the form that I originally proved, there is a rule that attaches a modular function to each commuting pair of elements of the monster, and these functions satisfy compatibility relations with modular transformations and conjugacy "up to a constant". My primary aim was to show that these constants are controlled by a 3-cocycle for the monster with coefficients in $\mathbb{C}^\times$.

(2) In order to identify the cohomology class of the cocycle, I intended to establish a theory of twisted conformal blocks on nodal curves. Such a theory would allow higher associativity data from fusing twisted modules to determine the class.

(3) During the course of this project, some new developments in the field led to a new goal: finding new constructions of the Moonshine module vertex operator algebra $V^\natural$. In particular, Abe, Lam, and Yamada showed in early 2017 that for primes p among 3,5,7, and 13, one could build $V^\natural$ by an intermediate orbifold construction using certain automorphisms of the Leech lattice of order $2p$. It was therefore natural to try to find all cyclic orbifold constructions of $V^\natural$. This is precisely the domain of Tuite's conjecture. The mathematical foundations of cyclic orbifold duals were completed by van Ekeren, Möller, and Scheithauer in 2015, so Tuite's conjecture had suddenly become feasible.

(4) One of the most interesting points in Abe, Lam, and Yamada's construction is that it requires very few computations. This suggested that with current methods, the construction of a self-dual integral form of $V^\natural$ over a small ring like $\mathbb{Z}[1/n, e^{\pi i/n}]$ may not need much explicit work. Therefore, the second goal I made after reading their result was finding out whether their method could be used to construct a self-dual integral form of the $V^\natural$.

(5) As a part of the construction of a self-dual integral form of $V^\natural$, it was necessary to understand the automorphisms of the Leech lattice vertex operator algebra over small commutative rings. The general question of understanding the automorphism group schemes of lattice vertex operator algebras naturally became an additional goal in this project. Following the work of Dong and Nagatomo, I conjectured that the automorphism group scheme has the form $G_L O(\hat{L})$, where G_L is a normal subgroup scheme that is a split reductive group over $\mathbb{Z}$ with root datum is determined in an explicit way by the lattice, and $O(\hat{L})$ is the automorphism group of a μ_2-cover of L. Here, when R is a commutative ring, $\mu_2(R) = \{x \in R | x^2 = 1\}$.

３．研究の方法

(1) For a complete solution of the strong form of Generalized Moonshine, it appears that regularity of the Monster-fixed point vertex operator subalgebra in $V^\natural$ may be necessary. This would require a suitable generalization of my work with M. Miyamoto. Many of the steps in our earlier work can be extended to this more general case.

(2) For the determination of fusion rules and higher associativity data for twisted modules, I attempted to generalize the theory of vertex algebras on algebraic curves that was developed by Frenkel and Ben-Zvi, and enhanced to allow twisted modules by Frenkel and Szczesny. For this purpose, I investigated logarithmic geometry, the theory of stacks, and the theory of crystals.

(3) For the cyclic orbifold constructions of $V^\natural$, I used the cyclic orbifold duality established by van Ekeren, Möller, and Scheithauer in 2015. Paquette, Persson, and Volpato showed in early 2017 that cyclic orbifolds of $V^\natural$ by Fricke elements yield $V^\natural$ again, so it suffices to show that cyclic orbifolds by non-Fricke elements yield the Leech lattice vertex operator algebra, and that the corresponding elements of the Conway group are precisely those that satisfy Tuite's "no massless states" condition. For the first claim, a uniqueness result of Dong and Mason implies it suffices to show the weight 1 space has dimension 24. This is done by analyzing the Weyl-Kac-Borcherds denominator formulas of the corresponding Monstrous Lie algebras. The second claim follows from Tuite's computations.

(4) For constructing a self-dual integral form, I needed to show that (i) order n cyclic orbifold duals could be defined over small rings - namely $\mathbb{Z}[1/n, e^{\pi i/n}]$. This is done using some direct manipulations with the axioms of abelian intertwining algebras. We find that the Abe-Lam-Yamada method yields a self-dual form of $V^\natural$ over such a ring. (ii) Next, I needed to show these forms have monster symmetry. The forms come with natural actions of large subgroups of the monster, so Wilson's work on maximal subgroups implies those subgroups generate the monster. (iii) Finally, I had to glue the various forms of $V^\natural$ over different small rings to get a self-dual

integral form. This is done by applying faithfully flat descent for vertex operator algebras. Gluing isomorphisms are given by Wilson's result on the existence of pB-free elementary subgroups of the monster.
(5) Working out the automorphism group scheme of a lattice vertex operator algebra V_L has two main stages: We need to show that G_L and $O(\hat{L})$ act faithfully on V_L, and we need to show that any automorphism of $V_L \otimes_{\mathbb{Z}} R$ as a vertex operator algebra over R is given by the product of an R-point in G_L and an R-point in $O(\hat{L})$. For the G_L action, it suffices to produce compatible actions of the maximal torus and root group schemes, and the $O(\hat{L})$ action naturally comes from the construction of V_L. In the reverse direction, it is necessary to prove that (i) for any automorphism, there is an element of G_L such that composition restricts to an automorphism of a Heisenberg subalgebra (ii) automorphisms of Heisenberg subalgebras induce uniquely defined automorphisms on the L-grading of V_L

４．研究成果
(1) I have constructed a self-dual integral form $V_{\mathbb{Z}}^{\natural}$ of $V^{\natural}$, with monster symmetry. This is a vertex operator algebra over the integers, with an invariant bilinear form. The most important consequence of this result is that the existence of such an integral form was the last open assumption in Borcherds and Ryba's solution to the modular moonshine conjecture. In particular, we now know that for any Fricke element g of prime order, the Tate cohomology $\hat{H}^0(g, V_{\mathbb{Z}}^{\natural})$ is a vertex algebra over $\mathbb{F}_p$ with an action of the centralizer of g, and for any p-regular centralizing element h, the graded Brauer character of h on $\hat{H}^0(g, V_{\mathbb{Z}}^{\natural})$ is a modular function equal to the McKay-Thompson series $T_{gh} = \sum_{n \geq 0} Tr(gh|V_n^{\natural})q^{n-1}$.

(2) As a second corollary to the existence of $V_{\mathbb{Z}}^{\natural}$, we find that as a graded abelian group, it is a direct sum of infinitely many positive-definite unimodular lattices with faithful monster actions. The existence of such a lattice in dimension 196884 was open until now. Conway and Norton had attempted to find such a lattice in 1985.

(3) Orbifold duality and the Tuite conjecture. I proved Tuite's 1993 conjecture that non-Fricke automorphisms of $V^{\natural}$ and fixed-point free automorphisms of the Leech lattice satisfying a "no massless states" condition are connected by cyclic orbifold duality. One unexpected corollary was a proof that the Monstrous Lie algebras attached to non-Fricke elements of the monster are Borcherds-Kac-Moody. This was previously only established for the Monstrous Lie algebras attached to Fricke elements, in the course of the proof of the Generalized Moonshine conjecture.

(4) From the Borcherds-Kac-Moody property of non-Fricke Monstrous Lie algebras, I was able to show that the constants relating the Hauptmoduls in Generalized Moonshine are roots of unity. This was an unexpected consequence of the solution to Tuite's conjecture, and is the first progress toward the strong form of the Generalized Moonshine conjecture.

(5) One corollary to the resolution of Tuite's conjecture is that any counterexample to Frenkel, Lepowsky, and Meurman's uniqueness conjecture cannot have any non-Fricke automorphisms. Borcherds's proof of the Monstrous Moonshine conjecture implies any automorphism of such a counterexample must have completely replicable McKay-Thompson series. Together with T. Komuro and S. Urano, I showed that at most 3 of the 157 non-monstrous completely replicable functions can arise as such a McKay-Thompson series.

(6) In joint work with H. Mochizuki, I have shown that the group schemes G_L and $O(\hat{L})$ act faithfully on V_L, and we have made substantial progress showing that all automorphisms come from these two groups.

５．主な発表論文等

〔雑誌論文〕　計6件（うち査読付論文　5件／うち国際共著　0件／うちオープンアクセス　4件）

１．著者名 Carnahan Scott	４．巻 12
２．論文標題 51 constructions of the Moonshine module	５．発行年 2018年
３．雑誌名 Communications in Number Theory and Physics	６．最初と最後の頁 305～334
掲載論文のDOI（デジタルオブジェクト識別子） 10.4310/CNTP.2018.v12.n2.a3	査読の有無 有
オープンアクセス オープンアクセスではない、又はオープンアクセスが困難	国際共著 －

１．著者名 Carnahan Scott, Komuro Takahiro、Urano Satoru	４．巻 14
２．論文標題 Characterizing Moonshine Functions by Vertex-Operator-Algebraic Conditions	５．発行年 2018年
３．雑誌名 Symmetry, Integrability and Geometry: Methods and Applications	６．最初と最後の頁 1-8
掲載論文のDOI（デジタルオブジェクト識別子） 10.3842/SIGMA.2018.114	査読の有無 有
オープンアクセス オープンアクセスとしている（また、その予定である）	国際共著 －

１．著者名 Carnahan Scott	４．巻 2086
２．論文標題 Moonshine over Z?	５．発行年 2018年
３．雑誌名 RIMS Kokyuroku	６．最初と最後の頁 91-98
掲載論文のDOI（デジタルオブジェクト識別子） なし	査読の有無 無
オープンアクセス オープンアクセスとしている（また、その予定である）	国際共著 －

１．著者名 Carnahan Scott	４．巻 373
２．論文標題 Building vertex algebras from parts	５．発行年 2020年
３．雑誌名 Communications in Mathematical Physics	６．最初と最後の頁 1-43
掲載論文のDOI（デジタルオブジェクト識別子） 10.1007/s00220-019-03607-0	査読の有無 有
オープンアクセス オープンアクセスではない、又はオープンアクセスが困難	国際共著 －

1．著者名 Carnahan Scott	4．巻 15:030
2．論文標題 A Self-Dual Integral Form of the Moonshine Module	5．発行年 2019年
3．雑誌名 SIGMA	6．最初と最後の頁 1-36
掲載論文のDOI（デジタルオブジェクト識別子） 10.3842/SIGMA.2019.030	査読の有無 有
オープンアクセス オープンアクセスとしている（また、その予定である）	国際共著 －

1．著者名 Carnahan Scott	4．巻 50:40
2．論文標題 Fricke Lie algebras and the genus zero property in Moonshine	5．発行年 2017年
3．雑誌名 Journal of Physics A: Mathematical and Theoretical	6．最初と最後の頁 1-19
掲載論文のDOI（デジタルオブジェクト識別子） 10.1088/1751-8121/aa781d	査読の有無 有
オープンアクセス オープンアクセスとしている（また、その予定である）	国際共著 －

〔学会発表〕　計13件（うち招待講演　13件／うち国際学会　9件）

1．発表者名 Carnahan Scott
2．発表標題 Extended Monstrous Moonshine
3．学会等名 Vertex operator algebras, number theory, and related topics, Sacramento State University（招待講演）（国際学会）
4．発表年 2018年

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4．発表年 2018年

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４．発表年
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１．発表者名
Scott Carnahan

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Moonshine over Z

３．学会等名
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４．発表年
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2019年

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３．学会等名
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４．発表年
2019年

〔図書〕　計0件

〔産業財産権〕

〔その他〕

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