Bruce, Andrew James; **Ibarguëngoytia, Eduardo**; **Poncin, Norbert**
Linear $\mathbb{Z}_2^n$-manifolds and linear actions. (English) Zbl 07425535
SIGMA, Symmetry Integrability Geom. Methods Appl. 17, Paper 060, 58 p. (2021)

$\mathbb{Z}_2^n$-manifolds are manifolds for which the structure sheaf has a $\mathbb{Z}_2^n$-grading and the commutation rules for the local coordinates comes from the standard scalar product [*A. J. Bruce* and *J. Grabowski*, "Riemannian structures on $\mathbb{Z}_2^n$-manifolds", Preprint, arXiv:2007.07666; *A. J. Bruce* et al., SIGMA, Symmetry Integrability Geom. Methods Appl. 16, Paper 002, 47 p. (2020; Zbl 1433.58011); *A. J. Bruce* and *N. Poncin*, Rev. Unión Mat. Argent. 60, No. 2, 611–636 (2019; Zbl 1431.58002); J. Nonlinear Math. Phys. 26, No. 3, 420–453 (2019; Zbl 1417.46034); *T. Covolo* et al., J. Math. Phys. 57, No. 7, 073503, 16 p. (2016; Zbl 1345.58003); J. Geom. Phys. 110, 393–401 (2016; Zbl 1404.58011); "Differential calculus on $\mathbb{Z}_2^n$-supermanifolds", Preprint, arXiv:1608.00949; "The Frobenius theorem for $\mathbb{Z}_2^n$-supermanifolds", Preprint, arXiv:1608.00961; *N. Poncin*, Banach Cent. Publ. 110, 201–217 (2016; Zbl 1416.58002)]. While for standard supermanifolds all functions are polynomials in the Grassmann odd variables, for $\mathbb{Z}_2^n$-geometry, due to the presence of formal variables that are not nilpotent, formal power series are indispensible. This paper shows the the category of finite-dimensional $\mathbb{Z}_2^n$-graded vector spaces and that of linear $\mathbb{Z}_2^n$-manifolds are isomorphic, which is exploited to define linear actions of $\mathbb{Z}_2^n$-Lie groups. The authors explicitly construct a *manifoldification* functor $\mathcal{M}$ associating a linear $\mathbb{Z}_2^n$-manifold to every finite-dimensional $\mathbb{Z}_2^n$-graded vector space as well as a *vectorification* function as its inverse.

The authors introduce an analogue of the *even rules* including extra odd parameters to render everything even [*P. Deligne* and *J. W. Morgan*, in: Quantum fields and strings: a course for mathematicians. Vols. 1, 2. Material from the Special Year on Quantum Field Theory held at the Institute for Advanced Study, Princeton, NJ, 1996–1997. Providence, RI: AMS, American Mathematical Society. 41–97 (1999; Zbl 1170.58302), §1.7] in this higher graded setting as the *zero degree rules*, making extensive use of $\mathbb{Z}_2^n$-Grassmann algebras Λ, Λ-points (i.e., functors of points from appropriate locally small categories C to a functor category whose source is the category G of $\mathbb{Z}_2^n$-Grassmann algebras Λ) and the Schwarz-Voronov embedding [*A. S. Schwarz*, Commun. Math. Phys. 87, 37–63 (1982; Zbl 0503.53048); Theoret. and Math. Phys. 60, 657–660 (1984); Theoret. and Math. Phys. 60, 660–664 (1984)], which is a fully faithful functor of points $\mathcal{S}$ from $\mathbb{Z}_2^n$-manifolds to a functor category with source G and the category of Fréchet manifolds [*R. S. Hamilton*, Bull. Am. Math. Soc., New Ser. 7, 65–222 (1982; Zbl 0499.58003)] over commutative Fréchet algebras as target [*A. J. Bruce* et al., SIGMA, Symmetry Integrability Geom. Methods Appl. 16, Paper 002, 47 p. (2020; Zbl 1433.58011)]. It is shown that the zero degree rules functor $\mathcal{F}$ understood as an assignment of a functor from G to the category of modules over commutative Fréchet algebras, given a finite-dimensional $\mathbb{Z}_2^n$-graded vector space, is fully faithful (Theorem 2.2 and Proposition 2.25). The "zero degree rules" allow of identifying a finite-dimensional $\mathbb{Z}_2^n$-graded vector space, considered as a functor, with the functor of points of its "manifoldification", which means that the functors $\mathcal{S} \circ \mathcal{M}$ and $\mathcal{F}$ are naturally isomorphic, being fundamental when describing linear group actions on $\mathbb{Z}_2^n$-graded vector spaces and linear $\mathbb{Z}_2^n$-manifolds.

The paper is also concerned with the category of $\mathbb{Z}_2^n$-Lie groups and its fully faithful functor of points valued in a functor category with G as source category and Fréchet Lie groups over commutative Fréchet algebras as target category. The general linear $\mathbb{Z}_2^n$-group being defined as a functor in this functor category, it is shown (Theorem 3.4) that it is representable, i.e., is a genuine $\mathbb{Z}_2^n$-manifold, leading to interesting insights into the computation of the inverse of an invertible degree zero $\mathbb{Z}_2^n$-graded square matrix of dimension $p\,|\underline{q}$ with entries in a $\mathbb{Z}_2^n$–commutative algebra. Besides, the approach making use of Λ-points and the zero rules allows of constructing a canonical smooth linear action of the general linear $\mathbb{Z}_2^n$-group on $\mathbb{Z}_2^n$-graded vector spaces and $\mathbb{Z}_2^n$-manifolds.

Although many of the statements in this paper are by no means surprising in themselves, their technical expositions are somewhat fastidious, mainly because one has to deal with formal power series in non-zero degree coordinates in $\mathbb{Z}_2^n$-geometry in place of polynomials in supergeometry, which coerces one into working with infinite-dimensional objects and the $\mathcal{J}$-adic topology with the ideal $\mathcal{J}$ generated by non-zero degree elements.

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

58A50 Supermanifolds and graded manifolds
58C50 Analysis on supermanifolds or graded manifolds
14A22 Noncommutative algebraic geometry
14L30 Group actions on varieties or schemes (quotients)
13F25 Formal power series rings
16L30 Noncommutative local and semilocal rings, perfect rings
17A70 Superalgebras

Keywords:

supergeometry; ringed spaces; functors of points; linear group actions

Full Text: DOI

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paper as accurately as possible without claiming the completeness or perfect precision of the matching.