Brown, Francis
Invariant differential forms on complexes of graphs and Feynman integrals. (English)
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SIGMA, Symmetry Integrability Geom. Methods Appl. 17, Paper 103, 54 p. (2021)

The principal objective in this paper is to study differential forms on a geometric incarnation of the graph complex [*M. Kontsevich*, in: The Gelfand Seminars, 1990-1992. Basel: Birkhäuser. 173–187 (1993; Zbl 0821.58018), where they are called the odd commutative graph complex; [*T. Willwacher*, Invent. Math. 200, No. 3, 671–760 (2015; Zbl 1394.17044)], where they are denoted by $\mathcal{GC}_N$]. To this end, the author consider a certain moduli space of metric graphs, which is related to both the moduli space of tropical cureves [Zbl 1218.14056] and Culler and Vogtmann's Outer spaces [*M. Culler* and *K. Vogtmann*, Invent. Math. 84, 91–119 (1986; Zbl 0589.20022)], and afterwards goes on to explain how to construct differential forms upon this space.

The invariant differential forms in question generate the stable real cohomology of the general linear group [*A. Borel*, Ann. Sci. Éc. Norm. Supér. (4) 7, 235–272 (1974; Zbl 0316.57026); *A. Borel*, Prog. Math. 14, 21–55 (1981; Zbl 0483.57026)]. By integrating such invariant forms over the space of metrics on a graph, the author defines canonical period integrals associated to graphs, which are shown always to be finite, taking the form of generalized Feynman integrals. The theory probes into the structure of the cohomology of the commutative graph complex, giving new connections between graph complexes, motivic Galois groups and quantum field theory, though it is not yet clear whether one can deduce a natural geometric action of the motivic Galois group $G^{\mathrm{mot}}_{\mathcal{MT}(\mathbb{Z})}$ on $H^0(\mathcal{GC}_2)$. Many of the constructions in the paper are valid more generally for certain classes of regular matroids.

The author conjectures

Conjecture. There is a non-canonical injective map of graded Lie algebras from the free LIe algebra on into graph cohomology

$$\mathbb{L}\left(\Omega^*_{\mathrm{can}}\right) \to \bigoplus_{n\in\mathbb{Z}} H^n\left(\mathcal{GC}_2\right)$$

with

$$\Omega^*_{\mathrm{can}} = \bigwedge\left(\bigoplus_{k\geqslant 1}\mathbb{Q}\omega^{4k+1}\right)$$

such that its restriction to the Lie subalgebras generated by primitive elements maps to the Lie subalgebra in degree zero

$$\mathbb{L}\left(\bigoplus_{k\geqslant 1}\omega^{4k+1}\right) \to H^0\left(\mathcal{GC}_2\right)$$

All other elements map to higher degree cohomology $\bigoplus_{n>0} H^n\left(\mathcal{GC}_2\right)$.

The author expects the above putative injective map not to be an isomorphism, because $H\left(\mathcal{GC}_N\right)$ is supposed to be too large.

Some related works go as follows.

- The work of *M. Berghoff* and *D. Kreimer* ["Graph complexes and Feynman rules", Preprint, `arXiv:arXiv:2008.09540`] studies properties of Feynman differential forms with respect to combinatorial operations on Outer space, differing from this paper in that the forms they consider have different degrees on the image of the cell, though raising intriguing possibility of constructing forms on moduli spaces of graphs with external legs whose dominator involves both the first and second Symanzik polynomials.
- *M. Kontsevich* [Astérisque 1126, 183–211 (2019)] suggested a possible relationship between the homology of the graph complex with a derived Grothendieck-Teichmüller Lie algebra defined from the moduli spaces $\mathcal{M}_{0,n}$ of curves of genus 0.
- *J. Alm* ["The Grothendieck-Teichmueller Lie algebra and Brown's dihedral moduli spaces", Preprint,

arXiv:1805.06684] introduced Stokes relations between multiple zeta values expressed as integrals over $\mathcal{M}_{0,n}$.

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

18G85 Graph complexes and graph homology
11F75 Cohomology of arithmetic groups
11M32 Multiple Dirichlet series and zeta functions and multizeta values
81Q30 Feynman integrals and graphs; applications of algebraic topology and algebraic geometry

Keywords:

graph complexes; outer space; tropical curves; motives; multiple zeta values; Feynman integrals; quantum field theory

Full Text: DOI

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