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AUTONOMOUS EQUATIONS OF MAHLER TYPE AND TRANSCENDENCE

By

Kumiko Nishioka and Seiji Nishioka

Abstract. In this paper, we study transcendence of values of Mahler functions satisfying first-order rational difference equations of Mahler type with constant coefficients.

1 Introduction and Result

Let K be an algebraic number field and d an integer greater than 1. For a formal power series $f(z) \in K[[z]]$ with radius of convergence $R > 0$ which satisfy the functional equation

$$f(z^d) = \frac{a_0(z) + a_1(z)f(z) + \cdots + a_m(z)f(z)^m}{b_0(z) + b_1(z)f(z) + \cdots + b_m(z)f(z)^m} \quad (m \geq 1), \tag{1}$$

K. Mahler proved the following theorem, where $a_i(z), b_i(z) \in K[z]$ satisfy $a_m(z) \neq 0$ or $b_m(z) \neq 0$, and that

$$a_0(z) + a_1(z)u + \cdots + a_m(z)u^m$$

and

$$b_0(z) + b_1(z)u + \cdots + b_m(z)u^m$$

are relatively prime as polynomials in u. Note that at least one of these polynomials is non-constant. Let $\Delta(z)$ be their resultant.

Theorem 1 (K. Mahler [1]). *Suppose $m < d$ and that $f(z)$ is transcendental over $K(z)$. If $\alpha \in \overline{\mathbf{Q}}$ satisfies*

$$0 < |\alpha| < \min\{1, R\}, \quad \Delta(\alpha^{d^k}) \neq 0 \quad (k \geq 0),$$

then $f(\alpha)$ is a transcendental number.

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However, we know few transcendental functions $f(z)$ satisfying the equation (1) in the case $m \geq 2$. In 1983, K. Mahler obtained a necessary and sufficient condition to exist a convergent power series

$$f(z) = f_0 + f_r z^r \left(1 + \sum_{j=1}^{\infty} \phi_j z^j\right), \quad r \geq 1,\ f_r \neq 0,$$

satisfying a functional equation $P(f(z), f(z^d)) = 0$, $P(u, v) \in K[u, v] \backslash \{0\}$, with constant coefficients. Here, we suppose that $P(u, v)$ is an irreducible polynomial with $\deg_u P = m \geq 1$ and $\deg_v P = n \geq 1$, and not a product of $u - v$ multiplied by constants. Choose an algebraic number f_0 such that $P(f_0, f_0) = 0$. Thinking of the algebraic function field of one variable defined by $P(u, v) = 0$, we find that there exists

$$U_0(v) = f_0 + \sum_{l=b}^{\infty} P_l (v - f_0)^{l/a}, \quad P_b \neq 0,\ 1 \leq a \leq m$$

such that $P(U_0(v), v) = 0$, where P_l $(l \geq b)$ are elements of a certain finite extension K' of K.

THEOREM 2 (K. Mahler [2]). *There exists a convergent power series*

$$f(z) = f_0 + f_r z^r \left(1 + \sum_{j=1}^{\infty} \phi_j z^j\right), \quad r \geq 1,\ f_r \neq 0,$$

satisfying the functional equation $P(f(z), f(z^d)) = 0$ *if and only if the following three conditions hold.*

(i) $bd = a$.

(ii) $r \geq 1$ *and for any* $l > b$ *with* $P_l \neq 0$, $(ldr)/a \in \mathbf{Z}$.

(iii) $f_r = P_b^{a/(a-b)}$.

Then $\phi_j \in K'(f_r)$ $(j \geq 1)$.

REMARK. Actually, K. Mahler introduced this theorem not over K' but over $\mathbf{C}$. However, we find that the proof implies the above.

We apply this theorem to the following functional equation with constant coefficients,

$$(2) \qquad f(z^d) = \frac{a_0 + a_1 f(z) + \cdots + a_m f(z)^m}{b_0 + b_1 f(z) + \cdots + b_m f(z)^m}, \quad a_i, b_i \in K,\ a_m \neq 0 \text{ or } b_m \neq 0,$$

where $a_0 + a_1 u + \cdots + a_m u^m$ and $b_0 + b_1 u + \cdots + b_m u^m$ are relatively prime. Let

$$P(u, v) = v(b_0 + b_1 u + \cdots + b_m u^m) - (a_0 + a_1 u + \cdots + a_m u^m).$$

We suppose that $P(u, v)$ is not a product of $u - v$ multiplied by constants. We think of a solution of the form

$$f(z) = f_0 + f_1 z + f_2 z^2 + \cdots.$$

Since f_0 is a root of $P(u, u) \in K[u] \backslash \{0\}$, we find that f_0 is an algebraic number. We may assume $f_0 = 0$ without loss of generality, for we are only interested in the transcendence of values of $f(z)$. Then we have

$$0 = P(f_0, f_0) = P(0, 0) = -a_0,$$

which implies $b_0 \neq 0$. Hence we may additionally assume $b_0 = 1$. Let $s \geq 1$ be the number such that $a_1 = \cdots = a_{s-1} = 0$ and $a_s \neq 0$. Then we find

$$U_0(v) = a_s^{-1/s} v^{1/s} + (\text{terms of higher degrees in } v),$$

which yields $b/a = 1/s$.

We will prove that it is possible to choose $a = s$ and $b = 1$. It is enough to prove

$$P_l \neq 0 \Rightarrow b | l.$$

Assume the contrary, and let $l_0 = nb + k$ $(0 < k < b)$ be the minimum such that $P_{l_0} \neq 0$ and $b \nmid l_0$. By $P(U_0(v), v) = 0$, we obtain

$$\begin{aligned} v(1 &+ b_1(P_b v^{b/a} + \cdots + P_{nb} v^{nb/a} + P_{nb+k} v^{(nb+k)/a} + \cdots) \\ &\quad + b_2(P_b v^{b/a} + \cdots)^2 + \cdots + b_m(P_b v^{b/a} + \cdots)^m) \\ &= a_s(P_b v^{b/a} + \cdots)^s + a_{s+1}(P_b v^{b/a} + \cdots)^{s+1} + \cdots + a_m(P_b v^{b/a} + \cdots)^m. \end{aligned}$$

For the right side, the first term whose exponent of $v^{1/a}$ is not divisible by b is

$$a_s s (P_b v^{b/a})^{s-1} (P_{nb+k} v^{(nb+k)/a}) = a_s s P_b^{s-1} P_{nb+k} v^{((s-1+n)b+k)/a},$$

and for the left side, the corresponding one is

$$v b_1 P_{nb+k} v^{(nb+k)/a} = b_1 P_{nb+k} v^{((s+n)b+k)/a}.$$

Comparing the exponents, we find a contradiction.

Hence the first condition in Theorem 2 is equivalent to $d = a = s$. Under this condition, the second condition holds for any $r \geq 1$, and so if we choose f_r satisfying the third condition, then there exists the convergent power series $f(z) \in K'(f_r)[[z]]$ such that $P(f(z), f(z^d)) = 0$. Thus we obtain a convergent power series

$$f(z) = f_r z^r + \cdots \in K'(f_r)[[z]], \quad r \geq 1, \, f_r \neq 0,$$

satisfying the following functional equation with constant coefficients,

$$f(z^d) = \frac{a_d f(z)^d + \cdots + a_m f(z)^m}{1 + b_1 f(z) + \cdots + b_m f(z)^m}, \tag{3}$$

where $a_i, b_i \in K$, $a_m \neq 0$ or $b_m \neq 0$, $a_d \neq 0$, and $a_d u^d + \cdots + a_m u^m$ and $1 + b_1 u + \cdots + b_m u^m$ are relatively prime.

Although Mahler's Theorem 1 is unsuitable for this $f(z)$ due to $d \leq m$, we have the following.

Theorem 3 (K. Nishioka [4]). *Theorem* 1 *still holds when* $m < d^2$.

Theorem 1 and Theorem 3 both require transcendence of $f(z)$ over $K(z)$. Generally, it is difficult to identify transcendence of functions. However, we can use the following in this situation.

Theorem 4 (S. Nishioka [6]). *Let* $f_1(z), \dots, f_n(z) \in \mathbf{C}((z))$ *satisfy the functional equations*,

$$f_i(z^d) = \frac{A_i(f_i(z))}{B_i(f_i(z))}, \quad i = 1, \dots, n,$$

where $A_i(u), B_i(u) \in \mathbf{C}[u] \backslash \{0\}$ *are relatively prime. If* $f_1(z), \dots, f_n(z)$ *are not constants and* $\max\{\deg A_i(u), \deg B_i(u)\}$ $(i = 1, \dots, n)$ *are distinct, then* $f_1(z), \dots, f_n(z)$ *are algebraically independent over* $\mathbf{C}$.

Since the independent variable z satisfies the functional equation $f(z^d) = f(z)^d$, we obtain the following as a corollary.

Corollary 5. *Let* $f(z) \in \mathbf{C}((z))$ *be a non-constant solution of the functional equation* (2)*. If* $m \neq d$*, then* $f(z)$ *and* z *are algebraically independent over* $\mathbf{C}$*, and so* $f(z)$ *is transcendental over* $\mathbf{C}(z)$.

Considering all the above results, we obtain the following.

THEOREM 6. *Let $d < m < d^2$. There exists a non-constant convergent power series $f(z) \in K''[[z]]$ satisfying the functional equation* (3), *where K'' is a certain finite extension of K. Let $R > 0$ be the radius of convergence. If $\alpha \in \overline{\mathbf{Q}}$ satisfies*

$$0 < |\alpha| < \min\{1, R\},$$

then $f(\alpha)$ is a transcendental number.

REMARK. The condition on the resultant $\Delta(z)$ is not needed, for the equation (3) is with constant coefficients. In this case, $\Delta(z)$ is a non-zero constant.

2 Another Example

In this section, we study the functional equations of the form (2) with $m = d$. Note that their non-constant solutions may be algebraic over $\mathbf{C}(z)$. For example, we look at $f(z) \in \mathbf{C}[[z]] \backslash \mathbf{C}$ satisfying

$$f(z^2) = \frac{f(z)^2}{1 + cf(z)^2}, \quad c \in \mathbf{C}.$$

The series $f(z)$ is related to the Mandelbrot set. It is proved that $f(z)$ is transcendental over $\mathbf{C}(z)$ if $c \neq 0$ and $c \neq -2$ in the lecture note [5] by K. Nishioka. On the other hand, $f(z) = z^r$ if $c = 0$, and $f(z) = (z^r + z^{-r})^{-1}$ if $c = -2$ (see the proof in [5]).

However, we obtain the following general result for similar functional equations.

THEOREM 7. *For $d \geq 3$, a non-constant solution $f(z) \in \mathbf{C}[[z]]$ of the functional equation,*

$$f(z^d) = \frac{f(z)^d}{1 + cf(z)^d}, \quad c \neq 0,$$

is transcendental over $\mathbf{C}(z)$.

PROOF. Assume that $f(z)$ is algebraic over $\mathbf{C}(z)$. We will derive a contradiction. By Theorem 1.3 in [5], we find $f(z) \in \mathbf{C}(z)$ (cf. Keiji Nishioka [3]). Let $g(z) = 1/f(z) \in \mathbf{C}(z)$. Then we obtain the following equation,

$$g(z^d) = g(z)^d + c.$$

Let

$$g(z) = \frac{a(z)}{b(z)},$$

where $a(z), b(z) \in \mathbf{C}[z]$ are relatively prime and $b(z)$ is monic. From the equation,

$$\frac{a(z^d)}{b(z^d)} = \frac{a(z)^d}{b(z)^d} + c,$$

we obtain

$$a(z^d)b(z)^d = (a(z)^d + cb(z)^d)b(z^d).$$

Since $a(z^d)$ and $b(z^d)$ are relatively prime, $b(z^d)$ divides $b(z)^d$. Comparing their degrees, we find $b(z^d) = b(z)^d$, and so $b(z) = z^n$. Hence

$$g(z) = c_1 z^{e_1} + \cdots + c_t z^{e_t}, \quad e_i \in \mathbf{Z},\ e_1 > \cdots > e_t,\ c_1 \cdots c_t \neq 0.$$

From the above equation, we obtain

$$c_1 z^{de_1} + \cdots + c_t z^{de_t} = (c_1 z^{e_1} + \cdots + c_t z^{e_t})^d + c.$$

In the case $t \geq 2$, the right side is

$$(c_1^d z^{de_1} + dc_1^{d-1} c_2 z^{(d-1)e_1 + e_2} + \cdots + dc_{t-1} c_t^{d-1} z^{e_{t-1} + (d-1)e_t} + c_t^d z^{de_t}) + c.$$

In this case, we have

$$(d-1)e_1 + e_2 > (d-2)e_1 + 2e_2 \geq e_1 + (d-1)e_2 \geq e_{t-1} + (d-1)e_t,$$

which implies that $(d-1)e_1 + e_2 \neq 0$ or $e_{t-1} + (d-1)e_t \neq 0$, and so the right side has a term whose exponent is one of them. However, the left side of the above equation does not have such a term, for the following hold,

$$de_1 > (d-1)e_1 + e_2 > de_2$$

and

$$de_{t-1} > e_{t-1} + (d-1)e_t > de_t.$$

Hence we conclude $t = 1$, which yields

$$c_1 z^{de_1} = c_1^d z^{de_1} + c.$$

This contradicts $c \neq 0$. □

By this theorem and Theorem 3, we obtain the following.

Corollary 8. *Let $d \geq 3$. There exists a non-constant convergent power series $f(z) \in K''[[z]]$ satisfying*

$$f(z^d) = \frac{f(z)^d}{1 + cf(z)^d}, \quad c \in K^{\times},$$

where K'' is a certain finite extension of K. Let $R > 0$ be the radius of convergence. If $\alpha \in \overline{\mathbf{Q}}$ satisfies

$$0 < |\alpha| < \min\{1, R\},$$

then $f(\alpha)$ is a transcendental number.

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References

[1] Mahler, K., Arithmetische Eigenschaften der Lösungen einer Klasse von Functionalgleichungen, Math. Ann., **101** (1929), 342–366.
[2] Mahler, K., On the analytic solution of certain functional and difference equations, Proc. Roy. Soc. London Ser. A, **389** (1983), 1–13.
[3] Nishioka, Ke., Algebraic function solutions of a certain class of functional equations, Arch. Math., **44** (1985), 330–335.
[4] Nishioka, Ku., On a problem of Mahler for transcendency of function values, J. Austral. Math. Soc. Ser. A, **33** (1982), 386–393.
[5] Nishioka, Ku., Mahler functions and transcendence, LNM Vol. 1631, Springer-Verlag, 1996.
[6] Nishioka, S., Algebraic independence of solutions of first-order rational difference equations, Results Math., **64** (2013), 423–433. doi:10.1007/s00025-013-0324-8

Kumiko Nishioka
Department of Mathematics, Hiyoshi Campus
Keio University
4-1-1 Hiyoshi, Kohoku-ku, Yokohama
223-8521, Japan

Seiji Nishioka
Faculty of Science, Yamagata University
Kojirakawa-machi 1-4-12, Yamagata-shi
990-8560, Japan
E-mail: nishioka@sci.kj.yamagata-u.ac.jp