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A St.Petersburg Market:

A Banker with a Budget and People with Cognitive Bounds

by

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A St.Petersburg Market: A Banker with a Budget and People with Cognitive Bounds*

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Abstract

We analyze the St.Petersburg paradox from the perspective of a real social context. We formulate a market for a coin-tossing gamble between a banker with a finite budget and people with cognitive bounds on probabilities. The budget for the banker alone removes the paradox in the original form, as argued by Shapley. However, the standard expected reward criterion for the banker and people leads to the other difficulty of vacuousness of trades between the banker and people. To consider this problem, we formulate explicitly the situation as a market with a banker and people. Introducing cognitive bounds on probabilities for the people, we show that the theory allows an affirmative possibility of some trades in the market. Specifically, we adopt Kaneko's theory of EU theory with cognitive bounds for the people. This theory leads us to the incomparability for people between buying a ticket and not, and we take one more step toward semi-rationalistic behavioral-probability for incomparable options. Then, some people behave with positive probabilities of buying tickets and the banker gets significantly positive net rewords. Thus, our approach avoids the difficulty mentioned above.

Key Words: St.Petersburg Paradox, Expected Utility Theory with Cognitive Bounds, Incomparability, Behavioral Probability, Monte Carlo Method

1 Introduction

The St.Petersburg (SP) paradox remains a conundrum since the time of Bernoulli [3]. In the literature, typically, the mathematical structures of the SP gamble or possible resolutions have been discussed (cf., Samuelson [20], Peterson [19]), but the question of in what sense it is a paradox has not been asked per se. One exception is Shapley [22], who condemns the SP gamble as a con game in that the paradox relies upon arbitrarily large prizes exceeding any feasible budgets in the real world. Taking his criticism seriously, we formulate the entire problem as a monopolistic market consisting of a banker with a budget for prizes and people with bounded cognitive abilities. We adopt the return on investment (ROI) index for the banker, and Kaneko's

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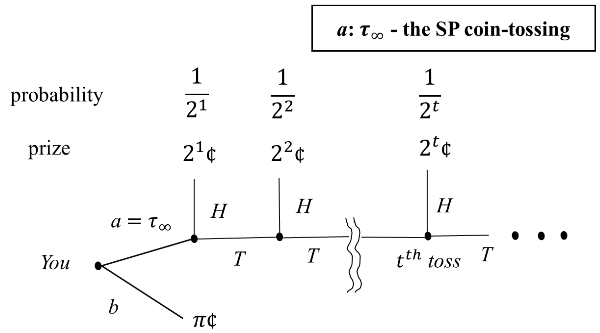

Figure 1: The original St.Petersburg game

[12] expected utility theory with cognitive bounds for the people. We argue that this theory allows an affirmative possibility of some trade in the market. In this section, we discuss various conceptual issues involved the development of the theory.

1.1 In what sense is the SP result a paradox?

The original SP coin-tossing gamble $g_\infty = [\tau_\infty, \pi]$ is as follows. A person decides either to participate or not in the coin-tossing τ_∞ with a fee π¢ (US cents) ($\pi \geq 0$); if he participates, a fair coin is tossed until it results in heads, and if it shows heads at the t-th coin toss, the prize 2^t¢ is attained and the gamble is concluded. The probability of having the prize 2^t¢ is $\frac{1}{2^t}$. The net reward from participation is $2^t - \pi$, and the reward from non-participation is 0. For simplicity, rather than net rewards, we consider the *opportunity* reward from non-participation b, depicted in Fig.1, which is π. The rewards from a are the prizes with the associated probabilities.

The expected reward from a is $\frac{1}{2} \cdot 2^1 + \cdots + \frac{1}{2^t} \cdot 2^t + \cdots = 1 + \cdots + 1 + \cdots = +\infty$, which is larger than any finite fee π. The *expected reward (ER) criterion* recommends to participate in the SP gamble whatever π is. On the other hand, an ordinary person tends to avoid the gamble unless π is very small. For instance, when $\pi = 50,000$¢ $= 500\$$ (USD), the probability of acquiring a prize greater than $50,000$¢ is $1 - (\frac{1}{2} + \cdots + \frac{1}{2^{18}}) = \frac{1}{2^{18}} = \frac{1}{262,144}$, since $2^{18} < 50,000 < 2^{19}$, and this probability is too small for him to take into account. We write this disagreement explicitly:

(*): For any large fee π, an ordinary person does not follow the ER recommendation.

We call (*) the *St.Petersburg paradox* in the original sense.

Shapley [22] criticizes the SP paradox (*) with the basic positions:

SH1: (*) is a paradox in an empirical sense but not in a mathematical sense.

SH2: (*) loses validity if a specific finite budget is introduced.

The first means that it is a paradox in a real context ([22], p.440) in that the recommendation of the ER criterion disagrees with our ordinary senses. The second states that when the budget is 2^{100}¢ and $\pi > 100$¢, the expected reward for a participant is $\frac{1}{2} \cdot 2 + \cdots + \frac{1}{2^{100}} \cdot 2^{100} = 100$¢ and the ER criterion recommends to avoid the SP gamble; but our ordinary senses quite possibly allow to participate in the gamble.

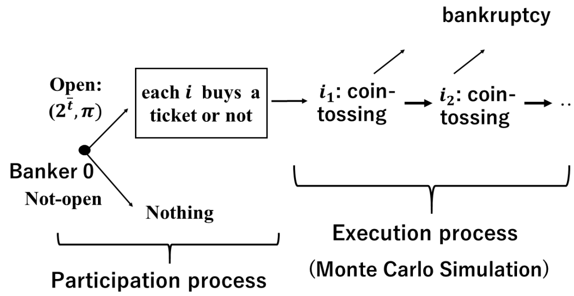

Figure 2: The SP gamble market (Γ_0, Γ_N)

Shapley succeeded in refuting the validity of (*)[1]. However, the budget 2^{100}¢ is absurd in a real context; the US federal annual budget for 2018 is about 4.2 trillion\$, roughly 2^{49}¢. Shapley called the SP gamble a *con game* and concluded that the SP paradox has little relevancy to game theory and economics ([22], p.442). The present author disagrees his conclusion and thinks that his critical argument has large potential. It could be applied to a class of problems where serious events take place with minuscule probabilities such as state lotteries, insurances, and more generally "black swan" events with very narrow possibilities but without good understanding of probabilities (cf., Teleb [25]). The other point is that there is no systematic theory to treat what the SP paradox is; SH1 and SH2 are interpreted to suggest to develop such a theory. The author in [12] developed a new EU theory incorporating cognitive bounds on probabilities, for which Shapley's critique provides a test. The present paper is written from these perspectives.

In SH1, we refer to ourselves as potential participants and observe a conflict between the EU recommendation and our *ordinary senses*. SH2 suggests some agent organizing and selling the SP gamble, whom we call the *banker*. We include the banker as a crucial player for an affirmative possibility of a trade of an SP gamble. We formulate the entire problem as a monopolistic market (Γ_0, Γ_N), called the *SP gamble market*, consisting of a banker and people, depicted in Fig.2.

In the market (Γ_0, Γ_N), the banker announces an SP gamble $(2^{\bar{t}}, \pi)$ to the people, where $2^{\bar{t}}$ is the maximum prize within the budget B_0 and π is the participation fee. Each person i evaluates this gamble to buy a ticket or not. The people with tickets go to the *execution process of coin-tossing;* each does coin-tossing one-by-one. The banker's evaluation criterion is the *ROI* (return on investment) index consisting of the ER revenue and the costs for the budget B_0 and facility. Our main concern is whether or not there are some *affirmative possibilities* for an SP gamble to attract people as well as to produce an enough revenue to have a positive *ROI* value.[2]

To have compatibility with the real world as well as our ordinary senses, we replace the budget 2^{100} or 2^{49}¢ and a participation fee π by socially viable ones:

$$\text{budget } B_0 = 4,700,000¢, \text{ and participation fee } \pi = 1000¢. \tag{1}$$

[1] Shapley [22] did not literally claim the resolution of the SP paradox, but Aumann [1], footnote 2, p.443 wrote that "it appears from his paper that he thinks that he has actually resolved the whole paradox". Shapley [23], p.446 explained moderately his thoughts on Aumann's comment.

[2] Participation in the case $\pi > \bar{t}$ was discussed as the problem of an intrinsic utility of a gamble for people (cf., Fishburn [8], Schmidt [21], and Diecidue, *et.al* [5]); the utility of gambling is formulated by modifying the classical expected utility theory. We do not deny the possibility that some people enjoy a gamble. However, this could allow too large possibilities for such preferences.

The rationales are: The average annual income of the 35 OECD countries is about $47,000$$ in 2018, and the maximum prize, not more than B_0, is $2^{\bar{t}} = 2^{22} = 4,194,304$¢; this is quite feasible for a trustworthy banker in these countries. The fee $\pi = 1000$¢ is also reasonable in these countries; a lottery ticket in Japan is typically 300¥ (1¢ $\doteqdot$ 1.13¥ in October 2021).

We eliminated the absurd elements, but as long as we keep the ER criterion for the banker and people, we meet another incompatibility with our real world experiences. The ER from a is $\frac{1}{2^1} \cdot 2^1 + \cdots + \frac{1}{2^{\bar{t}}} \cdot 2^{\bar{t}} = \bar{t}$, and the reward from b remains π. The ER criterion recommends the people to buy a ticket if and only if $\pi < \bar{t}$. In this case, since the revenue for the banker is negative, the banker should not open the market; also since the banker needs additional costs to prepare money for the SP gamble market, it does not open the market even in the case $\pi = \bar{t}$. Conversely, when $\pi > \bar{t}$, the ER criterion suggests the people not to participate in the market. Table 1.1 summarizes the situation; the market is vacuous in either $\pi \leq \bar{t}$ or $\pi > \bar{t}$. This holds for any maximum prize $2^{\bar{t}}$ and participation fee π. We call Table 1.1 *Shapley's vacuousness*, since it is a direct conclusions from Shapley's introduction of a banker with a budget.

Table 1.1; Shapley's vacuousness

	$\pi \leq \bar{t}$	$\pi > \bar{t}$
People	possibly participate	no participation
Banker	not open	possibly open

Table 1.1 tends to be incompatible with our ordinary senses or our experiences in the real world. Consider (1) with the replacement $\pi = 1000$¢ by $\pi = 50$¢. Table 1.1 still suggests no people to buy tickets, because $\bar{t} = 22 < 50 = \pi$. This is incompatible with the fact that national lotteries are common in many countries; one example of a state lottery in Japan will be described in Section 4.2. After all, SH2 removed the SP paradox in the original sense (*), but this removal leads to the new paradox without absurd elements.

As mentioned above, our concern is the affirmative possibility of a trade of an SP gamble in a market. Although the main idea for people's participations is based on Kaneko's [12] expected utility theory with cognitive bounds, we need more detailed treatments of his theory in that the context is much more specific. One example is a derivation of behavioral probability for a choice of a and b in Fig.1 when a and b are incomparable.

Our problem is not only to show the affirmative possibility but also to see the boundary between the affirmative and negative possibilities. Our ordinary senses evaluate a theoretical result, but our ordinary senses are not complete and are complemented by a theoretical result. Unless we see the boundary, the affirmative result may deceive our ordinary senses. We include this requirement in our study.

For SH1 and SH2, bounded rationality is crucial for people, but not for the banker. Budgets and how to be trusted by people are more important for the banker. To guarantee the maximum prize $2^{\bar{t}}$ to people, the banker needs to finance $2^{\bar{t}}$ by borrowing it from other sources, including the facility and operating costs. The above mentioned concept *ROI* represents this idea of how much return can be made to the investment. This idea is quite standard in accounting (cf., Needless, *et al.* [18]). In the economic literature, von Neumann [26] used it for the behavioral criterion for firms in his balanced growth model. We also emphasize that the possibility of bankruptcy is included in the above picture. Bankruptcy may be induced when some people hit large rewards. Bankruptcy itself is an important problem but causes a mathematical difficulty in the probability treatment of coin-tossing. Thus, we need an explicit treatment of the banker's financial moves.

1.2 Developments of the SP gamble market

The SP gamble market (Γ_0, Γ_N) is asymmetric between the banker and the people; the banker has a large budget while being accustomed to treating big and detailed amounts of money. The ER criterion is a natural idealization for the banker, except additional costs of preparing prizes. Each person is subject to a bounded cognitive ability over events with small probabilities, i.e., he has a *cognitive bound* $\frac{1}{2^{\rho_i}}$ ($\rho_i \geq 0$ is a natural number) so that the smaller probabilities than $\frac{1}{2^{\rho_i}}$ are not in his perspective. The *bound degree* ρ_i is typically smaller than the budget degree $\bar{t}$. For the people, the ER criterion is modified with cognitive bounds.

The banker announces an SP gamble $(2^{\bar{t}}, \pi)$ consisting of the maximum prize $2^{\bar{t}}$ and participation fee π to the people. The cognitive bound $\frac{1}{2^{\rho_i}}$ for person i plays two roles. He considers the subjective lottery restricted to the probabilities larger than or equal to $\frac{1}{2^{\rho_i}}$. This does not means that he totally ignores the prizes more than 2^{ρ_i}; instead, the announcement of the gamble $(2^{\bar{t}}, \pi)$ tells the maximal prize $2^{\bar{t}}$. Here, the probability is negligible but the prize remains in his mind. A sociological study reports that a great fortune appealed in the advertisement attracts people to buy tickets even though the probability is tiny (cf., Beckert-Lutter [2]). This plays an crucial role in our study of the SP gamble market.

We adopt expected utility theory (abbreviated as EU theory) with cognitive bounds due to Kaneko [12]. His concern was a general development of EU theory with cognitive bounds relative to the standard EU theory. On the other hand, our concern is specific to SP gambles; we meet problems caused by details of specific structures, which are new but not met in [12]. We will start with the 2-dimensional vector-valued expected reward functions with the *interval order* due to Fishburn [7], skipping some general foundations such as basic preferences,

The preferences derived by the interval order show *incomparability* between a (participation) and b (non-participation), particularly when ρ_i is small. Here, we take one more step that the people showing incomparability are forced to have *behavioral probability choices* between a and b.[3] When a and b are equally incomparable for a person, his behavioral probabilities of a and b are $\frac{1}{2}$ and $\frac{1}{2}$, but when these alternatives are incomparable to him but with inclination of a favor to a, the probability of a is larger than $\frac{1}{2}$. This step will be explained in an informal way in Section 3 and in an axiomatic manner in Section 7.

We emphasize the *locality* of our development, which is an explicit formulation of one aspect of Simon's [24] notion of bounded rationality.[4] Kaneko's [12] theory contains still a global tendency in the sense that it treats all lotteries with positive probabilities not smaller than the cognitive bound $\frac{1}{2^{\rho_i}}$, while our development targets comparisons between the two alternatives a and b, being based on the evaluations of substructures of the subjective understanding σ_{ρ_i} of a. This enables us to evaluate the complexity of i's choice behavior. Yet, our development has an enough perspective including Sharpley's vacuousness as well as discussing possibilities with different values of cognitive degrees for the people.

[3] It expresses a person's behavioral tendency and differs from Luce's [16] probabilistic preferences (cf., Gul, *et al.* [9], Echenique-Saito [6], and Loomis-Sugden [15]) where probabilities represents preferences over the alternatives.

[4] Simon [24] divided the notion of rationality into *substrantive* and *procedural*; the former is a property of a realized choice such as a "rational outcome" and the latter is an attribute of a performance of a system (person). The logical inference ability of a person is included in the latter.

A related remark is that the execution of coin-tossing is a difficult stochastic process; a reason for difficulty is that the process possibly includes bankruptcy, that is, the event of the banker becoming no longer able to pay the maximum prize $2^{\bar{t}}$. Bankruptcy is conceptually natural in the execution process, but it makes the process complicated for analytical treatments.[5] We adopt the method of *Monte Carlo* simulation for our study.

The present paper consists of seven sections including this introduction. Section 2 provides a theory of a monopolistic market with a banker and people. Section 3 gives sketches of derivations of the people utility functions and consequent behavioral probability. Postponing the formal treatments of these to Sections 6 and 7, Section 4.1 provides calculation tests for various cases of parameter values, and Section 4.2 discusses our approach as a possible resolution of the SP paradox. Section 5 closes the paper with a few remarks.

2 The SP Gamble Market

In this section, we describe the theory of an SP gamble market. Section 2.1 gives the basic components of the theory. Section 2.2 explains the temporal structure of the market and formulates the execution process of coin-tossing by the people with tickets. In the terms of these components, Section 2.3 provides the definition of an affirmative possibility and discuss its conceptual foundation for the resolution of the SP paradox.

2.1 Elements of the SP gamble market

The *SP gamble market* is given as $\Gamma = \langle \Gamma_0, \Gamma_N \rangle$, where Γ_0 is the list of elements for the *banker* 0 and Γ_N is the components for the *people* $N = \{1, ..., n\}$. The number of people n is assumed to be 1000 in most examples in the paper. The list Γ_0 is as follows:

$S1$: O (open the market) and NO (not open) are alternatives for the banker;

$S2$: $B_0 \geq 2$ is the banker's *initial budget,* $2^{\bar{t}} = \max\{2^t : 2^t \leq B_0\}$ is the *maximum prize*, and $\bar{t}$ is called the *budget degree*;

$S3$: π is the *participation fee* with $2^{\bar{t}} \geq \pi > 0$;

$S4$: $C(\ell)$ is the *cost* of facility for ℓ participants for coin-tossing;

$S5$: *ROI* is the *return on investment* index with a *desired level* $\alpha_0 > 0$; this is explained in Section 2.2.

The second list Γ_N is as follows:

$S6$: a, b are available choices for each $i \in N$; a is to purchase a ticket and b is not;

$S7$: $\frac{1}{2^{\rho_i}}$ is the *cognitive bound* of person $i \in N$, and ρ_i is called the *cognitive degree*;

$S8$: $\boldsymbol{u}_i = [\overline{u}_i; \underline{u}_i]$ is the vector-valued utility function of person $i \in N$ with the domain $\{a, b\}$, endowed with the interval order $\geq_I$ to be defined below;

$S9$: $\Pr_{\rho_i}(a)$ is a probability of choice a for person i and $\Pr_{\rho_i}(b) = 1-\ \Pr_{\rho_i}(a)$ is the probability of choice b.

[5] Bankruptcy may be ignored when the banker has a large amount of money in addition to the money prepared for the maximum prize. When it is ignored, the process can be reduced into that with the sum of binomial distributions.

The parameters, except for $S5$, $S8$, and $S9$, are natural numbers; ROI, α_0, $\overline{u}_i(f)$, $\underline{u}_i(f)$, $\Pr_{\rho_i}(f)$, are rational numbers for $f = a, b$.

In $S1$ to $S3$, the banker's alternative choices O, NO, the initial budget B_0, and the participation fee π are given. The maximum prize $2^{\overline{t}}$ is determined by B_0. The cost function $C(\ell)$ in $S4$ describes the cost needed for the facility running the SP coin-tossing for ℓ people in addition to a set-up cost. We assume the following specific function throughout the paper:

$$C(\ell) = \begin{cases} 50000 + 10000 \cdot (s+1) & \text{if } 100s < \ell \leq 100(s+1) \\ 0 & \text{if } \ell = 0. \end{cases} \tag{2}$$

It requires the set-up cost $50,000¢ = 500\$$ and the variable cost $10,000¢ = 100\$$ for each 100 people, e.g., when $\ell = 720$, $C(\ell) = C(720) = (50,000 + 10,000 \cdot 7)¢ = 1,200\$$.

For Γ_N, $S6$ gives the available choices a and b, and $S7$ gives the distribution of cognitive degrees over the people N. Table 2.1 gives an example of a distribution; φ_I has the median value $\rho_i = 15$, which has the 30% of people with cognitive abilities not to recognize the probability less than $\frac{1}{2^{15}} = \frac{1}{32,768}$.

Table 2.1; A distribution of cognitive degrees (%)

$\varphi \setminus \rho_i$	$\cdots$ 7	8	9	10	11	12	13	14	15	16	17	18	19	20 $\cdots$
φ_I	$\cdots$ 0	0	0	0	0	5	10	20	**30**	20	10	5	0	0 $\cdots$

The utility function $\boldsymbol{u}_{\rho_i} = [\overline{u}_{\rho_i}; \underline{u}_{\rho_i}]$ in $S8$ has the domain $\{a, b\}$ and takes a 2-dimensional vector value $[\overline{u}_{\rho_i}(f); \underline{u}_{\rho_i}(f)]$ for $f = a, b$ with $\overline{u}_{\rho_i}(f) \geq \underline{u}_{\rho_i}(f)$. The upper $\overline{u}_i(f)$ is the least upper bound of possible utilities and the lower $\underline{u}_i(f)$ is the greatest lower bound. Two values $[\overline{u}_i(f); \underline{u}_i(f)]$ and $[\overline{u}_i(g); \underline{u}_i(g)]$ are compared by the *interval order* $\geq_I$ (cf., Fishburn [7]) defined as

$$[\overline{u}_i(f), \underline{u}_i(f)] \geq_I [\overline{u}_i(g), \underline{u}_i(g)] \text{ if and only if } \underline{u}_i(f) \geq_I \overline{u}_i(g). \tag{3}$$

The properties of $\geq_I$ will be fully explained in Section 7. We denote $\boldsymbol{u}_{\rho_i}(f) \bowtie_I \boldsymbol{u}_{\rho_i}(g)$ iff neither $\boldsymbol{u}_{\rho_i}(f) \geq_I \boldsymbol{u}_{\rho_i}(g)$ nor $\boldsymbol{u}_{\rho_i}(g) \geq_I \boldsymbol{u}_{\rho_i}(f)$ holds; that is, f and g are incomparable. In this case, the behavioral probability $\Pr_{\rho_i}(a)$ of $S9$ plays a crucial role.

The utility function $\boldsymbol{u}_{\rho_i}$ for a and b will be explained in the context of EU theory with cognitive bounds in Section 3.1.

In the gamble market (Γ_0, Γ_N), the probability distributions over rewards induced by choice a appears in the objective and subjective manners. With the introduction of the budget degree $\overline{t}$, we truncate the original probability distribution τ_∞ to the set of rewards $Y_{\overline{t}} := \{2^1, ..., 2^{\overline{t}}, 0\}$, i.e., the *objective distribution* $\tau_{\overline{t}}$ over $Y_{\overline{t}}$ is defined by $\tau_{\overline{t}}(y) = \frac{1}{2^t}$ if $y = 2^t$ and $t \leq \overline{t}$ and $\tau_{\overline{t}}(0) = \frac{1}{2^{\overline{t}}}$. The prizes are the same as τ_∞ up to the $\overline{t}$-th toss, but after this, all prizes are reduced to 0 with the total probability $\frac{1}{2^{\overline{t}+1}} + \cdots = \frac{1}{2^{\overline{t}}}$. Now, we let $\widehat{\rho}_i := \min(\rho_i, \overline{t})$. We define the *subjective distribution* σ_{ρ_i} over $Y_{\rho_i} = \{2^1, ..., 2^{\widehat{\rho}_i}, 0\}$ by

$$\sigma_{\rho_i}(y) = \begin{cases} \frac{1}{2^t} & \text{if } y = 2^t \text{ and } t \leq \widehat{\rho}_i \\ \\ \frac{1}{2^{\widehat{\rho}_i}} & \text{if } y = 0. \end{cases} \tag{4}$$

Below the probability $\frac{1}{2^{\widehat{\rho}_i}}$, the reward is assumed to be 0 with the total probability $\frac{1}{2^{\widehat{\rho}_i+1}} + \cdots = \frac{1}{2^{\widehat{\rho}_i}}$. Person i with cognitive degree ρ_i takes this subjective distribution σ_{ρ_i} representing the alternative a. Following (4), we have $\sigma_0(0) = 1$; in this case, person i does not consider the SP coin-tossing. Thus, we assume $\rho_i > 0$ when we consider σ_{ρ_i}. Another note is that his choice is subjective but the coin-tossing is objective; that is, the domain $\{a, b\}$ of $\boldsymbol{u}_{\rho_i}$ is effectively $\{\sigma_{\rho_i}, \pi\}$, but when person i chooses σ_{ρ_i}, he mechanically follows the objective probability distribution $\tau_{\bar{t}}$ in the execution stages of coin-tossing. The explicit formula of $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ will be given in Section 3.

The comparison $\geq_I$ for $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ and $\boldsymbol{u}_{\rho_i}(\pi)$ defined by the interval order $\geq_I$ given by (3) often dictates incomparability between σ_{ρ_i} and π. Here, person i does not make a decision, but he is forced to choose either σ_{ρ_i} or π for some reason. This will be explained in Section 3.2. In this case, person i takes a probabilistic behavior. The *behavioral probability* $\Pr_{\rho_i}$ in $S9$ over the choices a and b is determined by $\rho_i, 2^{\bar{t}}$,and π, that is,

$$(\rho_i, 2^{\bar{t}}, \pi) \longmapsto \Pr_{\rho_i}(a). \tag{5}$$

The simple requirement is that for the comparable case, $\Pr_{\rho_i}(a)$ expresses the strict preference,

$$\Pr_{\rho_i}(a) = 1 \text{ if and only if } \boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) >_I \boldsymbol{u}_{\rho_i}(\pi), \tag{6}$$

where $>_I$ is the strict part of $\geq_I$. Our concern is the case where a and b are incomparable. The derivation of $\Pr_{\rho_i}(a)$ is sketched in Section 3.2 and is derived in an axiomatic way in Section 7.

We assume that $\Pr_{\rho_i}(a) = p_{\rho_i}$ is assumed to be independent over the people. The number of participants i with $\rho_i = k$ is given as the sum of random variables $\sum_{\rho_i=k} X_i$, where $X_i = 1$ means buying a ticket with probability p_k, and $X_i = 0$ not with $1-p_k$. The sum $\sum_{\rho_i=k} X_i$ obeys a binomial distribution with the expected value $\mu_k = \sum_{\rho_i} \Pr_{\rho_i}(a) = n\varphi(k) \cdot p_k$ and variance $\sigma_k^2 = n\varphi(k) \cdot p_k(1-p_k)$, where $n\varphi(k)$ is the number of people with $\rho_i = k$ (a natural number). The sum of binomial distributions $\sum_k (\sum_{\rho_i=k} X_i)$ is the total number of participants, where k varies over some finite domain. This is not a binomial distribution, but the expectation μ and variance σ^2 of $\sum_k \sum_{\rho_i=k} X_i$ are given simply as $\mu = \sum_k \mu_k$ and $\sigma^2 = \sum_k \sigma_k^2$. In Table 2.1 with $n = 1000$, $\bar{t} = 22$, and $\pi = 1000$, we have $\mu \doteqdot 443$ and $\sigma^2 \doteqdot (11.4)^2$; here, σ is small relative to μ.

The expectation μ may have decimal fractions; we round decimals to the nearest whole number, which is denoted by $\ell = \lceil \mu \rfloor$. We assume that given $\{\Pr_{\rho_i}(a)\}_{i \in N}$, the number of people $\ell = \lceil \mu \rfloor = \left\lceil \sum_k \sum_{\rho_i=k} \Pr_{\rho_i}(a) \right\rfloor$ buy tickets for coin-tossing.

2.2 Temporal structure of the SP gamble market

The temporal structure of the SP gamble market $\Gamma = \langle \Gamma_0, \Gamma_N \rangle$ is as follows:

1st stage: the banker chooses O with an announcement $(2^{\bar{t}}, \pi)$ to the people or No.
If O is chosen, the market goes to the 2nd stage, but if NO is chosen, the market is over.

2nd stage: each $i \in N$ receives the announcement $(2^{\bar{t}}, \pi)$ and decides to buy a ticket or not.

3rd stage: the people with tickets execute coin-tossing.

These are described in Fig.2. In the 2nd stage, each i reacts to $(2^{\bar{t}}, \pi)$ with the behavioral probability $\Pr_{\rho_i}(a) = p_{\rho_i}$. In the 2nd stage, person i uses his subjective understanding σ_{ρ_i} but if he purchases a ticket, his coin tossing follows the objective distribution $\tau_{\bar{t}}$. The banker's revenue is calculated by the results of coin-tossing. The choices by the banker and people are involved in the first two stages. The third stage is a mechanical process.

The 3rd stage of execution of coin-tossing is as follows:

EC1: the people with tickets are linearly ordered and make coin-tossing one by one;

EC2: the process stops when the banker goes into bankruptcy or when all people finish.

The people with tickets are ordered as $i_1, i_2, ..., i_\ell$ and execute coin-tossing one-by-one in this order. The initial budget of the banker is B_0. Let i_{ℓ_B} be a person with $\ell_B \leq \ell$, who is the person to cause bankruptcy for the banker by hitting a large prize. Let $(n_1, ..., n_{\ell_B})$ be a finite sequence with $n_t \in \{1, ..., \bar{t}\}$ for $t = 1, ..., \ell_B$; each n_t means that the coin-tossing of person i_t results in heads H in the n_t-th toss. We denote the banker's temporal budget $B(t)$ after i_t's coin-tossing.[6]

We define the sequence of temporary budgets $(B(0), B(1), ..., B(\ell_B))$ by

b0: $B(0) = B_0$;

b1: $B(t) = B(t-1) + \pi - 2^{n_t}$ for $t = 1, ..., \ell_B$;

b2: $B(t) + \pi \geq 2^{\bar{t}}$ for $t = 0, 1, ..., \ell_B - 1$;

b3: if $\ell_B < \ell$, then $B(\ell_B) + \pi < 2^{\bar{t}}$.

The banker has the initial budget $B(0) = B_0$ before the start. Person i_1 pays participation fee π to the banker, and makes coin-tossing. If i_1 goes to the n_1-th toss with the heads H, the banker pays 2^{n_1}¢ to i_1. After this payments, the resulting money amount becomes $B(1) = B(0) + \pi - 2^{n_1}$ and is brought to the second coin-tossing if $B(1) + \pi \geq 2^{\bar{t}}$. If $B(1) + \pi < 2^{\bar{t}}$, i.e., $\ell_B = 1$, the banker cannot guarantee the maximum prize $2^{\bar{t}}$ for person i_2; in this case, the banker is in *bankruptcy*. These are described in b0 to b3. In general, this process goes up to person ℓ_B with bankruptcy or to the last person ℓ. We include the case $B(\ell) + \pi < 2^{\bar{t}}$ in bankruptcy. Using the sequence $(n_1, ..., n_{\ell_B})$, $B(t)$ is expressed as

$$B(t) = B_0 + \sum_{s=1}^{t} (\pi - 2^{n_s}) \text{ for } t = 1, ..., \ell_B. \tag{7}$$

Thus, $B(t) - B_0 = \sum_{s=1}^{t} (\pi - 2^{n_s})$ is the sum of net revenue $\pi - 2^{n_s}$ up to t.

The above process is stochastic; ℓ_B, $(n_1, ..., n_{\ell_B})$, and $B(1), ..., B(\ell_B)$ are random variables. The execution stops at $\ell_B < \ell$ with bankruptcy. The expected value of $B(\ell_B)$ is denoted by $\overline{B(\ell_B)}$. To study the behavior of $(B(0), B(1), ..., B(\ell_B))$, we adopts the Monte Carlo simulation and simulate each person's coin-tossing by a random number generator. We calculate the *average revenue* $\overline{B(\ell_B)}$ over 10,000 runs in the Monte Carlo method.

The rules of the SP gamble market are set up to avoid strategic behavior between the banker and people. Each person i simply reacts to $(2^{\bar{t}}, \pi)$, i.e., the prizes up to the maximum prizes $2^{\bar{t}}$ and participation fee π, and no interactions between people are necessary to be take into account.

We are now in a state to consider the evaluation criterion (ROI, α_0) of the banker; the *return on investment ROI* is given as:

$$ROI = \frac{\overline{B(\ell_B)} - (B(0) + C(\ell))}{B(0) + C(\ell)} = \frac{(\overline{B(\ell_B)} - B_0) - C(\ell)}{B_0 + C(\ell)}. \tag{8}$$

As stated, the banker borrows $B_0 + C(\ell)$ from a financial institution and should return this amount with some interests after the SP gamble market; for example, if α_0 is the interest rate,

[6] If we ignore bankrupcy, e.g., enough money can be borrowed from the money market, the length ℓ_B is constant to be ℓ and we may focus on the sequences of results of coin tossing up to ℓ.

then the banker should return $(1+\alpha_0)(B_0+C(\ell))$. If the index ROI is greater than or equal to a desired level α_0, say 0.05, which is greater than the interest rate, then the banker generates a positive profit after paying the interest rate. We assume that the positivity of $ROI \geq \alpha_0$ is a necessary condition for the banker to open the market.

Suppose that the SP gamble market $\Gamma = (\Gamma_0, \Gamma_N)$ is given. Let B_0, π are given in $S2$, $S3$, and ℓ a natural number not greater than n. Recall that the maximal prize $2^{\bar{t}}$ is determined $S2$ and the behavioral probability $\mathrm{Pr}_{\rho_i}(a)$, depending upon $(2^{\bar{t}}, \pi)$, is given in $S9$ and (5). We say that a triple $\langle (B_0, \pi), \ell \rangle$ is an *affirmative possibility* iff

$$\ell = \left\lceil E \sum_k \sum_{\rho_i = k} \mathrm{Pr}_{\rho_i}(a) \right\rceil > 0; \tag{9}$$

$$ROI \geq \alpha_0. \tag{10}$$

Thus, the people follow the behavioral probabilities given above, and ℓ is the nearest whole number obtained by rounding $E \sum_k \sum_{\rho_i = k} \mathrm{Pr}_{\rho_i}(a)$, and the ROI index is not smaller than the desired level α_0, which is a necessary condition for the banker to open the market.

The main assertions of the paper are:

Assertion 0 (Shapley's vacuousness): For some $k_0 \leq \bar{t}$, if $\rho_i \geq k_0$ for all $i \in N$, then,

(1): $\pi > \bar{t}$ implies $\ell = \left\lceil E \sum_k \sum_{\rho_i = k} \mathrm{Pr}_{\rho_i}(a : (2^{\bar{t}}, \pi)) \right\rceil = 0$;

(2): $\pi \leq \bar{t}$ implies $ROI \leq 0$.

Assertion 1 (Affirmative possibility): In (1) with φ_I of Table 2.1 and $\alpha_0 = 0.05$,

(1): there is an affirmative possibility $\langle (B_0, \pi), \ell \rangle$;

(2): it remains affirmative for changes in parameter values around the above possibility.

This concept of an affirmative possibility is weaker than the monopolistic equilibrium which requires the "profit maximization" over various relevant $(B_0, \pi; \ell)$'s. It is even weaker than the "quantity demanded" in the micro-economics textbook meaning that a firm calculates who purchase tickets in response to the announcement $(2^{\bar{t}}, \pi)$ and verifies the ROI value reaches the desired level α_0. Two questions could be raised: First, why do we stop at this concept? Second, are the epistemic conditions involved in the concept compatible with our ordinary senses. These questions will be discussed after numerical examples in Section 4.2.

In the standard micro-economics, the monopolist (banker in our case) is assumed to be the profit maximizer, which requires the assumption that it knows the demand function from consumers. Here, this assumption is too demanding in that even the quantify demanded for $(2^{\bar{t}}, \pi)$ is difficult to know, which we calculate after some mathematical arguments as well as the Monte Carlo simulation.

3 Sketch of the Derivations of $\boldsymbol{u}_{\rho_i}$ and $\mathrm{Pr}_{\rho_i}(a)$

In Section 3.1, we describe, taking the results of Section 6 as the provisional starts, how to calculate the value $\boldsymbol{u}_{\rho_i}(\pi)$ for fee π and the value $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ for the subjective understanding σ_{ρ_i} of $\tau_{\bar{t}}$. To calculate $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$, we need to calculate $\boldsymbol{u}_{\rho_i}(x)$ for various pure alternatives x. As emphasized in Section 2, $\boldsymbol{u}_{\rho_i}(\pi)$ and $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ are often incomparable. We take one more step to derive the behavioral probabilities $\Pr_{\rho_i}(a)$ for σ_{ρ_i} from the vector values $\boldsymbol{u}_{\rho_i}(\pi)$ and $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$. For full understanding, axiomatic derivations of $\boldsymbol{u}_{\rho_i}(\pi)$, $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})\boldsymbol{u}_{\rho_i}$, and $\Pr_{\rho_i}(a)$ will be given in Sections 6 and 7.

3.1 The utility vectors $\boldsymbol{u}_{\rho_i}(\pi)$ and $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$

In this section, we take the following two provisional starts:

Provisional start M: We define $\boldsymbol{u}_{\rho_i}$ ($\rho_i \geq 0$) over possible payments $X_{\bar{t}} := \{0, 1, 2, ..., 2^{\bar{t}}\}$ by

$$\boldsymbol{u}_{\rho_i}(x) = \begin{cases} [x; x] & \text{if } x = \nu \cdot 2^{\bar{t}-\rho_i} \\ [(\nu+1) \cdot 2^{\bar{t}-\rho_i}; \nu \cdot 2^{\bar{t}-\rho_i}] & \text{if } (\nu+1) \cdot 2^{\bar{t}-\rho_i} > x > \nu \cdot 2^{\bar{t}-\rho_i}, \end{cases} \tag{11}$$

where $x \in X_{\bar{t}}$ and ν is a natural number with $0 \leq \nu \leq 2^{\rho_i}$. By (11), we have $\boldsymbol{u}_{\rho_i}(0) = [0; 0]$ and $\boldsymbol{u}_0(x) = [2^{\bar{t}}; 0]$ for x $(0 < x < 2^{\bar{t}})$. The other start is to extend the above $\boldsymbol{u}_{\rho_i}$ to an subjective SP gamble σ_{ρ_i}.

Provisional Start E: For $\rho_i > 0$, we define $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$, recalling $\widehat{\rho}_i := \min(\rho_i, \bar{t})$, by

$$\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) = \sum_{t=1}^{\widehat{\rho}_i} \frac{1}{2^t} \cdot \boldsymbol{u}_{\rho_i - t}(2^t) + \frac{1}{2^{\widehat{\rho}_i}} \cdot \boldsymbol{u}_{\rho_i - \widehat{\rho}_i}(0), \tag{12}$$

where the amount 0 in the last term follows from the assumption that when $\rho_i < \bar{t}$, person i treats those events reaching $t > \widehat{\rho}_i$ as having prize 0, which is taken into account (4). Also, $\boldsymbol{u}_{\rho_i - \widehat{\rho}_i}(0) = [0; 0]$ as noted after (11). Here, we assume $\rho_i > 0$ as remarked after (4), but the subscript $\rho_i - t$ of $\boldsymbol{u}_{\rho_i - t}(2^t)$ may take 0 when $t = \rho_i$. Hence, in start M, $\rho_i = 0$ is allowed.

The subscript $\rho_i - t$ of $\boldsymbol{u}_{\rho_i - t}$ decreases with the index t of the probability weight $\frac{1}{2^t}$. The degree ρ_i is person i's cognitive ability for the use of probabilities; two types of probability uses are involved in the recognition of the coefficient probability $\frac{1}{2^t}$ and the measurement of the prize 2^t. The latter is hidden in this section, but will be explicit in Section 6.1. Since he uses his total ρ_i to recognize $\frac{1}{2^t}$, he can use the remaining degree $\rho_i - t$ to measure the payoff 2^t.

The formulae of (11) and (12) look precise enough, but can be transformed into simpler and more convenient forms when we restrict our attention on the domains for π and σ_{ρ_i}.[7] The formulae in (11) express the results of measuring pure alternatives x's in terms of the *measurement scale*

$$\{\nu \cdot 2^{\bar{t}-\rho_i} : \nu = 0, 1, ..., 2^{\rho_i}\} \tag{13}$$

consisting of *scale grids* $\nu \cdot 2^{\bar{t}-\rho_i}$, $\nu = 0, 1, ..., 2^{\rho_i}$ on $X_{\bar{t}}$. The difference between each pair of adjacent grids is $2^{\bar{t}-\rho_i}$, with which $X_{\bar{t}}$ is divided into 2^{ρ_i} subintervals having the same length $2^{\bar{t}-\rho_i}$. The above definition (11) states that if x coincides with some scale grid $\nu \cdot 2^{\bar{t}-\rho_i}$, the utility value is uniquely determined to be x, i.e., $\boldsymbol{u}_{\rho_i}(x) = [x; x]$, and if x is in an interval, then $\boldsymbol{u}_{\rho_i}(x)$ is the vector of the adjacent scale grids. In Fig.3, the broken line segment from $\underline{x} = 0$ to $\overline{x} = 2^{\bar{t}}$ is the measurement scale with the $2^k + 1$ number of scale grids; x is exactly measured by the scale while x' is expressed by the two adjacent scale grids. This measurement is described in some axiomatic manner in Section 6.1.

[7] This is because our theory is local in that these steps are restricted on a choice between participation choice $a = \sigma_{\rho_i}$ and non-participation $b = \pi$, while Kaneko's [12] theory is global in that preferences are defined over the set of lotteries yet with some depth constraints.

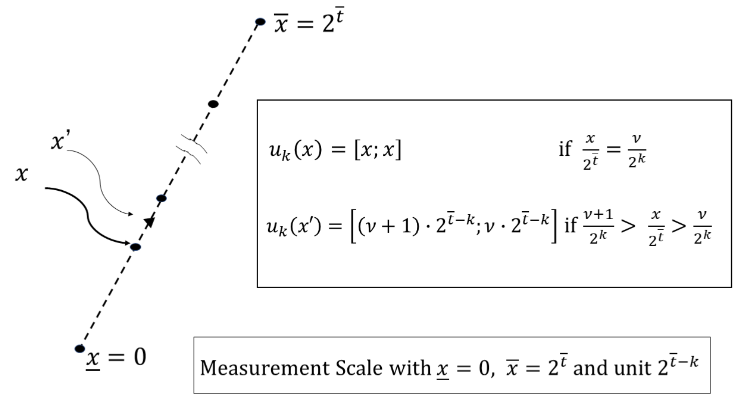

Figure 3: Measurement scale

The following lemma states that when ρ_i is large, π is exactly measured.

Lemma 3.1. There is some $k_0 \leq \bar{t}$ such that for all $i \in N$, if $\rho_i \geq k_0$, then $\boldsymbol{u}_{\rho_i}(\pi) = [\pi; \pi]$.

Proof. Let $k_0 = \bar{t}$. Then, for $\rho_i \geq k_0 = \bar{t}$, π is expressed as $\nu \cdot 2^{\bar{t}-\rho_i} = \nu \cdot \frac{1}{2^{\rho_i - \bar{t}}}$ and $\nu = \pi \cdot 2^{\bar{t}-\rho_i}$. By the first case of (11), we have $\boldsymbol{u}_{\rho_i}(\pi) = [\pi; \pi]$.■

In the above proof, $\bar{t}$ is taken for k_0. In general, k_0 may be smaller than $\bar{t}$. When π is an even number, k_0 can be $\bar{t} - 1$. When $\pi = 1000 = 2^3 \times 125$, k_0 can be $\bar{t} - 3$ and is the smallest to have the exact measurement.

The other part of (11) is still abstract; we need more concise forms for the purpose of practical calculations of $\boldsymbol{u}_{\rho_i}(\pi)$. It is enough to consider fees π with

$$0 < \pi < 2^{\bar{t}} \quad \text{and} \quad \pi \neq 2^t \tag{14}$$

for any t for our purpose. Under this condition, the second case of (11) is divided into the following:

$$\boldsymbol{u}_{\rho_i}(\pi) = \begin{cases} [2^{\bar{t}-\rho_i}; 0] & \text{if } 0 < \pi < 2^{\bar{t}-\rho_i} \\ [(\nu+1)\cdot 2^{\bar{t}-\rho_i}; \nu \cdot 2^{\bar{t}-\rho_i}] & \text{if } \nu \cdot 2^{\bar{t}-\rho_i} < \pi < (\nu+1)\cdot 2^{\bar{t}-\rho_i} \ \&\ \nu \geq 1. \end{cases} \tag{15}$$

The first case is the special case of the second for $\nu = 0$. We write these separate forms, because the first will appear more than the second in the examples. This will be seen in Table 3.1.

The formula in (12) for $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ can be expressed without using the sum operator. Since each term in the sum has the specific form $\boldsymbol{u}_{\rho_i - t}(2^t)$, we calculate it by (11).

Lemma 3.2 (Utility values for prizes with cognitive constraints)

(1): Let $\rho_i < \bar{t}$. Then $\boldsymbol{u}_{\rho_i - t}(2^t) = [2^{\bar{t}-(\rho_i - t)}; 0]$ for any t $(1 \leq t \leq \rho_i)$, and $\boldsymbol{u}_0(0) = [0; 0]$.

(2): Let $\rho_i \geq \bar{t}$. Then $\boldsymbol{u}_{\rho_i - t}(2^t) = [2^t; 2^t]$ for any $t \leq \bar{t}$.

Proof. **(1)**: Suppose $\rho_i < \bar{t}$. Let $t \leq \rho_i$. Since $0 \cdot 2^{\bar{t}-(\rho_i - t)} < 2^t < 2^t \cdot 2^{(\bar{t}-\rho_i)} = 1 \cdot 2^{\bar{t}-(\rho_i - t)}$, we have the second case of (11) with $\nu = 0$; thus, $\boldsymbol{u}_{\rho_i - t}(2^t) = [2^{\bar{t}-(\rho_i - t)}; 0]$. The formula $\boldsymbol{u}_0(0) = [0; 0]$ is a special case of the first assertion of (11).

(2): Let $\rho_i \geq \bar{t}$. Since $2^t = 2^{\rho_i - \bar{t}} \cdot 2^{\bar{t}-(\rho_i - t)}$, we have the first case of (11) with $\nu = 2^{\rho_i - \bar{t}}$; thus $\boldsymbol{u}_{\rho_i - t}(2^t) = [2^t; 2^t]$.■

The formula in (12) is calculated into simpler forms. Then, $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ is compared with $\boldsymbol{u}_{\rho_i}(\pi)$.

Theorem 3.1 (Value of $\boldsymbol{u}_{\rho_i}(\boldsymbol{\sigma}_{\rho_i})$)

(1): Let $\rho_i \leq \bar{t} - 1$. Then $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) = [\rho_i \cdot 2^{\bar{t}-\rho_i}; 0]$.

(2): Let $\rho_i \geq \bar{t}$. Then $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) = [\bar{t}; \bar{t}]$.

Proof.(1): Plugging the formulae of Lemma 3.2.(1) into (12), we have $\overline{u}_{\rho_i}(\sigma_{\rho_i}) = \sum_{t=1}^{\rho_i} \frac{1}{2^t} \cdot \overline{u}_{\rho_i - t}(2^t) + \frac{1}{2^{\rho_i}} \cdot \overline{u}_0(0) = \sum_{t=1}^{\rho_i} \frac{1}{2^t} \cdot 2^{\bar{t}-\rho_i} \cdot 2^t = \rho_i \cdot 2^{\bar{t}-\rho_i}$, and $\underline{u}_{\rho_i}(\sigma_{\rho_i}) = \sum_{t=1}^{\rho_i} \frac{1}{2^t} \cdot 0 + \frac{1}{2^{\rho_i}} \cdot 0 = 0$.

(2): Recall $\widehat{\rho}_i = \min(\rho_i, \bar{t})$. Plugging the the formula of Lemma 3.2.(2) to (12), we have $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) = \sum_{t=1}^{\widehat{\rho}_i} \frac{1}{2^t} \cdot \boldsymbol{u}_{\rho_i - t}(2^t) + \frac{1}{2^{\widehat{\rho}_i}} \cdot \boldsymbol{u}_{\rho_i - \widehat{\rho}_i}(0) = \sum_{t=1}^{\bar{t}} \frac{1}{2^t} \cdot \boldsymbol{u}_{\rho_i - t}(2^t) = \sum_{t=1}^{\bar{t}} \frac{1}{2^t} \cdot [2^t, 2^t] = [\bar{t}; \bar{t}]$.■

Now, we give comparisons between $\boldsymbol{u}_{\rho_i}(\pi)$ and $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ in a few cases. Only in the case $\rho_i \geq \bar{t}$, $\boldsymbol{u}_{\rho_i}$measures σ_{ρ_i} exactly, and in the case where all people have cognitive bounds $\rho_i \geq k_0$ (k_0 is given by Lemma 3.1), $\boldsymbol{u}_{\rho_i}$ measures π exactly. In these cases, if $\pi > \bar{t}$, some people stay at home. In the other case $\pi \leq \bar{t}$, people may want to buy tickets, but the revenue for the banker is less than or equal to the expenditure. Thus, the *ROI* index is negative, and the banker does not open the market. These are summarized as the following theorem precisely; it is Shaley's vacuousness in our theory.

Theorem 3.2 (Shapley's vacuousness). Let k_0 be the degree given by Lemma 3.1, and assume that $\rho_i \geq k_0$ for all $i \in N$.

(1): Let $\pi > \bar{t}$. Then, $\boldsymbol{u}_{\rho_i}(\pi) >_{\rho_i} \boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ for all for all $i \in N$, i.e., no people buy SP gambles.

(2): Let $\pi \leq \bar{t}$ and $B_0 \leq 2^{49}$. Then, the banker does not open the market.

Proof.(1): Let $\pi > \bar{t}$, and let i be arbitrary in N. Since $\boldsymbol{u}_{\rho_i}(\pi) = [\pi; \pi]$ by Lemma 3.1 and $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) = [\bar{t}; \bar{t}]$ by Theorem 3.1.(1), person i strictly prefers π to σ_{ρ_i}.

(2): We consider the two cases: (a) $\overline{B(\ell_B)} - B_0 \leq \ell \cdot (\pi - \bar{t})$ and (b) $\overline{B(\ell_B)} - B_0 > \ell \cdot (\pi - \bar{t})$.

Case (a): Since $\ell \cdot (\pi - \bar{t}) \leq 0$, we have $\overline{B(\ell_B)} - B_0 < 0$, which implies that the numerator of *ROI* is non-positive.

Case (b): Let $\overline{B(\ell_B)} - B_0 \geq \ell \cdot (\pi - \bar{t})$. It is impossible that $\ell = 0$, since no people participate in the coin-tossing, i.e., $\overline{B(\ell_B)} - B_0 = 0$, a contradiction. Now, let $\ell > 0$, that is, ℓ number of people make coin-tossing. The maximum net revenue for the banker occurs in the case where that coin-toss ends at the first toss for all of these people ℓ. In this case, the net revenue is $\ell \cdot (\pi - 2)$. Hence, $\overline{B(\ell_B)} - B_0 \leq \ell \cdot (\pi - 2)$. Since $\pi \leq \bar{t}$, the numerator of *ROI* is calculated as $(\overline{B(\ell_B)} - B_0) - C(\ell) \leq \ell \cdot (\pi - 2) - C(\ell) \leq \ell \cdot (\bar{t} - 2) - C(\ell)$. By the definition of $\bar{t}$ in $S2$, it holds that $2^{\bar{t}} \leq B_0 \leq 2^{49}$. Hence, $\bar{t} \leq 49$. By the specification of $C(\ell)$ of (2), we have $\ell \cdot (\bar{t} - 2) - C(\ell) \leq \ell \cdot 47 - C(\ell) \leq 47000 - C(1) = 47000 - 60000 < 0$. The numerator of *ROI* is negative.■

The above theorem assumes $B_0 \leq 2^{49}$ because of several specifications of the parameter values, particularly, the cost function $C(\cdot)$ has the specific form stated (2). We take 2^{49} since this number was referred in Section 1.1.

Let us focus purely on the comparisons between $\boldsymbol{u}_{\rho_i}(\pi)$ and $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$. The following theorem states that if $\pi > \bar{t}$, then π and σ_{ρ_i} are incomparable or π is strictly preferred, and that otherwise, they are incomparable for $\rho_i < \bar{t}$ or σ_{ρ_i} is strictly preferred for $\rho_i \geq \bar{t}$. Our concern is the case

$\pi > \bar{t}$, where no people want to buy tickets in the strict sense.

Theorem 3.3 (Almost incomparable) (1): Let $\pi > \bar{t}$. There is a $k_1 \leq \bar{t}$ such that $\boldsymbol{u}_{\rho_i}(\pi) \bowtie_{\rho_i} \boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ for all $\rho_i < k_1$ and $\boldsymbol{u}_{\rho_i}(\pi) >_{\rho_i} \boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ for all $\rho_i \geq k_1$.

(2): Let $0 < \pi \leq \bar{t}$. Then, $\boldsymbol{u}_{\rho_i}(\pi) \bowtie_{\rho_i} \boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ for all $\rho_i < \bar{t}$ and $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) >_{\rho_i} \boldsymbol{u}_{\rho_i}(\pi)$ for all $\rho_i \geq \bar{t}$.

Proof.(1): Let $\pi > \bar{t}$. By Lemma 3.1.(2) and Theorem 3.1.(2), we have $\boldsymbol{u}_{\rho_i}(\pi) = [\pi; \pi] >_{\rho_i} [\bar{t}; \bar{t}] = \boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$. We choose the minimum k_1 so that $\boldsymbol{u}_{k_1}(\pi) >_{k_1} \boldsymbol{u}_{k_1}(\sigma_{k_1})$. Observe $\underline{u}_t(\pi) \leq \underline{u}_{t+1}(\pi)$ and $\overline{u}_t(\sigma_t) \leq \overline{u}_t(\sigma_t)$ for any t, because of (11) and (12). By this, $\boldsymbol{u}_{k_1}(\pi) >_{k_1} \boldsymbol{u}_{k_1}(\sigma_{k_1})$ implies $\boldsymbol{u}_{\rho_i}(\pi) >_{\rho_i} \boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ for all $\rho_i \geq k_1$. By the definition of k_1, we have $\boldsymbol{u}_{\rho_i}(\pi) \bowtie_{\rho_i} \boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ for all $\rho_i < k_1$.

(2): Let $\rho_i \geq \bar{t}$. By Lemma 3.1.(2) and Theorem 3.1.(2), $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) = [\bar{t}; \bar{t}] >_{\rho_i} [\pi; \pi] = \boldsymbol{u}_{\rho_i}(\pi)$. Next, let $\rho_i < \bar{t}$. By Theorem 3.1.(1), $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) = [\rho_i \cdot 2^{\bar{t}-\rho_i}; 0]$ is a nondegenerate interval. It holds that $\rho_i \cdot 2^{\bar{t}-\rho_i} \geq \bar{t}$ for $\rho_i < \bar{t}$; if $\rho_i = \bar{t} - 1 > 0$, then $(\bar{t} - 1) \cdot 2^1 \geq \bar{t}$. Let $\rho_i \cdot 2^{\bar{t}-\rho_i} \geq \pi$ hold. Then, $(\rho_i + 1) \cdot 2^{\bar{t}-(\rho_i+1)} = (\rho_i + 1) \cdot 2^{\bar{t}-\rho_i} \cdot \frac{1}{2} = \frac{\rho_i+1}{2} \cdot 2^{\bar{t}-\rho_i} \leq \rho_i \cdot 2^{\bar{t}-\rho_i}$. By (11), $\boldsymbol{u}_{\rho_i}(\pi) = [(\nu + 1) \cdot 2^{\bar{t}-\rho_i}; \nu \cdot 2^{\bar{t}-\rho_i}]$ by (14) and (11), where $(\nu + 1) \cdot 2^{\bar{t}-\rho_i} > \pi > \nu \cdot 2^{\bar{t}-\rho_i}$. Hence, $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ and $\boldsymbol{u}_{\rho_i}(\pi)$ have an intersection. Hence, $\boldsymbol{u}_{\rho_i}(\pi) \bowtie_{\rho_i} \boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$■

Theorem 3.3 states that in the essential case $\pi > \bar{t}$, people have no strict preferences to buy tickets of SP gambles. Up to now, our theory does not reach an affirmative possibility. To see what are happening in the incomparability cases, we look at the calculation results for $\boldsymbol{u}_{\rho_i}(\pi)$ and $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$, which are given in Table 3.1. The results for $\boldsymbol{u}_{\rho_i}(\pi)$ follow from the first case of (15) and Lemma 3.1, and all the results for $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ are derived from Theorem 3.1. Then, we go to the sketch of the behavioral probability $\Pr_{\rho_i}(a)$ in Section 3.2.

Table 3.1; $\boldsymbol{u}_{\rho_i}(\pi)$, $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$, and $\Pr_{\rho_i}(a)$ for $\bar{t} = 22$

ρ_i	0*	1	2	3	4	5	6	7	8
$\boldsymbol{u}_{\rho_i}(\pi)$	$\begin{bmatrix}1\cdot2^{22}\\0\end{bmatrix}$	$\begin{bmatrix}1\cdot2^{21}\\0\end{bmatrix}$	$\begin{bmatrix}1\cdot2^{20}\\0\end{bmatrix}$	$\begin{bmatrix}1\cdot2^{19}\\0\end{bmatrix}$	$\begin{bmatrix}1\cdot2^{18}\\0\end{bmatrix}$	$\begin{bmatrix}1\cdot2^{17}\\0\end{bmatrix}$	$\begin{bmatrix}1\cdot2^{16}\\0\end{bmatrix}$	$\begin{bmatrix}1\cdot2^{15}\\0\end{bmatrix}$	$\begin{bmatrix}1\cdot2^{14}\\0\end{bmatrix}$
$\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}1\cdot2^{21}\\0\end{bmatrix}$	$\begin{bmatrix}2\cdot2^{20}\\0\end{bmatrix}$	$\begin{bmatrix}3\cdot2^{19}\\0\end{bmatrix}$	$\begin{bmatrix}4\cdot2^{18}\\0\end{bmatrix}$	$\begin{bmatrix}5\cdot2^{17}\\0\end{bmatrix}$	$\begin{bmatrix}6\cdot2^{16}\\0\end{bmatrix}$	$\begin{bmatrix}7\cdot2^{15}\\0\end{bmatrix}$	$\begin{bmatrix}8\cdot2^{14}\\0\end{bmatrix}$
$\Pr_{\rho_i}(a)$	0	1/2	3/4	5/6	7/8	9/10	11/12	13/14	15/16
ρ_i	9	10	11	12	13	14	15	16	17
$\boldsymbol{u}_{\rho_i}(\pi)$	$\begin{bmatrix}1\cdot2^{13}\\0\end{bmatrix}$	$\begin{bmatrix}1\cdot2^{12}\\0\end{bmatrix}$	$\begin{bmatrix}1\cdot2^{11}\\0\end{bmatrix}$	$\begin{bmatrix}1\cdot2^{10}\\0\end{bmatrix}$	$\begin{bmatrix}2\cdot2^{9}\\1\cdot2^{9}\end{bmatrix}$	$\begin{bmatrix}4\cdot2^{8}\\3\cdot2^{8}\end{bmatrix}$	$\begin{bmatrix}8\cdot2^{7}\\7\cdot2^{7}\end{bmatrix}$	$\begin{bmatrix}16\cdot2^{6}\\15\cdot2^{6}\end{bmatrix}$	$\begin{bmatrix}32\cdot2^{5}\\31\cdot2^{5}\end{bmatrix}$
$\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$	$\begin{bmatrix}9\cdot2^{13}\\0\end{bmatrix}$	$\begin{bmatrix}10\cdot2^{12}\\0\end{bmatrix}$	$\begin{bmatrix}11\cdot2^{11}\\0\end{bmatrix}$	$\begin{bmatrix}12\cdot2^{10}\\0\end{bmatrix}$	$\begin{bmatrix}13\cdot2^{9}\\0\end{bmatrix}$	$\begin{bmatrix}14\cdot2^{8}\\0\end{bmatrix}$	$\begin{bmatrix}15\cdot2^{7}\\0\end{bmatrix}$	$\begin{bmatrix}16\cdot2^{6}\\0\end{bmatrix}$	$\begin{bmatrix}17\cdot2^{5}\\0\end{bmatrix}$
$\Pr_{\rho_i}(a)$	17/18	19/20	21/22	23/24	23/26	21/28	15/30	1/32	0
ρ_i	18	19	20	21	22	23	$\cdots$		
$\boldsymbol{u}_{\rho_i}(\pi)$	$\begin{bmatrix}63\cdot2^{4}\\62\cdot2^{4}\end{bmatrix}$	$\begin{bmatrix}1000\\1000\end{bmatrix}$	$\begin{bmatrix}1000\\1000\end{bmatrix}$	$\begin{bmatrix}1000\\1000\end{bmatrix}$	$\begin{bmatrix}1000\\1000\end{bmatrix}$	$\begin{bmatrix}1000\\1000\end{bmatrix}$	$\cdots$		
$\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$	$\begin{bmatrix}18\cdot2^{4}\\0\end{bmatrix}$	$\begin{bmatrix}19\cdot2^{3}\\0\end{bmatrix}$	$\begin{bmatrix}20\cdot2^{2}\\0\end{bmatrix}$	$\begin{bmatrix}21\cdot2^{1}\\0\end{bmatrix}$	$\begin{bmatrix}22\\22\end{bmatrix}$	$\begin{bmatrix}22\\22\end{bmatrix}$	$\cdots$		
$\Pr_{\rho_i}(a)$	0	0	0	0	0	0			

Let us look at only five cases in Table 3.1 to show how to calculate the above results. When $\rho_i = 0$, we have $\boldsymbol{u}_{\rho_i}(\pi) = [1 \cdot 2^{22}; 0]$ by (15) and $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) = [0; 0]$ by Theorem 3.1. In this case, strict preference $\boldsymbol{u}_{\rho_i}(\pi) >_I \boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ holds; hence, the behavioral probability $\Pr_{\rho_i}(a)$ of choice a is 0. When $\rho_i = 20$, we have $\boldsymbol{u}_{\rho_i}(\pi) = [\pi; \pi] = [1000; 1000]$ by Lemma 3.1 and the remark after it, and we have $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) = [20 \cdot 2^2; 0]$ by Theorem 3.1.(1). Then, strict preference $\boldsymbol{u}_{\rho_i}(\pi) >_I \boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ holds. Again, we have the behavioral probability $\Pr_{\rho_i}(a) = 0$.

When $\rho_i = 1$, π is strictly between 0 and 2^{21}; i.e., $\boldsymbol{u}_{\rho_i}(\pi) = [2^{21}; 0]$, and $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) = [2^{21}; 0]$ is proved by Theorem 3.1.(1). These utility values are identical, and $\boldsymbol{u}_{\rho_i}(\pi) \bowtie_I \boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ by the definition (3) of the interval order $\geq_I$. Then, it is reasonable to put $\Pr_{\rho_i}(a) = \frac{1}{2}$, provided that person i is forced to make a choice. When $\rho_i = 10$, we have $\boldsymbol{u}_{\rho_i}(\pi) = [1 \cdot 2^{12}; 0]$ and $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) = [10 \cdot 2^{12}; 0]$. By (3), we have $\boldsymbol{u}_{\rho_i}(\pi) \bowtie_I \boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$. The vectors are different; we need a further consideration to fix $\Pr_{\rho_i}(a)$. This will be the subject of Section 3.2.

Table 3.2; $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$, $\boldsymbol{u}_{\rho_i}(\pi)$, and $\Pr_{\rho_i}(a)$

$\rho_i = 0$	$\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) = [0; 0]$	$\boldsymbol{u}_{\rho_i}(\pi) = [1 \cdot 2^{22}; 0]$	$\Pr_{\rho_i}(a) = 0$
$\rho_i = 1$	$\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) = [1 \cdot 2^{21}; 0]$	$\boldsymbol{u}_{\rho_i}(\pi) = [1 \cdot 2^{21}; 0]$	$\Pr_{\rho_i}(a) = 1/2$
$\rho_i = 10$	$\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) = [10 \cdot 2^{12}; 0]$	$\boldsymbol{u}_{\rho_i}(\pi) = [1 \cdot 2^{12}; 0]$	$\Pr_{\rho_i}(a) = 19/20$
$\rho_i = 13$	$\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) = [13 \cdot 2^{9}; 0]$	$\boldsymbol{u}_{\rho_i}(\pi) = [2 \cdot 2^9; 1 \cdot 2^9]$	$\Pr_{\rho_i}(a) = 23/26$
$\rho_i = 20$	$\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) = [20 \cdot 2^{2}; 0]$	$\boldsymbol{u}_{\rho_i}(\pi) = [1000; 1000]$	$\Pr_{\rho_i}(a) = 0$

3.2 Behavioral probability $\mathbf{Pr}_{\rho_i}(a)$

Table 3.1 tells that person i with $1 \leq \rho_i \leq 16$ does not make a choice between σ_{ρ_i} and π in the sense of incomparability, and that person i only with $\rho_i = 0$ or $\rho_i \geq 17$ chooses π. Thus, no people participate in the SP gamble market in the exact sense.

In general, people are often unable to make choices by rational thinking. One famous story is *Buridan's donkey* (cf., Zupko [28]); a donkey faces two carrots in the same distances (up to his cognitive ability), cannot choose the right or left carrot, and eventually dies of starvation.[8] In our problem, person i finds incomparability between σ_{ρ_i} and π, and our theory is silent about what his next behavior is. In the real world, majority of people make choices because they have been forced to make for survival or by social norm, authorities, etc.

Suppose that person i is incomparable between σ_{ρ_i} and π, but is forced to make a choice. He has already used his capacity for his rationalistic choice; now, he uses different sources to make a choice. People have had experiences and practices in similar situations. Person i digs his memories in past experiences of having choices between incomparable alternatives. Here, we explain the derivation of behavior probability $\Pr_i(a)$ in the three cases in Table 3.2; as explained, the cases $\rho_i = 0$ and $\rho_i = 20$ show strict preference $\boldsymbol{u}_{\rho_i}(\pi) >_I \boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$. A rigorous derivation will be provided in an axiomatic manner in Section 7.

The simplest incomparable case is $\rho_i = 1$. Here, $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) = \boldsymbol{u}_{\rho_i}(\pi) = [1 \cdot 2^{22}; 0]$; since the utility values are identical, person i has the same inclination to choose either σ_{ρ_i} or π. This inclination is interpreted as the equal probability $\frac{1}{2}$ for σ_{ρ_i} and π; we define $\mathbf{Pr}_{\rho_i}(a) = p_{\rho_i} = \frac{1}{2}$ for $\rho_i = 1$.

Consider the case $\rho_i = 10$. In this case, the interval $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) = [10 \cdot 2^{21}; 0]$ is 10 times larger than $\boldsymbol{u}_{\rho_i}(\pi) = [1 \cdot 2^{21}; 0]$ and includes $\boldsymbol{u}_{\rho_i}(\pi)$, but they are still incomparable with respect to $\geq_I$. We partition $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ into $A = [1 \cdot 2^{21}; 0]$ and $B = [10 \cdot 2^{21}; 1 \cdot 2^{21}]$. The reasoning for $\rho_i = 1$ is applied to the comparison between A and $\boldsymbol{u}_{\rho_i}(\pi)$; since they are identical, each of A and $\boldsymbol{u}_{\rho_i}(\pi)$

[8] In literature, there are many instances treating such problems, e.g., "Sophie's Choice" by William Styron and "Terror" by Ferdinand von Schirachon. In the literature of economics, however, people (experimental subjects) are simply assumed to answer to questionaires. In the game theory literature, Davis-Maschler [4] asked some game theorists about a questionaire related to their theory, only Martin Shubik made an explicit refusal to answer to it because of insufficiency of descriptions of the theory for answering.

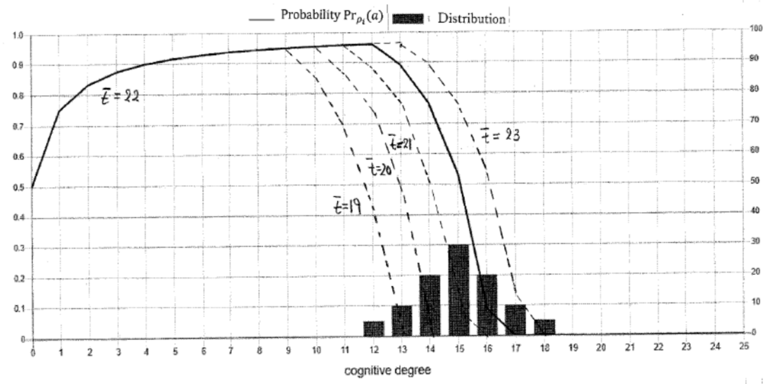

Figure 4: Behavioral probability $\Pr_{\rho_i}(a)$

happens with probability $\frac{1}{2}$. The second case is the comparison between the remaining B and $\boldsymbol{u}_{\rho_i}(\pi)$; the interval order $\geq_I$ gives a strict preference to B over $\boldsymbol{u}_{\rho_i}(\pi) = [1 \cdot 2^{21}; 0]$; this gives probability 1 to B. The weights for these two comparisons between

$$A \text{ vs. } \boldsymbol{u}_{\rho_i}(\pi) \text{ and } B \text{ vs. } \boldsymbol{u}_{\rho_i}(\pi)$$

are assumed to be proportional to the sizes, i.e., the $\frac{|A|}{|A|+|B|} = \frac{1}{10}$ and $\frac{|B|}{|A|+|B|} = \frac{9}{10}$, where $|A|$ and $|B|$ are the lengths of the intervals A and B. We calculate the weighted sum of the probabilities having σ_{ρ_i} as

$$\frac{|A|}{|A|+|B|} \cdot \frac{1}{2} + \frac{|B|}{|A|+|B|} \cdot 1 = \frac{19}{20}. \tag{16}$$

Here, a large portion of people with cognitive degree 10 choose participation.

Consider case $\rho_i = 13$. In this case, we partition $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) = [13 \cdot 2^9; 0]$ into $A = [1 \cdot 2^9; 0]$, $B = [2 \cdot 2^9; 1 \cdot 2^9]$, and $C = [13 \cdot 2^9; 2 \cdot 2^9]$, and consider the three comparisons A vs.$\boldsymbol{u}_{\rho_i}(\pi)$, B vs. $\boldsymbol{u}_{\rho_i}(\pi)$, and C vs. $\boldsymbol{u}_{\rho_i}(\pi)$. In the first comparison, $\boldsymbol{u}_{\rho_i}(\pi) = [2 \cdot 2^9; 1 \cdot 2^9] >_I A$, in the second, $\boldsymbol{u}_{\rho_i}(\pi)$ and B are identical, in the third, $C >_I \boldsymbol{u}_{\rho_i}(\pi)$. We take the weighted sum in the same way as above:

$$\frac{|A|}{|A|+|B|+|C|} \cdot 0 + \frac{|B|}{|A|+|B|+|C|} \cdot \frac{1}{2} + \frac{|C|}{|A|+|B|+|C|} \cdot 1 = \frac{1}{26} + \frac{22}{26} = \frac{23}{26}.$$

In the other cases of Table 3.1, we calculate $\Pr_{\rho_i}(a)$ in the manner as one of the above 5 case. Table 3.1 gives the exhaustive list of $\Pr_{\rho_i}(a)$, which is depicted with the solid line in Fig.4; the other broken lines are with $\bar{t} = 19, ..., 23$.

A methodological remark is: we will ratios of differences in the upper and lower utilities. This needs the those utility values to be cardinal in the sense that they are determined up to positive linear transformations, which will be stated in Lemma 6.1 and Remark 6.1.

4 Performance of the SP Gamble Market

We look at the performance of our theory in numerical examples. Our numerical calculations have the following tasks; one is to show an affirmative possibility in the sense of (9), (10), and the other task is to see the performance relative to the changes of parameter values of the SP gamble market. We employ the comparative statics method (sensitivity analysis) to have the second task, to see how the affirmative possibility works positively as well as negatively. We choose only some examples of changes for parameters. Still, these calculations show the rich performance of the theory. We mention briefly the possible effects of other parameters and the example of a Japanese state lottery.

4.1 Comparative statics: affirmative and negative possibilities

We adopt, as the central case, the following parameter values:

$$B_0 = 2^{\bar{t}} = 2^{22}¢,\ \bar{t} = 22,\ \pi = 1000¢ = 10\$,\ n = 1000,\ \text{and } \varphi_I, \tag{17}$$

where φ_I is given in Table 2.1. The behavior probability $\Pr_{\rho_i}(a)$ is fully determined and given in Table 3.1. The expected number of participants is given as

$$\ell = \left\lceil \sum_{\rho_i} \varphi_I(\rho_i) \cdot n \cdot \Pr_{\rho_i}(a) \right\rceil. \tag{18}$$

We take the nearest whole number $\lceil \cdot \rceil$. We assume that these people ℓ go to the execution stage of coin-tossing. Using Tables 2.1 and 3.1, we calculate $\ell = 443$. These people buy the ticket of the SP gamble; (18) has a positive number of people, but the requirement (10): $ROI \geq \alpha_0 = 0.05$ should still be considered. We adopt by the Monte Carlo method; the coin-tossing of each person is simulated using a random number generator. The coin-tossing stage for 443 people is repeated $10,000$ times, and the average is adopted to calculate the total net (t.n.) revenue $\overline{B(\ell_B)} - B_0$, bankruptcy rate, and ROI.

We focus on four parameters to have comparative statics, which are listed in Table 4.1. We see the effects of changes of one parameter, provided that the other parameters are fixed; the central case is (17) with the other parameters fixed. The first two are simple and are easily interpreted. Then, (c) and (d) are more subtle and essential in our theory.

Table 4.1; the list of comparative statics

	#participants	t.n.revenue	bankruptcy	ROI
(a): Additional: $B_0 - 2^{\bar{t}}$	$\longrightarrow$	$\searrow$	$\searrow$	$\nearrow$ $\searrow$
(b): participation fee π	$\searrow$	$\nearrow$ $\searrow$	$\searrow$	$\nearrow$ $\searrow$
(c): budget degree $\bar{t}$	$\nearrow$ $\searrow$	$\nearrow$	$\nearrow$	$\nearrow$ $\searrow$
(d): cognitive degrees	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$

(a): **Additional** $B_0 - 2^{\bar{t}}$ $(\bar{t} = 22)$: In Section 1.1, we adopted the budget $B_0 = 47,000\$$ from the OECD data, and chose $2^{\bar{t}} = 2^{22} = 4194304¢$ for the maximum prize within the budget B_0. In our theory, the maximum prize $2^{\bar{t}}$ works effectively to attract people to purchase tickets, but the additional $B_0 - 2^{\bar{t}}$ has no effects on the behavior of people. This additional amount has two effects on the banker; it decreases the bankruptcy possibility and so increases the t.n.revenue. This additional amount is a borrowing from some financial source, and its interest payment is a cost to the banker, which is counted in the ROI index. Let us see Table 4.2. Recall $\lceil \ell \rceil = 443$, which is constant over the five cases. As stated, those people go to the execution stage of coin-tossing one-by-one until the banker gets bankruptcy or all the people finish coin-tossing. The calculation results are given as Table 4.2.

Table 4.2; Changes of the additional money

budget B_0	#participants	t.n.revenue \$	bankruptcy %		ROI
$B_0 = 2^{22}¢$	443	4284	1.08		.076
$B_0 = (2^{22}+1000)¢$	443	4313	0.79		.077
$B_0 = (2^{22}+10000)¢$	443	4320	0.52		.077
$B_0 = (2^{22}+100000)¢$	443	4332	0.23		.076
$B_0 = 47,000\$$	443	4351	0.15		.069

The initial budget B_0 is increased from $2^{22}¢$ to $47,000\$$ by adding some amounts of money. The total net revenue $\overline{B_{\ell_B}} - B_0$ is increasing with additional amounts, since bankruptcy is avoided by additional amounts. This increases the ROI in the beginning but decreases after some amount, because the additional amount yields larger interests. In the last row of $B_0 = 47,000\$$, the total net revenue is higher than the first row, but the ROI value is smaller.

In the following, we assume $B_0 = 2^{\bar{t}}$.

(***b***): **Changes in the participation fee** π: As expected, the number of participants is decreasing with the fee π. The bankruptcy rate is decreasing because a higher fee avoids bankruptcy by increasing the budget for the next person. The total net revenue is increasing by $\pi = 10\$$, and then it decreases since the number of participants decreases. A similar behavior is observed on the ROI; it takes the maximum at $\pi = 20\$$. The choice between 10\$ and 20\$ needs another criterion.

Table 4.3; Changes of π

Cases \ Values	# participants	revenues \$	bankruptcy %		ROI
$\pi = 3\$$	761	2067	2.95		.018
$\pi = 5\$$	668	3140	1.99		.046
$\pi = 10\$$	443	4284	1.08		.076
$\pi = 20\$$	210	4142	0.40		.078
$\pi = 30\$$	133	3950	0.24		.076
$\pi = 40\$$	78	3096	0.21		.059

(c): **Budget degree** $\bar{t}$: As stated in the end of Section 1.1, the maximum prize $2^{\bar{t}}$ has the role

Table 4.4; Changes of the maximum prize $2^{\bar{t}}¢$

Maximum prize	#participants	t.n.revenue \$	bankruptcy %		ROI
$2^{20}¢$	78	759	0.88		.014
$2^{21}¢$	210	2032	1.04		.057
$2^{22}¢$	443	4284	1.08		.076
$2^{23}¢$	668	6452	1.32		.062
$2^{24}¢$	826	7964	1.14		.039

to attract people to SP gamble market. This is shown in Table 4.4; the larger is $\bar{t}$, the larger is the participation number. The reason is as follows: for a small budget degree $\bar{t}$, the cognitive ability works to evaluate precisely prizes, but for a larger $\bar{t}$, it fades out. The exact argument of this effect will be discussed in Section 6.1.

The bankruptcy rate is also increasing with $\bar{t}$ up to $\bar{t} = 23$. When $B_0 = 2^{24}¢$, more people are attracted, by even larger prize $2^{24}¢ \doteqdot 167,772\$$. Here, the bankruptcy rate gets smaller, 1.14%, than in the case $B_0 = 2^{23}¢$. The reason is, perhaps; the maximum prize $2^{24}¢$ is much larger than 2^{23} and the probability of going beyond the $23th$ coin-tossing is small. This large prize causes the decrease of ROI with higher interest payments, from the previous cases. For the same reason, ROI is decreased from the case $B_0 = 2^{22}¢$ to $B_0 = 2^{23}¢$.

(***d***): **Cognitive bounds**: The concept of cognitive bound $\frac{1}{2^{\rho_i}}$ plays a crucial role in our theory, but being different from the other variables in (a) to (c), the banker cannot control the distribution φ on ρ_i's. Still, it would be an important question for our study how φ is distributed. Here, we consider the following four distributions from φ_{III} to φ_0; it is shifted to the left by degree 1, and the median case of each is denoted by the bold **30**%. The distribution φ_{III} has more people with lower cognitive degrees, and as the distribution goes down, more people have higher cognitive degrees.

Table 4.4; distributions of cognitive bounds

$\varphi \setminus \rho_i$	$\cdots$ 7	8	9	10	11	12	13	14	15	16	17	18	19	20 $\cdots$
φ_0	$\cdots$ 0	0	0	0	0	0	5	10	20	**30**	20	10	5	0 $\cdots$
φ_I	$\cdots$ 0	0	0	0	0	5	10	20	**30**	20	10	5	0	0 $\cdots$
φ_{II}	$\cdots$ 0	0	0	0	5	10	20	**30**	20	10	5	0	0	0 $\cdots$
φ_{III}	$\cdots$ 0	0	0	5	10	20	**30**	20	10	5	0	0	0	0 $\cdots$

It is found in Table 4.5 that the number of participants is larger with a lower distribution. The bankruptcy rate is larger from φ_0 to φ_{III}, but the ROI rate is also larger in the same direction. Both are caused by the increase of participants ℓ, though the increase of bankruptcy rate has an effect of decreasing ROI. In this table, we should not forget that the banker cannot choose φ directly; thus, the ROI rate is a reference for the banker to think about opening the market or not.

Table 4.5; Changes of cognitive abilities

Distribution φ	# participants ℓ	t.n.revenues \$	bankruptcy %		ROI
φ_0	229	2223	1.05		.033
φ_I	443	4286	1.08		.076
φ_{II}	668	6465	1.09		.122
φ_{III}	802	7751	1.15		.147

4.2 Further possible studies

In Section 4.1, we have seen the performance of our theory in examples with various parameter changes from the central case in (17). These results seem compatible with our ordinary senses, but we have fixed some elements of the theory. Here, we indicate a few other possibilities of parameter changes.

(e): **Number of people n and cost function**: This is a control variable for the banker when the banker can choose a place or a community. In this sense, this differs from (a) to (c). If n is much larger than $1,000$, the fixed and variable costs of $C(\cdot)$ should be reconsidered taking the budget degree $\bar{t}$ into account. When $\bar{t}$ is much larger than $\bar{t} = 23$, the bankruptcy rate may be kept small, but once it happens, the banker would have a large damage. In this case, the question to avoid bankruptcy in (a) is relevant.

The opposite case is with small n. This is relevant to an experimental study, since a large budget cannot be feasible for a laboratory experiment and π should be small, too. For example, when $n = 100 - 200$, perhaps, the relevant cognitive degrees treated in (d) are too large. Test subjects may be students, and they may live under low income situations. It may be possible to arrange experimental designs in the perspective of the present paper.

(f): **Risk attitudes**: We have assumed risk neutrality for the utility functions of people. This simplifies the calculation of $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$, specifically, as Theorem 3.1. Without risk neutrality, we can adopt the risk-averse utility function $\boldsymbol{u}_{\rho_i}(\cdot)$, but should return to the general formula in (12). An implementation with a computer program could not be very difficult; it would provide examples appealing more to our ordinary senses. We have now large freedom of a choice of a utility function, which requires a lot of studies. This extension to allow risk aversion may be relevant for (e).

(g): **Meanings of simulations**: We do not use fully the simulation method in the two senses. The SP gamble market $\Gamma = (\Gamma_0, \Gamma_N)$ has two separated processes depicted in Fig.2. The calculation of the number of participants is based on the theory given in Section 3.1; it is not based on a simulation method. Once the number of participants is determined, we adopted the Monte Carlo method to pursue the dynamics of the banker's budget. We use the Monte Carlo method to calculate the statistical distribution of the outcomes, because it is theoretically difficult to calculate the probability distribution of the outcome.

The other possible simulation study is fully to implement the Monte Carlo method both in two stages. Now, we see the behavior for a smaller number of runs, e.g., $25 - 50$ runs/year, instead of a huge number, such as $10,000$, of runs to calculate statistical distributions. This is closer to social environments, but for this kind of studies, we need the analysis of the present paper as a benchmark case.

The following is does not belong to comparability statics in the SP gamble market. However, it is still a variant of the SP gamble market and shows a perspective of our analysis.

(h): **Application to a state lottery**: A state lottery is a variant of the SP gamble market (Γ_0, Γ_N) and helps us consider what should be emphasized for the SP gamble market.[9] Table 4.6 describes one called the *year-end Jumbo Lottery* (2020) in Japan. The 1st prize could give the huge amount of money about 7 million\$, yet with the tiny probability $\frac{5}{10^8}$.

Table 4.6; the Year-end Jumbo (YEJB) Lottery

prizes	1st	1st^{-1}	$1\text{st}^{-2})^*$	2nd	3rd	4-th	5-th	6-th	7-th	miss
rewards	$7 \cdot 10^8$¥	$1.5 \cdot 10^8$	10^5	10^7	10^6	$5 \cdot 10^4$	10^4	3000	300	0
probabilities	$\frac{5}{10^8}$	$\frac{1}{10^7}$	$\frac{1}{10^5}$	$\frac{5}{10^7}$	$\frac{5}{10^6}$	$\frac{1}{10^4}$	$\frac{2}{10^4}$	$\frac{1}{10^2}$	$\frac{1}{10}$	.889685*

The expected revenue from this lottery is 128¥, and the price of the ticket is 300¥. Here, Shapley's vacuousness could hold as long as the ER criterion is assumed for people and the banker. The YEJB lottery still attracts some people.[10] This difficulty could be eliminated

[9] In Japan, the business of lottery is organized by the non-profit organization "Japan Lottery Association". It is legally determined that the expected value of rewards from a ticket is at most a half of a ticket. No one would buy the tickets.as long as they follow the expected value. However, since the association was founded in 1964, it has run lotteries constantly with large revenues, and has attracted a large number of people.

[10] Here, the probability of the 1st^{-2} prize is taken as $\frac{5}{10^7}$ as an approximation for simplicity, and the that of miss .889685 is an approximation.

by introducing the cognitive bound for people. It would be interesting to make comparisons between the SP gamble market and this YEJP lottery from the taking the points (e) to (g) into account.

We note that Theorem 3.1 cannot be applied to this problem, and thus, we should return to the summation formula (12).

5 Discussions

This paper started with Shapley's critique formulated as SH1 and SH2 in Section 1.1; the original SP paradox is removed by introducing a finite budget. Following SH1 and SH2, however, it is changed into Shapley's vacuousness as long as the ER criterion is kept for the banker and people. This vacuousness is a variant of the SP paradox but is more serious in that it includes a larger class of societal problems including state lotteries, insurances, and Black swan events; the classical EU theory fails to treat them in a natural manner. We formulated the SP problem, taking SH1 and SH2 into account, as the SP gamble market with a banker and people. We applied the EU theory with bounded cognitive abilities developed in Kaneko [12] to people. We have succeeded in showing a quite rich resolution of Shapley's vacuousness.

The results in Section 4.1 look compatible with our ordinary senses. This is true because the parameter values of the theory are taken following our ordinary senses. Also, since the theoretical structure has enough complexity to calibrate the theory to avoid possible incompatibilities. Because our ordinary senses are partial, we cannot make precise judgments on numerical results. Perhaps, this is a limit of our method to resolve the SP paradox. To go beyond this conclusion, what should we do?

One possibility is to analyze what our ordinary senses are. The inductive game theory, initiated by Kaneko-Matsui [14], Kaneko-Kline [13], look for experiential sources for a player's understanding of a game structure. This theory may give a good hint for a study of our ordinary senses and where we should go.

Although there are similar markets to the SP gamble market, there are no actual markets having the structure of of the SP gamble markets. It is impossible to carry out field experiments on the SP gamble markets. There are two ways to overcome this difficulty. The first is to conduct a field study: a hypothetical SP gamble market is designed and ask people if they participate in the market with a given fee; and we may target similar state lottery markets, such as mentioned in (h), Section 4.2. We can study such markets with our theory, and a field research is possible, too. The second is to target laboratory experiments; here, the exact SP gamble markets can be designed such as the miniature gamble markets mentioned in Section 4.2.

The theory may be modified to apply it to similar but different problems where some events take place with very small probabilities; one example, investments and/or insurances. Here, we can discuss how such an insurance is designed by a company to attract customers. In a "black swan problem" such as an accident of a nuclear plants, the control of the probability of an severe accident is a true problem, but how to show (hide) it is "small" is an actual problem for an electricity company. This may be in the perspective of our theory and can be a good contribution to understanding of such a "black swan problem".

6 Axiomatic Bases for the Provisional Starts for $\boldsymbol{u}_{\rho_i}(\pi)$ and $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$

In Section 3, we took the provisional starts (11) for $\boldsymbol{u}_{\rho_i}(x)$ ($x \in X_{\bar{t}} = \{0, 1, ..., 2^{\bar{t}}\}$) and (12) for $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$. Kaneko [12] separated explicitly these two steps for the development of the EU theory with cognitive bounds, while the separation could be implicit in classical EU theory (cf., von Neumann-Morgenstern [27], Herstein-Milnor [10]). In this section, we derive (11) and (12) from the bases in axiomatic manners. We emphasize that the following development has two differences from [12]. Our theory is very local in that they aim to make comparisons between only $\boldsymbol{u}_{\rho_i}(\pi)$ and $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$, while [12] is global in that the comparisons are made for the pairs in the set of possible lotteries. In this sense, the present treatment is more faithful to bounded rationality of Simon [24]. The other difference is that here risk neutrality is assumed, while [12] treated general preferences.

We prepare various mathematical concepts, some of which are already given. Person i's cognitive degree ρ_i defines his cognitive bound $\frac{1}{2^{\rho_i}}$; he can think about probabilities expressed only as *probability grids* $\frac{\nu}{2^{\rho_i}}$ $(0 \leq \nu \leq 2^{\rho_i})$, and the set of probability grids is given as $\Pi_k = \{\frac{\nu}{2^k} : 0 \leq \nu \leq 2^k\}$ for $k = 0, ..., \rho_i$. Then, $\Pi_0 = \{0, 1\} \subsetneq \Pi_1 \subsetneq \cdots \subsetneq \Pi_{\rho_i}$. We define the *depth* $\delta(\alpha) = t$ of $\alpha \in \Pi_t$ iff $\alpha \in \Pi_t - \Pi_{t-1}$. Then, $\delta(\alpha) = t$ if and only if α is expressed as $\frac{\nu}{2^t}$ for some odd ν. Thus, each $\lambda \in \Pi_k$ has at most depth k. The depth $\delta(\cdot)$ plays a crucial role in Section 6.2.

6.1 Measurement of pure alternatives to the Provisional Start M

The goal is to show the formula (11) for $\boldsymbol{u}_{\rho_i}(x)$ ($x \in X_{\bar{t}}$) from the viewpoint of EU theory with cognitive bounds. Person i is conscious of the *upper reference point* $2^{\bar{t}}$ and the *lower reference point* 0. Each pure alternative $x \in X_{\bar{t}}$ is measured by the *measurement scale* $B_k(\overline{x}; \underline{x})$ *of depth* $k = 0, 1, ..., \rho_i$ consisting of $\overline{x} = 2^{\bar{t}}$, $\underline{x} = 0$, and probability grids $\frac{\nu}{2^k}$ $(0 \leq \nu \leq 2^k)$, i.e.,

$$B_k(\overline{x}; \underline{x}) = \{[\overline{x}, \alpha; \underline{x}] : \alpha \in \Pi_k\}, \tag{19}$$

where expression $[\overline{x}, \alpha; \underline{x}]$ is a *scale lottery*, meaning that the reference points $\overline{x} = 2^{\bar{t}}$ and $\underline{x} = 0$ happen with probabilities α and $1-\alpha$. The scale $B_k(\overline{x}; \underline{x})$ consists of all scale lotteries of cognitive depth at most k. As k increases, the scale $B_k(\overline{x}; \underline{x})$ is getting more precise, up to the most precise scale $B_{\rho_i}(\overline{x}; \underline{x})$.

Let k be a depth with $0 \leq k \leq \rho_i$. We let $\mathbb{Q}^2(\geq) := \{\boldsymbol{\lambda} = (\overline{\lambda}, \underline{\lambda}) \in \mathbb{Q} \times \mathbb{Q} : \overline{\lambda} \geq \underline{\lambda}\}$, where $\mathbb{Q}$ is set of rationals. A *base utility function* $\boldsymbol{u}_k$ is a function over $B_k(\overline{x}; \underline{x}) \cup X_{\bar{t}}$ having a 2-dimensional vector value $\boldsymbol{u}_k(f) = [\overline{u}_k(f); \underline{u}_k(f)]$ in $\mathbb{Q}^2(\geq)$ for each $f \in B_k(\overline{x}; \underline{x}) \cup X_{\bar{t}}$. As stated above, $\overline{u}_k(f)$ and $\underline{u}_k(f)$ are interpreted as the least upper of bounds and greatest lower bounds of possible utilities from f. We assume two axioms on $\boldsymbol{u}_k$ for $k = 0, ..., \rho_i$. The first axiom states that $\boldsymbol{u}_k$ is based on the measurement scale of precision k.

Axiom M0(Measurement scale):(1)(Upper and lower benchmarks):

$$\overline{u}_k([\overline{x}, 1; \underline{x}]) = \underline{u}_k([\overline{x}, 1; \underline{x}]) > \overline{u}_k([\overline{x}, 0; \underline{x}]) = \underline{u}_k([\overline{x}, 0; \underline{x}]), \tag{20}$$

and these values are independent of k;
(2)(Expected utility for scale lotteries): for all $[\overline{x}, \alpha; \underline{x}] \in B_k(\overline{x}; \underline{x})$,

$$\boldsymbol{u}_k([\overline{x}, \alpha; \underline{x}]) = \alpha \boldsymbol{u}_k([\overline{x}, 1; \underline{x}]) + (1 - \alpha)\boldsymbol{u}_k([\overline{x}, 0; \underline{x}]). \tag{21}$$

We stipulate that when $\boldsymbol{u}_k(f) = [\overline{u}_k(f); \underline{u}_k(f)]$ satisfies $\overline{u}_k(f) = \underline{u}_k(f)$, we write $\boldsymbol{u}_k(f) = [u_k(f); u_k(f)]$ dropping the upper and lower bars. Using this, Axioms M0.(1) states that the components of $\boldsymbol{u}_k([\overline{x}, 1; \underline{x}])$ and $\boldsymbol{u}_k([\overline{x}, 0; \underline{x}])$ are expressed without the upper and lower bars. The reference points $[\overline{x}, 1; \underline{x}] = \overline{x} = 2^{\bar{t}}$ and $[\overline{x}, 0; \underline{x}] = \underline{x} = 0$ are precisely measured, and $\overline{x}$ is strictly better than $\underline{x}$. Axiom M0.(2) is the *expected utility property* over the scale lotteries; that is, the value $\boldsymbol{u}_k([\overline{x}, \alpha; \underline{x}])$ is the expected values of $\boldsymbol{u}_k([\overline{x}, 1; \underline{x}])$ and $\boldsymbol{u}_k([\overline{x}, 0; \underline{x}])$ with the probability weights α and $1 - \alpha$. Hence, $\boldsymbol{u}_k([\overline{x}, \alpha; \underline{x}])$ is the vector of components without the upper and lower bars. Since $\{\boldsymbol{u}_k([\overline{x}, \alpha; \underline{x}]) : [\overline{x}, \alpha; \underline{x}] \in B_k(\overline{x}; \underline{x})\}$ with $\geq_I$ is isomorphic to the set Π_k, $\boldsymbol{u}_k$ represents $B_k(\overline{x}; \underline{x})$ as well as Π_k.

The next axiom, M1, describes how a pure alternative $x \in X_{\bar{t}}$ is measured. In this paper, we assume the *risk-neutral* utility function $\boldsymbol{u}_k(x)$. In the axiom, we use $\frac{\nu}{2^k}\overline{x} = \frac{\nu}{2^k}\overline{x} + (1 - \frac{\nu}{2^k})\underline{x}$, because $\underline{x} = 0$.

Axiom M1 (Risk-neutrality): for all $x \in X_{\bar{t}}$

(1): $x = \frac{\nu}{2^k}\overline{x}$ if and only only if $\boldsymbol{u}_k(x) = \boldsymbol{u}_k([\overline{x}, \frac{\nu}{2^k}; \underline{x}])$;

(2): $\frac{\nu+1}{2^k}\overline{x} > x > \frac{\nu}{2^k}\overline{x}$ if and only if $\boldsymbol{u}_k(x) = [u_k([\overline{x}, \frac{\nu+1}{2^k}; \underline{x}]); u_k([\overline{x}, \frac{\nu}{2^k}; \underline{x}])]$.[11]

The left-hand of M1.(1) is a representation of pure alternative x in terms of convex combinations, $\frac{\nu}{2^k}\overline{x} = \frac{\nu}{2^k}\overline{x} + (1 - \frac{\nu}{2^k})\underline{x}$, of the upper and lower reference points with the weights $\frac{\nu}{2^k}$ and $1 - \frac{\nu}{2^k}$, and the right-hand side is the expected utility of $\overline{x}$ and $\underline{x}$ with the same weights. M1.(2) is the same assertion except with the replacement of the exact weights by the adjacent weights, where the upper and lower bars are unnecessary by Axiom M0.

We take a specific representation $\{\boldsymbol{u}_k\}_{k=0}^{\rho_i}$ satisfying;

$$u_k([\overline{x}, 1; \underline{x}]) = \overline{x} = 2^{\bar{t}} \text{ and } u_k([\overline{x}, 0; \underline{x}]) = \underline{x} = 0. \tag{22}$$

By Axiom M0.(1), these values are independent of $k \leq \rho_i$. This is a normalization and causes no conceptual problem, which will be shown in Lemma 6.2.

We have the following theorem; the second assertion is the goal, i.e., the formula (11), of this subsection.

Theorem 6.1(Measurement by the scale): **(1)**: For any $k = 0, 1, ..., \rho_i$ and $x \in X_{\bar{t}}$,

$$\boldsymbol{u}_k(x) = \begin{cases} [\frac{\nu}{2^k}\overline{x}; \frac{\nu}{2^k}\overline{x}] & \text{if } x = \frac{\nu}{2^k}\overline{x} \\ [\frac{\nu+1}{2^k}\overline{x}; \frac{\nu}{2^k}\overline{x}] & \text{if } \frac{\nu+1}{2^k}\overline{x} > x > \frac{\nu}{2^k}\overline{x}; \end{cases} \tag{23}$$

(2): Under (22), (23) is (11).

Proof. We consider (1) and (2) both only in the case where $x = \frac{\nu}{2^k}\overline{x}$. The other case is parallel. Let $x = \frac{\nu}{2^k}\overline{x}$. By Axiom B1.(1) and B0.(1), we have $\boldsymbol{u}_k(x) = \boldsymbol{u}_k([\overline{x}, \frac{\nu}{2^k}; \underline{x}]) = [\frac{\nu}{2^k}\overline{x}; \frac{\nu}{2^k}\overline{x}]$, i.e., (1). Then, plugging (22), we have $\boldsymbol{u}_k(x) = [\nu \cdot 2^{\bar{t}-k}; \nu \cdot 2^{\bar{t}-k}]$, i.e., (2).■

An important result in the classical EU theory is the uniqueness of a derived utility function up to a positive linear transformation (cf., von Neumann-Morgenstern [27], Chap.1, Herstein-Milnor [10]). Our theory shares this property; the structure of vector-value utility functions is preserved under a positive linear transformation. It guarantees that the normalization (22) is meaningful. However, the property needs to be described specifically in our context; two streams

[11] The risk-aversion may be introduced by modifying Axiom M1. There are a few manners for this modification.

$\{\boldsymbol{u}_k\}_{k=0}^{\rho_i}$ and $\{\boldsymbol{u}'_k\}_{k=0}^{\rho_i}$ of utility functions are affine-transformed to each other if and only if they are identical with respect to intended preferences including incomparabilities. The following lemma states one direction in each of (i) and (ii), but it implies both directions.

Lemma 6.1 (Uniqueness up to affine transformations). Let $\{\boldsymbol{u}_k\}_{k=0}^{\rho_i}$ and $\{\boldsymbol{u}'_k\}_{k=0}^{\rho_i}$ be given with Axioms M0 and M1. Then, for some rational $\alpha > 0$ and β, $\boldsymbol{u}'_k(x) = \alpha\boldsymbol{u}_k(x) + \beta\cdot[1;1]$ for all $x \in X_{\rho_i}$ if and only if for all $k \leq \rho_i$, and $x \in X_{\overline{t}}$,

(i): if $\boldsymbol{u}_k(x) = [u_k([\overline{x}, \frac{\nu+1}{2^k}; \underline{x}]); u_k([\overline{x}, \frac{\nu}{2^k}; \underline{x}])]$, then $\boldsymbol{u}'_k(x) = [u'_k([\overline{x}, \frac{\nu+1}{2^k}; \underline{x}]); u'_k([\overline{x}, \frac{\nu}{2^k}; \underline{x}])]$;

(ii): if $\boldsymbol{u}_k(x) = \boldsymbol{u}_k([\overline{x}, \frac{\nu}{2^k}; \underline{x}])$, then $\boldsymbol{u}'_k(x) = \boldsymbol{u}'_k([\overline{x}, \frac{\nu}{2^k}; \underline{x}])$.

Proof (Only-If): Let $\boldsymbol{u}'_k(\cdot) = \alpha\boldsymbol{u}_k(\cdot) + \beta\cdot[1;1]$. Then, $\boldsymbol{u}'_k(\overline{x}) = \alpha\boldsymbol{u}_k(\overline{x}) + \beta\cdot[1;1]$ and $\boldsymbol{u}'_k(\underline{x}) = \alpha\boldsymbol{u}_k(\underline{x}) + \beta\cdot[1;1]$. We consider (i). We show only that if $\underline{u}_k(x) = u_k([\overline{x}, \frac{\nu}{2^k}; \underline{x}])$, then $\underline{u}'_k(x) = u'_k([\overline{x}, \frac{\nu}{2^k}; \underline{x}])$. By Axiom M0.(1), $\boldsymbol{u}'_k([\overline{x}, \frac{\nu+1}{2^k}; \underline{x}])$ is decomposed into $\frac{\nu}{2^k}\boldsymbol{u}'_k(\overline{x}) + (1-\frac{\nu}{2^k})\boldsymbol{u}'_k(\underline{x}) = \frac{\nu}{2^k}(\alpha\boldsymbol{u}_k(\overline{x}) + \beta\cdot[1;1]) + (1-\frac{\nu}{2^k})(\alpha\boldsymbol{u}_k(\underline{x}) + \beta\cdot[1;1]) = \alpha(\frac{\nu}{2^k}\boldsymbol{u}_k(\overline{x}) + (1-\frac{\nu}{2^k})\boldsymbol{u}_k(\underline{x})) + \beta\cdot[1;1] = \alpha\boldsymbol{u}_k([\overline{x}, \frac{\nu}{2^k}; \underline{x}]) + \beta\cdot[1;1]$. It follows from this and $\overline{u}'_k(x) = \alpha\overline{u}_k(x) + \beta$ that $\overline{u}'_k(x) = u'_k([\overline{x}, \frac{\nu}{2^k}; \underline{x}])$. Similarly, we have (ii).

(If): We take $\alpha = \frac{\boldsymbol{u}'_k(\overline{x}) - \boldsymbol{u}'_k(\underline{x})}{\boldsymbol{u}_k(\overline{x}) - \boldsymbol{u}_k(\underline{x})}$ and $\beta = \boldsymbol{u}'_k(\overline{x}) - \frac{u'_k(\overline{x}) - u'_k(\underline{x})}{\boldsymbol{u}_k(\overline{x}) - \boldsymbol{u}_k(\underline{x})}\cdot u_k(\overline{x})$. It also holds that $\beta = \boldsymbol{u}'_k(\underline{x})$ $- \frac{\boldsymbol{u}'_k(\overline{x}) - \boldsymbol{u}'_k(\underline{x})}{\boldsymbol{u}_k(\overline{x}) - \boldsymbol{u}_k(\underline{x})}\cdot u_k(\underline{x})$. Then, $\boldsymbol{u}'_k(x) = \alpha\boldsymbol{u}_k(x) + \beta\cdot[1;1]$ by Axiom B0.(2).■

6.2 Extension to σ_{ρ_i} to the provisional start E

The goal is to show the formula (12) for $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ from the viewpoint of EU theory with cognitive bounds; in other words, $\{\boldsymbol{u}_k\}_{k=0}^{\rho_i}$ given in Section 6.1 is extended to σ_{ρ_i}. Let us explain the basic idea of the derivation using Fig.5. Let $\sigma^0_{\rho_i} = \sigma_{\rho_i}$. To avoid confusions, we use a new symbol $\boldsymbol{U}_{\rho_i}(\cdot)$ rather than $\boldsymbol{u}_{\rho_i}(\cdot)$, since the domain of $\boldsymbol{U}_{\rho_i}(\cdot)$ differs from that of $\boldsymbol{u}_{\rho_i}(\cdot)$. We consider $\boldsymbol{U}_{\rho_i}(\sigma_{\rho_i})$. By one axiom, we return from $\boldsymbol{U}_{\rho_i}(\cdot)$ to $\boldsymbol{u}_{\rho_i}(\cdot)$

The process of derivation starts separating the smallest prize 2^1 in $\sigma^0_{\rho_i}$ from the remaining part $\sigma^1_{\rho_i}$; it is expressed as the subtree in the second broken-line rectangular in Fig.5. The outermost $\sigma^0_{\rho_i} = \sigma_{\rho_i}$ is expressed as the compound lottery $\frac{1}{2}2^1 * \frac{1}{2}\sigma^1_{\rho_i}$, meaning that each of 2^1 and $\sigma^1_{\rho_i}$ happens with probability $\frac{1}{2}$. These components are is evaluated by $\boldsymbol{U}_{\rho_i-1}$; thus, $\boldsymbol{U}_{\rho_i}(\sigma_{\rho_i})$ is decomposed into $\boldsymbol{U}_{\rho_i}(\sigma_{\rho_i}) = \frac{1}{2}\cdot\boldsymbol{U}_{\rho_i-1}(2^1) + \frac{1}{2}\cdot\boldsymbol{U}_{\rho_i-1}(\sigma^1_{\rho_i})$. As mentioned after the formula (12), the person spends the cognitive resource to measure the outer probability $\frac{1}{2}$; his cognitive resource decreases to $\rho_i - 1$, which is the subscripts of $\boldsymbol{U}_{\rho_i-1}(2^1)$ and $\boldsymbol{U}_{\rho_i-1}(\sigma^1_{\rho_i})$. The term $\boldsymbol{U}_{\rho_i-1}(\sigma^1_{\rho_i})$ is further decomposed to the third rectangular $\frac{1}{2}\cdot\boldsymbol{U}_{\rho_i-2}(2^2)+ \frac{1}{2}\cdot\boldsymbol{U}_{\rho_i-2}(\sigma^2_{\rho_i})$. Repeating this decomposition and evaluation, person i goes to the innermost rectangular to finish this process.

We need to define the elements occurring in Fig.5. Recall $\widehat{\rho}_i := \min(\overline{t}, \rho_i)$. The domain (support) of $\sigma^0_{\rho_i} = \sigma_{\rho_i}$ is denoted by $Y_{\rho_i} = \{y_1, ..., y_{\widehat{\rho}_i}, y_{\widehat{\rho}_i+1}\}$, where $y_t = 2^t$ for $t \leq \widehat{\rho}_i$ and $y_{\widehat{\rho}_i+1} = 0$. We restrict Y_{ρ_i} to $Y^k_{\rho_i} = \{y_{k+1}, ..., y_{\widehat{\rho}_i}, y_{\widehat{\rho}_i+1}\}$ for $k = 0, ..., \widehat{\rho}_i - 1$; for example, $Y^0_{\rho_i} = Y_{\rho_i}$ and $Y^{\widehat{\rho}_i-1}_{\rho_i} = \{2^{\widehat{\rho}_i}, 0\}$. For $k = 0, ..., \widehat{\rho}_i - 1$, we define $\sigma^k_{\rho_i}$ over $Y^k_{\rho_i}$ by

$$\sigma^k_{\rho_i}(y_t) = \begin{cases} \frac{1}{2^{t-k}} & \text{if } k+1 \leq t \leq \widehat{\rho}_i \\ \\ \frac{1}{2^{t-k}} & \text{if } t = \widehat{\rho}_i + 1. \end{cases} \tag{24}$$

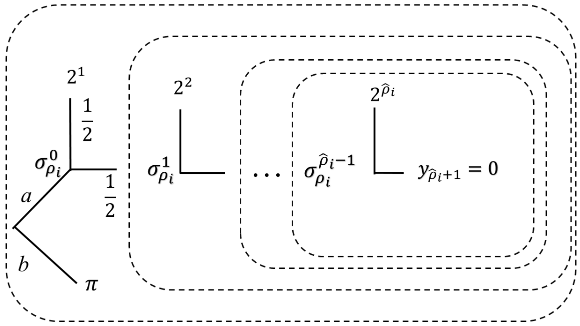

Figure 5: Thought process into the inner lotteries

The first $\sigma^0_{\rho_i}$ is σ_{ρ_i} itself, and the last $\sigma^{\widehat{\rho}_i-1}_{\rho_i}$ has the support $Y^{\widehat{\rho}_i-1}_{\rho_i} = \{2^{\widehat{\rho}_i}, 0\}$ with $\sigma^{\widehat{\rho}_i-1}_{\rho_i}(2^{\widehat{\rho}_i}) = \sigma^{\widehat{\rho}_i-1}_{\rho_i}(0) = \frac{1}{2}$. In the above thought process, $\sigma^k_{\rho_i}$ is decomposed into $\frac{1}{2}2^{k+1} * \frac{1}{2}\sigma^{k+1}_{\rho_i}$ for $k < \widehat{\rho}_i - 1$, the last one $\sigma^{\widehat{\rho}_i-1}_{\rho_i}$ has the domain $Y^{\widehat{\rho}_i-1}_{\rho_i} = \{2^{\widehat{\rho}_i}, 0\}$ and is no longer decomposed.

We describe the above idea of the decomposed process in an axiomatic manner. For this purpose, we need a few more definitions. Let $L_l(X_{\overline{t}}) = \{f : X_{\overline{t}} \to \Pi_l : \sum_{x \in X_{\overline{t}}} f(x) = 1\}$, where $X_{\overline{t}} = \{0, 1, \cdots, 2^{\overline{t}}\}$ and l is a depth, i.e., a nonnegative integer. We say that $(\boldsymbol{u}, \Delta)$ is a *legitimate pair* iff $\boldsymbol{u} = (\overline{u}, \underline{u})$ is defined over a subset Δ of $L_l(X_{\overline{t}})$ for some l and it takes the values in $\mathbb{Q}^2(\geq)$. We say that $\langle(\boldsymbol{U}_{\rho_i}, \Delta^0), (\boldsymbol{U}_{\rho_i-1}, \Delta^1), ..., (\boldsymbol{U}_{\rho_i-\widehat{\rho}_i}, \Delta^{\widehat{\rho}_i})\rangle$ is a *trajectory of the though process of decomposition* iff $(\boldsymbol{U}_{\rho_i-k}, \Delta^k)$ is a legitimate pair for each $k = 0, 1, ..., \widehat{\rho}_i$. The length of a trajectory can be shown to be $\widehat{\rho}_i = \min(\overline{t}, \rho_i)$ (without counting $(\boldsymbol{U}_{\rho_i}, \Delta^0)$) as a result from our axioms for the process, but to avoid unnecessary complications, we use the length $\widehat{\rho}_i$ as given.

The process starts with evaluation of utility values from participation fee π and his subjective understanding σ_{ρ_i} of $\tau_{\overline{t}}$. This is made by decomposing σ_{ρ_i} into subdistributions of shallower depths. Then, we make connections between adjacent pairs $(\boldsymbol{U}_{\rho_i-k}, \Delta^k)$ and $(\boldsymbol{U}_{\rho_i-(k+1)}, \Delta^{k+1})$ in a trajectory. For these connections, we need two concepts: Let f, f^1, f^2 be three lotteries in $L_l(X_{\overline{t}})$ with some $l \geq 1$. We say that $\{f^1, f^2\}$ is a *decomposition* of f iff

$$f(x) = \frac{1}{2} \cdot f^1(x) + \frac{1}{2} \cdot f^2(x) \quad \text{for all } x \in X_{\overline{t}}; \tag{25}$$

$$\text{for } t = 1, 2, \ \delta(f^t(x)) < \delta(f(x)) \quad \text{for all } x \in X_{\overline{t}} \text{ with } \delta(f(x)) > 0. \tag{26}$$

In Fig.5, subjective lottery $\sigma^k_{\rho_i}$ is decomposed to $y_{k+1} = 2^{k+1}$ and $\sigma^{k+1}_{\rho_i}$. In general, a lottery may have multiple decompositions. We say that $f \in L_l(X_{\overline{t}})$ is *mixed* iff $0 < \delta(f(x))$ for some $x \in X_{\overline{t}}$.

Let $\langle(\boldsymbol{U}_{\rho_i}, \Delta^0), (\boldsymbol{U}_{\rho_i-1}, \Delta^1), ..., (\boldsymbol{U}_{\rho_i-\widehat{\rho}_i}, \Delta^{\widehat{\rho}_i})\rangle$ be any trajectory.

Axiom E0 (Comparison target): The domain of $\boldsymbol{U}_{\rho_i}$ is $\Delta^0 = \{\pi, \sigma^0_{\rho_i}\}$, where $\sigma^0_{\rho_i} = \sigma_{\rho_i}$.

Axiom E1 (Decomposition): Let $k = 0, ..., \widehat{\rho}_i - 1$ and let f be mixed. Then, $f \in \Delta^k$ if and only if f has a decomposition $\{f^1, f^2\}$ such that $\{f^1, f^2\} \subseteq \Delta^{k+1}$.

Axiom E2 (Reduction of utility value): For $k = 0, ..., \widehat{\rho}_i - 1$, if $f \in \Delta^k$ has a decomposition

$\{f^1, f^2\}$, then

$$\boldsymbol{U}_{\rho_i - k}(f) = \tfrac{1}{2} \cdot \boldsymbol{U}_{\rho_i-(k+1)}(f^1) + \tfrac{1}{2} \cdot \boldsymbol{U}_{\rho_i-(k+1)}(f^2). \tag{27}$$

Axiom E3 (Connection to the Measurement Process): $\boldsymbol{U}_{\rho_i - k}(y) = \boldsymbol{u}_{\rho_i-k}(y)$ for any $y \in \Delta^k \cap X_{\overline{t}}$ and $k \leq \widehat{\rho}_i$.

Axiom E0 states that the axioms are about the evaluations of π and $\sigma^0_{\rho_i}$. Axiom E3 implies that the evaluation of π is the utility $\boldsymbol{u}_{\rho_i}(\pi)$ given in Section 6.1. The remaining part of E0 is $\sigma^0_{\rho_i} = \sigma_{\rho_i}$ in Δ^0. This is decomposed to lotteries of shallower depths by E1 to E3.

Axiom E1 requires that when $f \in \Delta^k$ is mixed, f is decomposed into two lotteries in Δ^{k+1}, and conversely, if f is obtained by combining two lotteries f^1, f^2 in Δ^{k+1}, then f belongs to Δ^k. It follows from the depth constraint (26) that the depths of lotteries in Δ^k are strictly decreasing, and the process stops when Δ^k has no mixed lottery. It will be shown that this happens for $k = \widehat{\rho}_i$; thus the length of a trajectory is uniquely determined to be $\widehat{\rho}_i$.

Axiom E2 connects the utility values of $(\boldsymbol{u}_{\rho_i-k}, \Delta^k)$ to those of $(\boldsymbol{u}_{\rho_i-(k+1)}, \Delta^{k+1})$ via decompositions. The subscripts of the utility functions are numbered along the decomposition process, and contain the information of permissible depths. In the beginning of the process, person i evaluates the utility value $\boldsymbol{U}_{\rho_i}(f)$ from the viewpoint of cognitive degree ρ_i. Then, he enters the scope of lower cognitive degree $\rho_i - 1$ with probability weight $\frac{1}{2}$, considers a decomposition $\{f^1, f^2\}$ of f, and evaluates each of $\{f^1, f^2\}$ is from the viewpoint of cognitive degree $\rho_i - 1$. Thus, $\boldsymbol{U}_{\rho_i-1}$ is used with the outer weights $\frac{1}{2}$ in (27).

The first $\sigma^0_{\rho_i}$ is σ_{ρ_i}, and in general, $\sigma^k_{\rho_i}$ is defined over the support $Y^k_{\rho_i}$. The last $\sigma^{\widehat{\rho}_i-1}_{\rho_i}$ has the support $Y^{\widehat{\rho}_i-1}_{\rho_i} = \{2^{\widehat{\rho}_i}, y_{\widehat{\rho}_i+1}\} = \{2^{\widehat{\rho}_i}, 0\}$, and the decomposition is applied to neither $2^{\widehat{\rho}_i}$ nor $2^{\widehat{\rho}_i+1}$, and the process does not go any further. We have the following theorem from Axioms E0 to E2: the domains $\Delta^0, \Delta^1, ..., \Delta^{\widehat{\rho}_i}$ are uniquely determined, but the utility functions $\boldsymbol{U}_{\rho_i}, \boldsymbol{U}_{\rho_i-1}, ..., \boldsymbol{U}_{\rho_i-\widehat{\rho}_i}$ are determined within some freedom, which will be explained below. A proof will be given in the end of this section.

Theorem 6.2 (Decomposition trajectory): Suppose $\Delta^0 = \{\pi, \sigma^0_{\rho_i}\}$. A trajectory $\langle(\boldsymbol{U}_{\rho_i}, \Delta^0)$, $(\boldsymbol{U}_{\rho_i-1}, \Delta^1)$, ..., $(\boldsymbol{U}_{\rho_i-\widehat{\rho}_i}, \Delta^{\widehat{\rho}_i})\rangle$ satisfies Axioms E1 and E2 if and only if ,

$$\Delta^k = \{2^k, \sigma^k_{\rho_i}\} \text{ for each } k \leq \widehat{\rho}_i - 1 \text{ and } \Delta^{\widehat{\rho}_i} = \{2^{\widehat{\rho}_i}, 0\}; \tag{28}$$

$$\boldsymbol{U}_{\rho_i}(\sigma^0_{\rho_i}) = \sum_{t=1}^{k} \frac{1}{2^t} \cdot \boldsymbol{U}_{\rho_i-t}(y_t) + \frac{1}{2^k} \cdot \boldsymbol{U}_{\rho_i-k}(\sigma^k_{\rho_i}) \text{ for each } k \leq \widehat{\rho}_i - 1. \tag{29}$$

Under Axioms E0, Axioms E1 and E2 form necessary and sufficient conditions for a trajectory to be the same as what is described in Fig.5. As remarked stated, the domain Δ^k in (28) is uniquely determined and $\Delta^{\widehat{\rho}_i} = \{2^{\widehat{\rho}_i}, y_{\widehat{\rho}_i+1}\} = \{2^{\widehat{\rho}_i}, 0\}$. (29) is an intermediate expression obtained in the decomposition process up to $k \leq \widehat{\rho}_i - 1$. When $k = \widehat{\rho}_i - 1$, the residue $\frac{1}{2^{\widehat{\rho}_i-1}} \cdot \boldsymbol{U}_{\rho_i-(\widehat{\rho}_i-1)}(\sigma^{\widehat{\rho}_i-1}_{\rho_i})$ becomes $\frac{1}{2^{\widehat{\rho}_i-1}} \cdot (\frac{1}{2}\boldsymbol{U}_{\rho_i-\widehat{\rho}_i}(2^{\widehat{\rho}_i}) + \frac{1}{2}\boldsymbol{U}_{\rho_i-\widehat{\rho}_i}(0))$, we obtain, under Axiom E3,

$$\boldsymbol{U}_{\rho_i}(\sigma^0_{\rho_i}) = \sum_{k=1}^{\widehat{\rho}_i} \frac{1}{2^k} \cdot \boldsymbol{u}_{\rho_i-k}(y_k) + \frac{1}{2^{\widehat{\rho}_i}} \cdot \boldsymbol{u}_{\rho_i-\widehat{\rho}_i}(y_{\widehat{\rho}_i+1}). \tag{30}$$

This formula is the goal (12), by denoting $\boldsymbol{U}_{\rho_i}(\sigma^0_{\rho_i})$ by $\boldsymbol{u}_{\rho_i}(\sigma^0_{\rho_i})$.

Remark 6.1. The uniqueness result mentioned in Lemma 6.1 is extended to the formula (30).

Now, we go to the proof of Theorem 6.2. In general, a lottery $f \in L_k(X_{\rho_i})$ may have multiple decompositions; this multiplicity makes an application of Kaneko's [12] theory difficult. In the case of the SP lottery $\sigma^k_{\rho_i}$, however, we can prove uniqueness, which simplifies the application of Axiom E2, which is the following lemma.

Lemma 6.2 (Unique decomposition). For each $k < \widehat{\rho}_i - 1$, lottery $\sigma^k_{\rho_i}$ has the unique decomposition consisting of y_k and $\sigma^{k+1}_{\rho_i}$.

Proof. It is easy to see that $\{y_k, \sigma^{k+1}_{\rho_i}\}$ is a decomposition of $\sigma^k_{\rho_i}$. We show its uniqueness. Suppose that $\{\tau_1, \tau_2\}$ is a decomposition of $\sigma^k_{\rho_i}$. Since the support of $\sigma^k_{\rho_i}$ is $Y^k_{\rho_i} = \{y_{k+1}, ..., y_{\widehat{\rho}_i}, y_{\widehat{\rho}_i+1}\}$, it holds by (25) that

$$\frac{1}{2} \cdot \tau_1(y_t) + \frac{1}{2} \cdot \tau_2(y_t) = \sigma^k_{\rho_i}(y_t) \text{ for } t = k+1, ..., \widehat{\rho}_i + 1. \tag{31}$$

Since $\sigma^k_{\rho_i}(y_{k+1}) = \frac{1}{2}$, we have $\frac{1}{2}\tau_1(y_{k+1}) + \frac{1}{2}\tau_2(y_{k+1}) = \frac{1}{2}$. By (26), $\delta(\tau_1(y_{k+1})) < \delta(\sigma^k_{\rho_i}(y_{k+1})) = 1$ and $\delta(\tau_2(y_{k+1})) < \delta(\sigma^k_{\rho_i}(y_{k+1})) = 1$, which implies that each of $\tau_1(y_{k+1})$ and $\tau_2(y_{k+1})$ is 0 or 1. Since $\frac{1}{2}\tau_1(y_{k+1}) + \frac{1}{2}\tau_2(y_{k+1}) = \frac{1}{2}$, at least one of $\tau_1(y_{k+1})$ and $\tau_2(y_{k+1})$ is 0. Consider the case $\tau_1(y_{k+1}) = 0$. Then, $\tau_2(y_{k+1}) = 1$; thus, $\tau_2(y_{k+1}) = 0$ for all $t = k+2, ..., \widehat{\rho}_i + 1$. So, τ_2 is pure outcome y_{k+1}. It remains to show $\tau_1 = \sigma^{k+1}_{\rho_i}$. By (31), τ_1 is obtained from $\sigma^k_{\rho_i}$ by restricting the support to $Y^k_{\rho_i}$ with normalization by multiplying by 2. The resulting lottery is $\tau_1 = \sigma^{k+1}_{\rho_i}$. Thus, $\{\tau_1, \tau_2\} = \{\sigma^{k+1}_{\rho_i}, y_{k+1}\}$. The other case $\tau_2(y_{k+1}) = 0$ is parallel.■

Proof of Theorem 6.2 (Only-if): Suppose that $\langle(\boldsymbol{U}_{\rho_i}, \Delta^0), (\boldsymbol{U}_{\rho_i - 1}, \Delta^1), ..., (\boldsymbol{U}_{\rho_i - \widehat{\rho}_i}, \Delta^{\widehat{\rho}_i})\rangle$ satisfies Axioms E1 and E2. Repeating Axiom E1 and Lemma 6.2, we have $\Delta^k = \{y_k, \sigma^k_{\rho_i}\}$ for $k = 1, ..., \widehat{\rho}_i - 1$, and $\Delta^{\widehat{\rho}_i} = \{y_{\widehat{\rho}_i}, y_{\widehat{\rho}_i+1}\} = \{y_{\widehat{\rho}_i}, 0\}$. Applying Axioms E2 and E1 to each $\sigma^k_{\rho_i}$, we have (29) for $k = 1, ..., \widehat{\rho}_i$.

(If): Axiom E1 follows from Lemma 6.2. Axiom E2 is proved as follows: by (29), $\boldsymbol{U}_{\rho_i - k}(\sigma^k_{\rho_i}) = 2^k \cdot \boldsymbol{U}_{\rho_i}(\sigma^0_{\rho_i}) - \sum_{t=1}^{k} 2^{k-t} \cdot \boldsymbol{U}_{\rho_i - t}(2^t) = \frac{1}{2} \cdot [2^{k+1} \cdot \boldsymbol{U}_{\rho_i}(\sigma^0_{\rho_i}) - \sum_{t=1}^{k+1} 2^{k+1-t} \cdot \boldsymbol{U}_{\rho_i - t}(2^t)] + \frac{1}{2} \cdot 2^0 \cdot \boldsymbol{U}_{\rho_i - (k+1)}(2^{k+1}) = \frac{1}{2} \cdot \boldsymbol{U}_{\rho_i - (k+1)}(\sigma^{k+1}_{\rho_i}) + \frac{1}{2} \cdot \boldsymbol{U}_{\rho_i - (k+1)}(y_{k+1})$.■

7 Semi-rationalistic Choice for Incomparable Alternatives

We described utility values, $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ and $\boldsymbol{u}_{\rho_i}(\pi)$, of person i with cognitive degree ρ_i over σ_{ρ_i} and π. In many cases, $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ and $\boldsymbol{u}_{\rho_i}(\pi)$ are incomparable. As discussed in Section 3.2, people are often forced to choose an alternative. Here, we develop a probabilistic interpretation of incomparability.

The interval order $\geq_I$ due to Fishburn [7] is adopted to make a comparison between two utility vectors. Here, we introduce $\geq_I$ in a general manner and state its properties. Recall $\mathbb{Q}^2(\geq) = \{\boldsymbol{\lambda} = (\overline{\lambda}, \underline{\lambda}) \in \mathbb{Q} \times \mathbb{Q} : \overline{\lambda} \geq \underline{\lambda}\}$. The first component $\overline{\lambda}$ of $\boldsymbol{\lambda} = (\overline{\lambda}, \underline{\lambda})$ is the least upper bound of possible utility values, and the second $\underline{\lambda}$ is the greatest lower bound. We say that $\boldsymbol{\lambda} = (\overline{\lambda}, \underline{\lambda})$ is *singular* iff $\overline{\lambda} = \underline{\lambda}$, and *non-singular* iff $\overline{\lambda} > \underline{\lambda}$. For two vectors $\boldsymbol{\lambda} = (\overline{\lambda}, \underline{\lambda})$ and $\boldsymbol{\mu} = (\overline{\mu}, \underline{\mu})$ in $\mathbb{Q}^2(\geq) = \{[\overline{\alpha}; \underline{\alpha}] \in \mathbb{Q}^2 : \overline{\alpha} \geq \underline{\alpha}\}$, we define the *interval order* $\geq_I$ over $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ by

$$\boldsymbol{\lambda} \geq_I \boldsymbol{\mu} \text{ if and only if } \underline{\lambda} \geq \overline{\mu}. \tag{32}$$

The comparison $\boldsymbol{\lambda} \geq_I \boldsymbol{\mu}$ means that any possible utility value from $\boldsymbol{\lambda}$ is larger than or equal to any possible utility value from $\boldsymbol{\mu}$. The strict part of $\geq_I$ is denoted by $>_I$. Recall that $\geq_I$ may

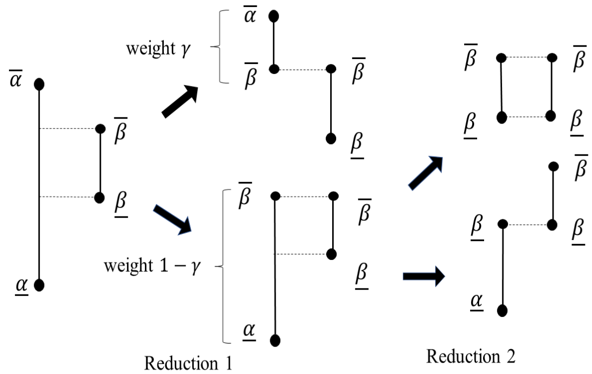

Figure 6: Reduction steps

not be complete, i.e., neither $\boldsymbol{\lambda} \geq_I \boldsymbol{\mu}$ nor $\boldsymbol{\mu} \geq_I \boldsymbol{\lambda}$, which is denoted by $\boldsymbol{\lambda} \bowtie_I \boldsymbol{\mu}$. This $\bowtie_I$ is the *incomparability* relation. The following properties hold:

(a)*: singular case*: when $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are singular, $\boldsymbol{\lambda} \geq_I \boldsymbol{\mu}$ if and only if $\underline{\lambda} \geq \overline{\mu}$;

(b)*: strict preference*: if $\boldsymbol{\lambda}$ or $\boldsymbol{\mu}$ is non-singular, then $\boldsymbol{\lambda} \geq_I \boldsymbol{\mu}$ implies $\boldsymbol{\lambda} >_I \boldsymbol{\mu}$;

(c) *incomparability*: neither $\boldsymbol{\lambda} \geq_I \boldsymbol{\mu}$ nor $\boldsymbol{\mu} \geq_I \boldsymbol{\lambda}$ if and only if $\underline{\lambda} < \overline{\mu} < \overline{\lambda}$ or $\underline{\mu} < \overline{\lambda} < \overline{\mu}$.

In general, the indifference $\boldsymbol{\lambda} \geq_I \boldsymbol{\mu}$ & $\boldsymbol{\mu} \geq_I \boldsymbol{\lambda}$ holds if and only if both are singular and identical.

7.1 An axiomatic system determining the semi-rationalistic choice-function

Let Λ be a finite subset of $\mathbb{Q}^2(\geq)$. We impose the *decomposition closedness* condition on Λ : for any $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Lambda$ and $\xi = \overline{\beta}, \underline{\beta}$,

$$\text{if } \overline{\alpha} \geq \xi \geq \underline{\alpha}, \text{ then } [\overline{\alpha}; \xi] \text{ and } [\xi; \underline{\alpha}] \text{ are in } \Lambda. \tag{33}$$

Here, $\boldsymbol{\alpha}$ is decomposed into $[\overline{\alpha}; \xi]$ and $[\xi; \underline{\alpha}]$ by $\xi = \overline{\beta}$ or $\underline{\beta}$, such as Reduction 1 in Fig.6. When $\boldsymbol{\beta} = \boldsymbol{\alpha}$, (33) implies $[\overline{\alpha}; \overline{\alpha}] \in \Lambda$ and $[\underline{\alpha}; \underline{\alpha}] \in \Lambda$. However, the essential case of (33) is $\overline{\alpha} > \xi > \underline{\alpha}$; in this case, new non-degenerated intervals are $[\overline{\alpha}; \xi]$ and $[\xi; \underline{\alpha}]$. This condition generates a finite number of subintervals.

For given $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathbb{Q}^2(\geq)$, we say that Λ is the set *generated* by $\boldsymbol{\lambda}, \boldsymbol{\mu}$, denoted by $\Lambda(\boldsymbol{\lambda}; \boldsymbol{\mu})$, iff Λ is the smallest among Λ's having $\boldsymbol{\lambda}, \boldsymbol{\mu}$ and satisfying (33). We will target $\boldsymbol{\lambda} = \boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$, $\boldsymbol{\mu} = \boldsymbol{u}_{\rho_i}(\pi)$, and the generated intervals. This $\Lambda(\boldsymbol{\lambda}; \boldsymbol{\mu})$ is the uniquely determined. For example, when $\overline{\lambda} > \overline{\mu} > \underline{\lambda} > \underline{\mu}$, we have

$$\Lambda(\boldsymbol{\lambda}; \boldsymbol{\mu}) = \{[\overline{\lambda}; \overline{\mu}], [\overline{\mu}; \underline{\lambda}], [\underline{\lambda}; \underline{\mu}]\} \cup \{[\xi; \xi] : \xi = \overline{\lambda}, \underline{\lambda}, \overline{\mu}, \underline{\mu}\} \cup \{\boldsymbol{\lambda}, \boldsymbol{\mu}\}. \tag{34}$$

Here, $[\overline{\lambda}; \underline{\mu}]$ is not in $\Lambda(\boldsymbol{\lambda}; \boldsymbol{\mu})$. This $\Lambda(\boldsymbol{\lambda}; \boldsymbol{\mu})$ has only 9 vectors. In general, it has at most 9 vectors.

Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathbb{Q}^2(\geq)$. A *behavioral-probability function* is given as $\eta : \Lambda(\boldsymbol{\lambda}; \boldsymbol{\mu})^2 \to \mathbb{Q}_{[0,1]}$. The value $\eta(\boldsymbol{\alpha}; \boldsymbol{\beta})$ means that person i chooses $\boldsymbol{\alpha}$ with probability $\eta(\boldsymbol{\alpha}; \boldsymbol{\beta})$ from $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. We focus on a person i for $\eta(\boldsymbol{\alpha}; \boldsymbol{\beta})$, but the information about i is included in $\boldsymbol{\alpha}, \boldsymbol{\beta}$. Our final target is to calculate $\eta(\boldsymbol{\lambda};\boldsymbol{\mu})$ rather than $\{\eta(\boldsymbol{\alpha};\boldsymbol{\beta}) : (\boldsymbol{\alpha};\boldsymbol{\beta}) \in \Lambda(\boldsymbol{\lambda};\boldsymbol{\mu})^2\}$, but the decomposition method requires $\eta(\cdot;\cdot)$ over the entire set $\Lambda(\boldsymbol{\lambda};\boldsymbol{\mu})^2$.

We assume the following three axioms on $\eta(\boldsymbol{\alpha},\boldsymbol{\beta})$.

Axiom S1(Probability): $\eta(\boldsymbol{\alpha};\boldsymbol{\beta}) + \eta(\boldsymbol{\beta};\boldsymbol{\alpha}) = 1$ for $\boldsymbol{\alpha},\boldsymbol{\beta} \in \Lambda(\boldsymbol{\lambda};\boldsymbol{\mu})$.

Axiom S2 (Preservation of the interval order): Let $\boldsymbol{\alpha},\boldsymbol{\beta} \in \Lambda(\boldsymbol{\lambda};\boldsymbol{\mu})$. If $\boldsymbol{\alpha} \geq_I \boldsymbol{\beta}$ and $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$, then $\eta(\boldsymbol{\alpha};\boldsymbol{\beta}) = 1$.

Axiom S3 (Proportional reduction): Let $\boldsymbol{\alpha},\boldsymbol{\beta} \in \Lambda(\boldsymbol{\lambda};\boldsymbol{\mu})$.

(1): If $\overline{\alpha} > \overline{\beta} \geq \underline{\alpha}$, then $\eta(\boldsymbol{\alpha};\boldsymbol{\beta}) = \frac{\overline{\alpha}-\overline{\beta}}{\overline{\alpha}-\underline{\alpha}} \cdot \eta([\overline{\alpha};\overline{\beta}];\boldsymbol{\beta}) + \frac{\overline{\beta}-\underline{\alpha}}{\overline{\alpha}-\underline{\alpha}} \cdot \eta([\overline{\beta};\underline{\alpha}];\boldsymbol{\beta})$.

(2): If $\overline{\alpha} \geq \underline{\beta} > \underline{\alpha}$, then, $\eta(\boldsymbol{\alpha};\boldsymbol{\beta}) = \frac{\overline{\alpha}-\underline{\beta}}{\overline{\alpha}-\underline{\alpha}} \cdot \eta([\overline{\alpha};\underline{\beta}];\boldsymbol{\beta}) + \frac{\underline{\beta}-\underline{\alpha}}{\overline{\alpha}-\underline{\alpha}} \cdot \eta([\underline{\beta};\underline{\alpha}];\boldsymbol{\beta})$.

Axiom S1 expresses the intended interpretation that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is chosen with probability $\eta(\boldsymbol{\alpha};\boldsymbol{\beta})$ and $1-\eta(\boldsymbol{\alpha};\boldsymbol{\beta})$, respectively. Axiom S2 states that the certainty case, $\eta(\boldsymbol{\alpha};\boldsymbol{\beta}) = 1$, coincides with the interval order $\geq_I$. Axiom S3 reduces $\eta(\boldsymbol{\alpha};\boldsymbol{\beta})$ into the probability values determined by S1 and S2.

Let us see two simple observations: first, since $\eta(\boldsymbol{\alpha};\boldsymbol{\alpha}) + \eta(\boldsymbol{\alpha};\boldsymbol{\alpha}) = 1$ by Axiom S1, we have

$$\eta(\boldsymbol{\alpha};\boldsymbol{\alpha}) = \frac{1}{2} \text{ for } \boldsymbol{\alpha} \in \Lambda(\boldsymbol{\lambda};\boldsymbol{\mu}). \tag{35}$$

The second observation is the dual of Axiom S2; it is obtained by Axioms S2 and S1:

$$\text{if } \boldsymbol{\alpha} \geq_I \boldsymbol{\beta} \text{ but } \boldsymbol{\alpha} \neq \boldsymbol{\beta}, \text{ then } \eta(\boldsymbol{\beta};\boldsymbol{\alpha}) = 0. \tag{36}$$

The statements (35) and (36) with S2 will be used in the terminal cases of the reduction step.

The other simple observation is: Axiom S3 states a partition with respect to the first component in $\eta(\cdot;\cdot)$, but in either case of Axiom S3, it can be partitioned with respect to the second component, i.e., for case (1), we have

$$\eta(\boldsymbol{\beta};\boldsymbol{\alpha}) = \frac{\overline{\alpha}-\overline{\beta}}{\overline{\alpha}-\underline{\alpha}}\eta(\boldsymbol{\beta};[\overline{\alpha};\overline{\beta}]) + \frac{\overline{\beta}-\underline{\alpha}}{\overline{\alpha}-\underline{\alpha}} \cdot \eta(\boldsymbol{\beta};[\overline{\beta};\underline{\alpha}]). \tag{37}$$

This will be proved in Section 7.2.

We have many cases for detailed representations of $\eta(\boldsymbol{\alpha};\boldsymbol{\beta})$. They are summarized into three cases in Theorem 7.1. We denote $\ell[\boldsymbol{\alpha}] = \ell[\overline{\alpha};\underline{\alpha}] = \overline{\alpha} - \underline{\alpha}$ for an interval $\boldsymbol{\alpha} = [\overline{\alpha};\underline{\alpha}]$; it is the length of the interval $\boldsymbol{\alpha}$. Theorem 7.1 states that $\eta(\boldsymbol{\alpha};\boldsymbol{\beta})$ and $\eta(\boldsymbol{\beta};\boldsymbol{\alpha})$ are described explicitly by $\ell[\cdot]$ with $0, \frac{1}{2}, 1$ in each of the three cases. A proof of Theorem 7.1 will be given in Section 7.2.

Theorem 7.1 (Semi-rationalistic behavioral-probability). A function $\eta : \Lambda(\boldsymbol{\lambda};\boldsymbol{\mu})^2 \to \mathbb{Q}_{[0,1]}$ satisfies Axioms S1 to S3 if and only if for each $(\boldsymbol{\alpha};\boldsymbol{\beta}) \in \Lambda(\boldsymbol{\lambda};\boldsymbol{\mu})^2$, $\eta(\boldsymbol{\alpha};\boldsymbol{\beta})$ and $\eta(\boldsymbol{\beta};\boldsymbol{\alpha})$ are given as

(A): if $\underline{\alpha} \geq \overline{\beta}$, then $\eta(\boldsymbol{\alpha};\boldsymbol{\beta}) = \begin{cases} 1 & \text{if } \boldsymbol{\alpha} \neq \boldsymbol{\beta} \\ \frac{1}{2} & \text{if } \boldsymbol{\alpha} = \boldsymbol{\beta} \end{cases}$ and $\eta(\boldsymbol{\beta};\boldsymbol{\alpha}) = \begin{cases} 0 & \text{if } \boldsymbol{\alpha} \neq \boldsymbol{\beta} \\ \frac{1}{2} & \text{if } \boldsymbol{\alpha} = \boldsymbol{\beta}; \end{cases}$

(B): if $\overline{\alpha} \geq \overline{\beta} \geq \underline{\beta} \geq \underline{\alpha}$ and $\boldsymbol{\alpha}$ is non-singular, then

$$\eta(\boldsymbol{\alpha};\boldsymbol{\beta}) = \frac{\ell[\overline{\alpha};\overline{\beta}]}{\ell[\boldsymbol{\alpha}]} + \frac{1}{2} \cdot \frac{\ell[\boldsymbol{\beta}]}{\ell[\boldsymbol{\alpha}]} \quad \text{and} \quad \eta(\boldsymbol{\beta};\boldsymbol{\alpha}) = \frac{1}{2} \cdot \frac{\ell[\boldsymbol{\beta}]}{\ell[\boldsymbol{\alpha}]} + \frac{\ell[\underline{\beta};\underline{\alpha}]}{\ell[\boldsymbol{\alpha}]}. \tag{38}$$

(C): if $\overline{\alpha} \geq \overline{\beta} \geq \underline{\alpha} \geq \underline{\beta}$ and both $\boldsymbol{\alpha};\boldsymbol{\beta}$ are non-singular, then

$$\eta(\boldsymbol{\alpha};\boldsymbol{\beta}) = \frac{\ell[\overline{\alpha};\overline{\beta}]}{\ell[\boldsymbol{\alpha}]} + \frac{1}{2} \cdot \frac{\ell[\overline{\beta};\underline{\alpha}]}{\ell[\boldsymbol{\alpha}]} \cdot \frac{\ell[\overline{\beta};\underline{\alpha}]}{\ell[\boldsymbol{\beta}]} + \frac{\ell[\overline{\beta};\underline{\alpha}]}{\ell[\boldsymbol{\alpha}]} \cdot \frac{\ell[\underline{\alpha};\underline{\beta}]}{\ell[\boldsymbol{\beta}]} \quad \text{and} \quad \eta(\boldsymbol{\beta};\boldsymbol{\alpha}) = \frac{1}{2} \cdot \frac{\ell[\overline{\beta};\underline{\alpha}]}{\ell[\boldsymbol{\alpha}]} \cdot \frac{\ell[\overline{\beta};\underline{\alpha}]}{\ell[\boldsymbol{\beta}]}. \tag{39}$$

The above (A) to (C) are exhaustive, which is proved in Section 7.2, but are not exclusive; for example, the case $\overline{\alpha} > \overline{\beta} = \underline{\beta} = \underline{\alpha}$ is in (A) and (C). Non-exclusiveness is allowed to avoid complication of presentations.

Theorem 7.1 implies that Axioms S1 to S3 determine the behavioral-probability function $\eta(\cdot;\cdot)$ uniquely. Then, this axiomatic system is connected to the axiomatic system M0 to M2 and the system E0 to E3; $\boldsymbol{\lambda} = \boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ and $\boldsymbol{\mu} = \boldsymbol{u}_{\boldsymbol{\rho_i}}(\pi)$ and the function $\eta(\cdot;\cdot)$ over $\Lambda(\boldsymbol{\lambda};\boldsymbol{\mu})^2$ is expressed by Theorem 7.1. The behavioral probability $\Pr_{\rho_i}(a)$ in Seection 3.2 is determined as $\eta(\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}); \boldsymbol{u}_{\boldsymbol{\rho_i}}(\pi))$. Table 4.1 is covered by cases (A) and (B) of Theorem 7.1. In general, for comparisons between $\boldsymbol{\lambda} = \boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ and $\boldsymbol{\mu} = \boldsymbol{u}_{\boldsymbol{\rho_i}}(\pi)$, case (C) of Theorem 7.1 is unnecessary because of Theorem 3.1.

7.2 Proof of Theorem 7.1

Let us see that (A), (B), and (C) are exhaustive. Since both $\eta(\boldsymbol{\alpha};\boldsymbol{\beta})$ and $\eta(\boldsymbol{\beta};\boldsymbol{\alpha})$ are given in the three cases, it suffices to consider the case $\overline{\alpha} \geq \overline{\beta}$. Thus, we consider the following three subcases:

(a): $\underline{\alpha} \geq \overline{\beta}$; (b): $\overline{\alpha} \geq \overline{\beta} \geq \underline{\beta} \geq \underline{\alpha}$; and (c): $\overline{\alpha} \geq \overline{\beta} \geq \underline{\alpha} \geq \underline{\beta}$

In (a), (A) covers all the cases where $\boldsymbol{\alpha}$ and/or $\boldsymbol{\beta}$ are singular or not. In (b), if $\boldsymbol{\alpha}$ is singular, then this case is included in (A). In (c), if $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ is singular, then this is included in (A). Thus, (A), (B), and (C) cover all the cases.

Proof of Theorem 7.1 (If part): We show that $\eta(\cdot;\cdot)$ satisfies Axioms S1, S2, and S3.

Axiom S1: $\eta(\boldsymbol{\alpha};\boldsymbol{\beta}) + \eta(\boldsymbol{\beta};\boldsymbol{\alpha}) = 1$ for all $\boldsymbol{\alpha};\boldsymbol{\beta} \in \Lambda(\boldsymbol{\lambda};\boldsymbol{\mu})^2$. This is verified for the cases (A), (B), and (C). This is straightforward for (A). For (C), $\eta(\boldsymbol{\alpha};\boldsymbol{\beta}) + \eta(\boldsymbol{\beta};\boldsymbol{\alpha}) = \frac{\ell[\overline{\alpha};\overline{\beta}]}{\ell[\boldsymbol{\alpha}]} + \frac{\ell[\overline{\beta};\underline{\alpha}]}{\ell[\boldsymbol{\alpha}]} \cdot \frac{\ell[\overline{\beta};\underline{\alpha}]}{\ell[\boldsymbol{\beta}]} + \frac{\ell[\overline{\beta};\underline{\alpha}]}{\ell[\boldsymbol{\alpha}]} \cdot \frac{\ell[\underline{\alpha};\underline{\beta}]}{\ell[\boldsymbol{\beta}]} = \frac{\ell[\overline{\alpha};\overline{\beta}]}{\ell[\boldsymbol{\alpha}]} + \frac{\ell[\overline{\beta};\underline{\alpha}]}{\ell[\boldsymbol{\alpha}]} \cdot (\frac{\ell[\overline{\beta};\underline{\alpha}]}{\ell[\boldsymbol{\beta}]} + \frac{\ell[\underline{\alpha};\underline{\beta}]}{\ell[\boldsymbol{\beta}]}) = \frac{\ell[\overline{\alpha};\overline{\beta}]}{\ell[\boldsymbol{\alpha}]} + \frac{\ell[\overline{\beta};\underline{\alpha}]}{\ell[\boldsymbol{\alpha}]} = 1$. (B) is similar.

Axiom S2: This follows from (A) of the theorem.

Axiom S3: It holds by (A) that $\eta([\overline{\alpha};\overline{\beta}];\boldsymbol{\beta}) = 1$, $\eta(\boldsymbol{\beta};\boldsymbol{\beta}) = \frac{1}{2}$, and $\eta([\underline{\beta};\underline{\alpha}];\boldsymbol{\beta}) = 0$. Consider case (1) of S3; Case (2) is similar. Then, by (38),

$$\begin{aligned}
\eta([\overline{\alpha};\overline{\beta}];\boldsymbol{\beta}) &= \tfrac{\ell[\overline{\alpha};\overline{\beta}]}{\ell[\boldsymbol{\alpha}]} \cdot 1 + \tfrac{\ell[\boldsymbol{\beta}]}{\ell[\boldsymbol{\alpha}]} \cdot \tfrac{1}{2} + \tfrac{\ell[\underline{\beta};\underline{\boldsymbol{\alpha}}]}{\ell[\boldsymbol{\alpha}]} \cdot 0 \\
&= \tfrac{\ell[\overline{\alpha};\overline{\beta}]}{\ell[\boldsymbol{\alpha}]} \cdot \eta([\overline{\alpha};\overline{\beta}];\boldsymbol{\beta}) + \tfrac{\ell[\boldsymbol{\beta}]}{\ell[\boldsymbol{\alpha}]} \cdot \eta(\boldsymbol{\beta};\boldsymbol{\beta}) + \tfrac{\ell[\underline{\beta};\underline{\boldsymbol{\alpha}}]}{\ell[\boldsymbol{\alpha}]} \cdot \eta([\underline{\beta};\underline{\alpha}];\boldsymbol{\beta}) \\
&= \tfrac{\ell[\overline{\alpha};\overline{\beta}]}{\ell[\boldsymbol{\alpha}]} \cdot \eta([\overline{\alpha};\overline{\beta}];\boldsymbol{\beta}) + \tfrac{\ell[\overline{\beta};\underline{\alpha}]}{\ell[\boldsymbol{\alpha}]} \cdot \eta([\overline{\beta};\underline{\alpha}];\boldsymbol{\beta}).
\end{aligned}$$

Only-If part: It suffices to show (A), (B), and (C) of Theorem 7.1.

(A): Let us see the left statement; the right follows from Axiom S1. Suppose $\underline{\alpha} \geq \overline{\beta}$. If $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$, then $\eta(\boldsymbol{\alpha};\boldsymbol{\beta}) = 1$ by Axiom S2, and if $\boldsymbol{\alpha} = \boldsymbol{\beta}$, then $\eta(\boldsymbol{\alpha};\boldsymbol{\beta}) = \frac{1}{2}$ by (35).

To prove (B) and (C) of Theorem 7.1, first we prove the following lemma.

Lemma 7.2. Let $\boldsymbol{\zeta} = [\overline{\zeta};\underline{\zeta}]$ and $\boldsymbol{\zeta}' = [\overline{\zeta}';\underline{\zeta}']$ with $\overline{\zeta} = \overline{\zeta}'$. Then

(1): if $\overline{\zeta} > \underline{\zeta}' \geq \underline{\zeta}$, then $\eta(\boldsymbol{\zeta};\boldsymbol{\zeta}') = \frac{1}{2} \cdot \frac{\ell[\overline{\zeta};\underline{\zeta}']}{\ell[\boldsymbol{\zeta}]}$;

(2): if $\overline{\zeta} > \underline{\zeta} \geq \underline{\zeta}'$, then $\eta(\boldsymbol{\zeta};\boldsymbol{\zeta}') = \frac{1}{2} \cdot \frac{\ell[\boldsymbol{\zeta}]}{\ell[\boldsymbol{\zeta}']} + \frac{\ell[\underline{\zeta};\underline{\zeta}']}{\ell[\boldsymbol{\zeta}]}$.

Proof. **(1)**: By Axiom S3, we have $\eta(\boldsymbol{\zeta};\boldsymbol{\zeta}') = \frac{\ell[\overline{\zeta};\underline{\zeta}']}{\ell[\boldsymbol{\zeta}]} \cdot \eta([\overline{\zeta};\underline{\zeta}'];\boldsymbol{\zeta}') + \frac{\ell[\underline{\zeta}';\underline{\zeta}]}{\ell[\boldsymbol{\zeta}]} \cdot \eta([\underline{\zeta}';\underline{\zeta}];\boldsymbol{\zeta})$. Since

$[\overline{\zeta};\underline{\zeta}'] = \boldsymbol{\zeta}'$, we have $\eta([\overline{\zeta};\underline{\zeta}'];\boldsymbol{\zeta}) = \frac{1}{2}$ by (35). Since $\boldsymbol{\zeta}' \geq_I [\underline{\zeta}';\underline{\zeta}]$, we have $\eta([\underline{\zeta}';\underline{\alpha}];\boldsymbol{\zeta}) = 0$ by Axioms S1 and S2. Hence, $\eta(\boldsymbol{\zeta};\boldsymbol{\zeta}') = \frac{1}{2} \cdot \frac{\ell(\overline{\zeta};\zeta')}{\ell(\boldsymbol{\zeta})}$.

(2): Switching $\boldsymbol{\zeta}$ with $\boldsymbol{\zeta}'$, (2) is reduced to (1).■

Now, we return to the proof of the only-if part. To prove (B) and (C) of Theorem 7.1.

Consider (B): $\overline{\alpha} \geq \overline{\beta} \geq \underline{\beta} \geq \underline{\alpha}$, and $\boldsymbol{\alpha}$ is non-singular. By Axiom S3, we have

$$\eta(\boldsymbol{\alpha};\boldsymbol{\beta}) = \frac{\ell[\overline{\alpha};\overline{\beta}]}{\ell[\boldsymbol{\alpha}]} \cdot \eta([\overline{\alpha},\overline{\beta}];\boldsymbol{\beta}) + \frac{\ell[\overline{\beta};\underline{\alpha}]}{\ell[\boldsymbol{\alpha}]} \cdot \eta([\overline{\beta};\underline{\alpha}];\boldsymbol{\beta}). \tag{40}$$

By Axiom S2, $\eta([\overline{\alpha},\overline{\beta}];\boldsymbol{\beta}) = 1$. When $\boldsymbol{\beta}$ is singular, we $\eta([\overline{\beta};\underline{\alpha}];\boldsymbol{\beta}) = 0$ by Axiom S2; thus, $\eta(\boldsymbol{\alpha};\boldsymbol{\beta}) = \frac{\ell[\overline{\alpha};\overline{\beta}]}{\ell[\boldsymbol{\alpha}]} = \frac{\ell[\overline{\alpha};\overline{\beta}]}{\ell[\boldsymbol{\alpha}]} + \frac{\ell[\boldsymbol{\beta}]}{\ell[\boldsymbol{\alpha}]}$. Suppose that $\boldsymbol{\beta}$ is non-singular. The second term is decomposed into

$$\eta([\overline{\beta};\underline{\alpha}];\boldsymbol{\beta}) = \frac{\ell[\overline{\beta};\underline{\beta}]}{\ell[\boldsymbol{\beta}]} \cdot \eta([\overline{\beta};\underline{\beta}];\boldsymbol{\beta}) + \frac{\ell[\underline{\beta};\underline{\alpha}]}{\ell[\boldsymbol{\alpha}]} \cdot \eta([\underline{\beta};\underline{\alpha}];\boldsymbol{\beta}).$$

By (35), $\eta([\overline{\beta};\underline{\beta}];\boldsymbol{\beta}) = \frac{1}{2}$. If $\underline{\beta} = \underline{\alpha}$, then $\frac{\ell[\underline{\beta};\underline{\alpha}]}{\ell[\boldsymbol{\alpha}]} = 0$, and if $\underline{\beta} > \underline{\alpha}$, by Axiom S2, $\eta([\underline{\beta};\underline{\alpha}];\boldsymbol{\beta}) = 0$. Summing up these, we have $\eta([\overline{\beta};\underline{\alpha}];\boldsymbol{\beta}) = \frac{1}{2}$. Plugging this to (40), we have $\eta(\boldsymbol{\alpha};\boldsymbol{\beta}) = \frac{\ell[\overline{\alpha};\overline{\beta}]}{\ell[\boldsymbol{\alpha}]} = \frac{\ell[\overline{\alpha};\overline{\beta}]}{\ell[\boldsymbol{\alpha}]} + \frac{\ell[\boldsymbol{\beta}]}{\ell[\boldsymbol{\alpha}]}$.

Consider (C): $\overline{\alpha} \geq \overline{\beta} \geq \underline{\alpha} \geq \underline{\beta}$, and both $\boldsymbol{\alpha};\boldsymbol{\beta}$ are non-singular. Then,

$$\eta(\boldsymbol{\alpha};\boldsymbol{\beta}) = \frac{\ell(\overline{\alpha};\overline{\beta})}{\ell(\boldsymbol{\alpha})} \cdot \eta([\overline{\alpha},\overline{\beta}];\boldsymbol{\beta}) + \frac{\ell(\overline{\beta};\underline{\alpha})}{\ell(\boldsymbol{\alpha})} \cdot \eta([\overline{\beta},\underline{\alpha}];\boldsymbol{\beta}). \tag{41}$$

In the first term of the right-hand side, we have $\eta([\overline{\alpha},\overline{\beta}];\boldsymbol{\beta}) = 1$ by S2. The second term is reduced by Lemma 7.2.(1) to $\frac{\ell(\overline{\beta};\underline{\alpha})}{\ell(\boldsymbol{\alpha})} \cdot \frac{1}{2} \cdot \frac{\ell([\underline{\alpha};\underline{\beta}])}{\ell(\boldsymbol{\beta})}$. Thus, $\eta(\boldsymbol{\alpha};\boldsymbol{\beta}) = \frac{\ell(\overline{\alpha};\overline{\beta})}{\ell(\boldsymbol{\alpha})} + \frac{1}{2} \cdot \frac{\ell(\overline{\beta};\underline{\alpha})}{\ell(\boldsymbol{\alpha})} \cdot \frac{\ell([\underline{\alpha};\underline{\beta}])}{\ell(\boldsymbol{\beta})}$, which is (C).■

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