Saldanha, Nicolau; Shapiro, Boris; Shapiro, Michael
Grassmann convexity and multiplicative Sturm theory. Revisited. (English) Zbl 07406858
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A linear ordinary homogeneous differential equation

$$y^{(n)}+p_1(x)y^{(n-1)}+\cdots+p_n(x)y=0$$

of order n with real-valued continuous coefficients $p_i(x)$ defined on interval $I\subseteq\mathbb{R}$ is called *disconjugate on* I provided that its nontrivial solutions has at most $n-1$ zeros on I counting multiplicities, where the interval I is open or closed. The second author of this paper proposed the following conjecture in the previous century [*B. Z. Shapiro*, Izv. Akad. Nauk SSSR, Ser. Mat. 54, No. 1, 173–187 (1990; Zbl 0702.34032)].

Conjecture. (Upper bound on the number of real zeros of a Wronskian) Given any differential equation of the above form disconjugate on I, a positive integer $1\le k\le n-1$, and an arbitrary k-tuple $(y_1(x),y_2(x),\dots y_k(x))$ of its linearly independent solutions, the number of real zeros of

$$\det(W(y_1(x),y_2(x),\dots y_k(x)))=\det\begin{pmatrix} y_1(x) & y_1'(x) & \dots & y_1^{(k-1)}(x)\\ y_2(x) & y_2'(x) & \dots & y_2^{(k-1)}(x)\\ \dots & \dots & \dots & \dots\\ y_k(x) & y_k'(x) & & y_k^{(k-1)}(x)\end{pmatrix}$$

on I counting multiplicities does not exceed $k(n-k)$.

The cases of $k=1$ and $k=n-1$ in the above conjecture are straightforward. The simplest non-trivial case that $k=2$ and $n=4$ has been settled in [*B. Shapiro* and *M. Shapiro*, Int. J. Math. 11, No. 4, 579–588 (2000; Zbl 1110.53301)]. The first main result in this paper is

Theorem. The conjecture obtains for $k=2$ and $k=n-2$ for any $n\ge 3$.

The above conjecture has a reformulation. Let Lo_n^1 be the nilpotent Lie group of lower triangular $n\times n$ matrices whose diagonal entries equal 1. Let $\mathcal{J}$ be the set of $n\times n$ real matrices A such that the entry A_{ij} is positive if $i=j+1$ and 0 otherwise. A smooth curve

$$\Gamma:[0,1]\to\mathrm{Lo}_n^1$$

is called *flag-convex* providing that

$$(\Gamma(t))^{-1}\Gamma'(t)\in\mathcal{J}$$

for all $t\in[0,1]$. Given a flag-convex curve Γ and an integer k with $0<k<n$, we define $m_k:[0,1]\to\mathbb{R}$ by

$$m_k(t)=\det(\mathrm{swminor}(\Gamma(t),k))$$

for any $t\in[0,1]$, where $\mathrm{swminor}(L,k)$ is the $k\times k$ submatrix of L formed by its last k rows and its first k columns. The above conjecture is now equivalent to

Conjecture. The number of zeros, counting multiplicies, of $t\in[0,1]$ of the smooth function $m_k:[0,1]\to\mathbb{R}$ is at most $k(n-k)$.

The above theorem is of the following reformulation.

Theorem. For any flag-convex function $\Gamma:[0,1]\to\mathrm{Lo}_n^1$, the functions m_2 and m_{n-2} have at most $2(n-2)$ zeros each.

The second main result in this paper goes as follows.

Theorem. Consider a smooth flag-convex curve $\Gamma.:I\to\mathrm{Lo}_n^1$ with a non-degenerate interval I. Then, for any open subinterval $I_1\subset I$, there exists a matrix $L_1\in\mathrm{Lo}_n^1$ such that, for $\Gamma_1(t)=L_1\Gamma.(t)$ and $m_k=m_{\Gamma_1,k}$, the following properties obtain:

(1) all roots of each m_k in I are simple and belong to I_1;

(2) if $k_1 \neq k_2$, then m_{k_1} and m_{k_2} have no common roots;

(3) for each k, the function m_k admits precisely $k(n-k)$ roots in I.

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MSC:

51M35 Synthetic treatment of fundamental manifolds in projective geometries (Grassmannians, Veronesians and their generalizations)
34B05 Linear boundary value problems for ordinary differential equations
52A55 Spherical and hyperbolic convexity

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