Leinster, Tom; **Shulman, Michael**
Magnitude homology of enriched categories and metric spaces. (English) Zbl 07432508
Algebr. Geom. Topol. 21, No. 5, 2175-2221 (2021)

The first author [Doc. Math. 13, 21–49 (2008; Zbl 1139.18009); *C. Berger* and *T. Leinster*, Homology Homotopy Appl. 10, No. 1, 41–51 (2008; Zbl 1132.18007); Doc. Math. 18, 857–905 (2013; Zbl 1284.51011); *N. Gigli*, Measure theory in non-smooth spaces. Warsaw: De Gruyter Open (2017; Zbl 1388.28001), pp.156–193] introduced the notion of *magnitude* as a numerical invariant of enriched categories, particularly including metric spaces as $[0,\infty]$-enriched categories. The principal objective in this paper is to generalize this homology theory to a large class of enriching category $\boldsymbol{V}$, and in particular to arbitrary metric spaces. Specifically, when $\boldsymbol{V}$ is a *semicartesian* monoidal category (i.e. the monoidal unit is the terminal object) and $\Sigma : \boldsymbol{V} \to \boldsymbol{A}$ is a strong symmetric monoidal functor to an abelian category, the *magnitude homology* $H_*^{\Sigma}(X)$ of any $\boldsymbol{V}$-category X is defined, turning out to be no other than the *Hochschild homology* of X with constant coefficients. It is shown that given any rank function rk: $\mathrm{ob}(\boldsymbol{A}) \to k$ with k the multiplicative monoid of a (semi)ring, the composite $\mathrm{rk} \circ \Sigma$ is a size $\# : \mathrm{ob}(\boldsymbol{A}) \to k$ and any sufficiently finite $\boldsymbol{V}$-category X is of a magnitude computable as the Euler characteristic of $H_*^{\Sigma}(X)$.

If it is unwound in the case of metric spaces, a calculable algebraic invariant with the help of $\mathbb{R}$-graded chain complexes is obtained [*Y. Asao* and *K. Izumihara*, Homology Homotopy Appl. 23, No. 1, 297–310 (2020; Zbl 1472.55005); *Y. Asao*, Algebr. Geom. Topol. 21, No. 2, 647–664 (2021; Zbl 07362536); *K. Gomi*, Forum Math. 32, No. 3, 625–639 (2020; Zbl 1472.55006); *B. Jubin*, "On the magnitude homology of metric spaces", Preprint, `arXiv:1803.05062`; *R. Kaneta* and *M. Yoshinaga*, Bull. Lond. Math. Soc. 53, No. 3, 893–905 (2021; Zbl 1472.55007)]. It is shown that $H_1^{\Sigma}(X) = 0$ iff X is Menger convex (i.e., for any two distinct points, there is another point strictly between them), which implies in particular that a closed subset $X \subseteq \mathbb{R}^n$ abides by $H_1^{\Sigma}(X) = 0$ iff it is convex in the usual sense.

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

18G90 Other (co)homology theories (category-theoretic aspects)
16E40 (Co)homology of rings and associative algebras (e.g., Hochschild, cyclic, dihedral, etc.)
51F99 Metric geometry
55N31 Persistent homology and applications, topological data analysis

Keywords:

magnitude; magnitude homology; Euler characteristic; enriched category; metric space; categorification; Hochschild homology

Full Text: DOI