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A categorification of the Cartan-Eilenberg formula. (English) Zbl 07466440

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The classical Cartan-Eilenberg formula [*H. Cartan* and *S. Eilenberg*, Homological algebra. Princeton, NJ: Princeton University Press (1956; Zbl 0075.24305), Theorem 10.1] presents the mod p cohomology of a finite group G as a subalgebra of the cohomology for any p-Sylow subgroup S of G. The formalism of fusion systems provides a concise formula expressing the result in terms of the fusion category $\mathcal{F}_S(G)$ as follows:

$$H^*(G, \mathbb{F}_p) \simeq \lim_{\mathrm{P} \in \mathcal{F}_\mathrm{S}(\mathrm{G})^{\mathrm{op}}} H^*(P, \mathbb{F}_p)$$

The formula really holds for any cohomological Mackey functor $\mathcal{M}$ [*S. Park*, Commun. Algebra 45, No. 4, 1409–1415 (2017; Zbl 1372.20050)]:

$$\mathcal{M}(G) \simeq \lim_{\mathrm{P} \in \mathcal{F}_\mathrm{S}(\mathrm{G})^{\mathrm{op}}} \mathcal{M}(P)$$

The principal objective in this paper is to establish a categorical Cartan-Eilenberg formula (Theorem 3.9) for any p-monadic Mackey 2-functor $\mathcal{M}$:

$$\mathbb{M}(G) \cong \operatorname*{bilim}_{P \in \widehat{\mathcal{T}}_S(G)^{\mathrm{op}}} \mathbb{M}(P)$$

The left-hand is to stand, by way of example, for the categories of group representations

$$\mathbb{M}(G) = \mathrm{mod}(\mathbb{R}G)$$

the stable categories of group representations

$$\mathbb{M}(G) = \mathrm{stmod}(\mathbb{R}G)$$

or the derived categories of the group algebras

$$\mathbb{M}(G) = \mathrm{D}(kG)$$

$\mathbb{M}$ can be regarded as a 2-functor

$$\mathbb{M} : (\boldsymbol{gpd}^{\mathrm{f}})^{\mathrm{op}} \to \boldsymbol{Cat}$$

where $\boldsymbol{gpd}^{\mathrm{f}}$ is the 2-category of finite groupoids with faithful functors as 1-morphisms and $\boldsymbol{Cat}$ is the 2-category of small categories.

This categorification demands to replace the fusion category $\mathcal{F}_S(G)$ by an extended transporter category $\widehat{\mathcal{T}}_S(G)$ and the classical limit by a pseudo bilimit taken in the 2-category of categories. The bilimit in Theorem 3.9 is reinterpretable, through a 2-finality argument with recourse to the criterion in [*J. Maillard*, "On 2-final 2-functors", Preprint, `arXiv:2101.08727`], as a descent-shaped bilimit, claiming that the functor $\mathbb{M}$ is a 2-sheaf, which admits of recovering the main theorem in [*P. Balmer*, J. Eur. Math. Soc. (JEMS) 17, No. 1, 189–228 (2015; Zbl 1351.20004)].

The paper is based upon the author's PhD thesis under the supervision of *I. Dell'Ambrogio* [J. Homotopy Relat. Struct. 5, No. 1, 319–358 (2010; Zbl 1278.19010)].

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MSC:

18A05 Definitions and generalizations in theory of categories
18A30 Limits and colimits (products, sums, directed limits, pushouts, fiber products, equalizers, kernels, ends and coends, etc.)
18C15 Monads (= standard construction, triple or triad), algebras for monads, homology and derived functors for monads
18C20 Eilenberg-Moore and Kleisli constructions for monads
20C05 Group rings of finite groups and their modules (group-theoretic aspects)
20L05 Groupoids (i.e. small categories in which all morphisms are isomorphisms)

Keywords:

categorification; fusion of finite groups; bilimits

Full Text: DOI

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