Cox, Sean

Maximum deconstructibility in module categories. (English) Zbl 07455911

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For a class of $\mathcal{F}$ of modules, *homological algebra relative to* $\mathcal{F}$ attempts to employ similar methods to classical homological algebra with $\mathcal{F}$ playing the same role that the class $\mathcal{P}_0$ of projective modules does in the classical setting. It turned out that, in order for them to work well (e.g. for the relative Ext independent of the $\mathcal{F}$-resolutions), $\mathcal{F}$ should be a *precovering class*, aka *right-approximating* class. Around the turn of the millennium [*P. C. Eklof* and *J. Trlifaj*, Bull. Lond. Math. Soc. 33, No. 1, 41–51 (2001; Zbl 1030.16004); *L. Bican* et al., Bull. Lond. Math. Soc. 33, No. 4, 385–390 (2001; Zbl 1029.16002)], the notion of a deconstructible class grew out of the solution of the *Flat Cover Conjecture* [*E. E. Enochs*, Isr. J. Math. 39, 189–209 (1981; Zbl 0464.16019); *H. Bass*, Trans. Am. Math. Soc. 95, 466–488 (1960; Zbl 0094.02201)]. Deconstructible classes are always precovering [*R. Göbel* and *J. Trlifaj*, Approximations and endomorphism algebras of modules. Volume 1: Approximations. Volume 2: Predictions. Berlin: Walter de Gruyter (2012; Zbl 1292.16001), Theorem 7.21], to show that a class is deconstructible having become one of the main tools in demonstration that the class is precovering.

The principal objective in this paper is to establish a new top-down characterization of deconstructibility with two main applications below which are relative consistency results.

(I) Vopěnka's Principle (VP) [*R. M. Solovay* et al., Ann. Math. Logic 13, 73–116 (1978; Zbl 0376.02055); *A. Kanamori*, Stud. Logic Found. Math. 96, 145–153 (1978; Zbl 0453.03055)] implies that for any class $\mathfrak{X}$ of modules, the class $\mathfrak{X}-\mathcal{GP}$ of $\mathfrak{X}$-Gorenstein projective modules is deconstructible.

(II) VP implies that every class of modules, or of complexes of modules, that could conceivably be deconstructible, is in fact deconstructible.

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MSC:

18G25 Relative homological algebra, projective classes (category-theoretic aspects)
03E75 Applications of set theory
16E30 Homological functors on modules (Tor, Ext, etc.) in associative algebras
16D40 Free, projective, and flat modules and ideals in associative algebras
16D90 Module categories in associative algebras
16B70 Applications of logic in associative algebras

Keywords:

Gorenstein projective; deconstructible; precovering; stationary logic; elementary submodel

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