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Weak topologies on toposes. (English summary)
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It is well known [P. T. Johnstone, *Sketches of an elephant: a topos theory compendium. Vol. 1*, Oxford Logic Guides, 43, Oxford Univ. Press, New York, 2002; MR1953060] that the Lawvere-Tierney (LT) topologies on an elementary topos $\mathcal{E}$ is not closed under composition, the composition failing to be idempotent in general. In the realm of topological spaces, a pretopological space was considered [E. Čech, *[Topological spaces]* (Czech), Nakladatelství Československé Akademie Věd, Prague, 1959; MR0104205; W. Tholen, Rend. Istit. Mat. Univ. Trieste **25** (1993), no. 1-2, 451–465 (1994); MR1346340; D. N. Dikranjan and W. Tholen, *Categorical structure of closure operators*, Math. Appl., 346, Kluwer Acad. Publ., Dordrecht, 1995; MR1368854]. In topos theory an analogous notion is a weak LT-topology or a weak topology for short [S. N. Hosseini and S. S. Mousavi, Appl. Categ. Structures **14** (2006), no. 2, 99–110; MR2247446; S. N. Hosseini and A. Ilaghi-Hosseini, J. Mahani Math. Res. Cent. **1** (2012), no. 2, 137–145, doi:10.22103/JMMRC.2012.513]. The paper under review is concerned with some properties of weak LT-topologies. A synopsis of the paper goes as follows:

- §2 is concerned with weak topologies on a topos $\mathcal{E}$. A class of weak topologies, called weak ideal topology, is introduced on the topos M-**Sets**, consisting of all representations
$$X \times M \to X$$
of a fixed monoid M on a variable set X, by means of the left ideals of the monoid. It is also shown that, for a productive weak topology j on $\mathcal{E}$, the full subcategory $\mathbf{Sh}_j\mathcal{E}$ of all j-sheaves of $\mathcal{E}$ is a topos.
- §3 is concerned with a class of weak topologies on $\mathcal{E}$ induced by natural transformations of the identity functor $\mathrm{id}_{\mathcal{E}}$ on $\mathcal{E}$. It is shown in the special case of the topos M-**Sets** that they correspond to weak ideal topologies with respect to certain left ideals of M.
- §4 shows that the weak topologies on a (co)complete topos constitute a complete residuated lattice, calculating joins of topologies.
- §5 establishes, for a productive weak topology j on $\mathcal{E}$, a left adjoint to the inclusion functor from the category $\mathbf{Sep}_j\mathcal{E}$ of all separated objects of $\mathcal{E}$ to the full subcategory C_j of $\mathcal{E}$ consisting of all objects E of $\mathcal{E}$ for which the closure of the diagonal subobject Δ_E of $E \times E$ is closed. It is also shown that the former category $\mathbf{Sep}_j\mathcal{E}$ is in fact a quasitopos whenever the topos $\mathcal{E}$ is complete and co-well-powered.
- §6 is devoted to finding the associated sheaf to any separated object of $\mathcal{E}$ with respect to productive weak topology j on $\mathcal{E}$. A restricted associated sheaf functor to a productive weak topology j on $\mathcal{E}$ is constructed.

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.