Rogers, Morgan

On supercompactly and compactly generated toposes. (English) Zbl 07420164

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This paper presents and thoroughly invesigates the class of supercompactly generated toposes as well as that of compactly generated toposes. The former class includes all regular toposes as well as all presheaf toposes. The author is motivated by the fact (Theorem 1.8.5) that any topos admitting a hyperconnected morphism from a supercompactly generated Grothendieck topos is always such a topos. This notably includes any topos of continuous actions of a topological monoid on sets [*M. Rogers*, "Toposes of topological monoid actions", Preprint, `arXiv:2105.00772`]. The author's developments of relevant classes of geometric morphisms stems from proper and relative proper geometric morphisms [*I. Moerdijk* and *J. J. C. Vermeulen*, Proper maps of toposes. Providence, RI: American Mathematical Society (AMS) (2000; Zbl 0961.18003), Chapters I and V]. It is shown how these notions of compactness for geometric morphisms interact with the notions for objects inside a topos. The author relies on [*O. Caramello*, Theories, sites, toposes. Relating and studying mathematical theories through topos-theoretic 'bridges'. Oxford: Oxford University Press (2018; Zbl 1461.03002)] for site-theoretical issues.

This paper consists of two sections, the first section being divided into 8 subsections while the second section being divided into 9 subsections. A synopsis of the paper goes as follows.

§1.1 recalls the definitions of supercompact and compact objects in a topos, leading to the notions of *supercompactly and compactly generated toposes* in §1.2. §1.3 turns to the full subcategories of supercompact and compact objects in a topos with a focus on monomorphisms, epimorphisms and the classes of *funneling* and *multifunneling* colimits (Definition 1.3.3 and Definition 1.3.6). §1.4 aims to present these subcategories as canonical sites for supercompactly and compactly generated toposes (Theorem 1.4.3). §1.5 investigates some extra conditions on a supercompactly generated topos guaranteeing further properties of its category of supercompact objects. §1.6 examines some classes of geometric morphisms whose inverse image functors preserve supercompact or compact objects, introducing the notions of *pristine* and *polished* geometric morphisms in analogy with proper geometric morphisms (Definition 1.6.2) and then focusing on relative versions of these properties (Definition 1.6.5). The remaining two sections investigate how some more familiar classes of geometric morphisms interact with supercompactly and compactly generated toposes, namely, surjections and embeddings in §1.7 as well as hyperconnected morphisms in §1.8.

§2 is engaged in a broader site-theoretic investigation. §2.1 exhibits the categorical data of *principal* and *finitely generated* sites forming the categories of sheaves as supercompactly and compactly generated toposes respectively. §2.2 investigates morphisms between representable sheaves, leading to how a general such site may be reduced via a canonical congruence without the resulting topos in §2.3. §2.4 characterizes the categories of supercompact and compact objects as *reductive* and *coalescent* categories respectively (Definition 2.4.2), abiding by additional technical conditions. Theorem 2.4.12 claims the correspondence between such categories and the toposes they generate, leading to a brief examination of the special cases producing localic toposes in §2.5. §2.6 compares reductive and coalescent categories with the familiar classes of (locally) regular and coherent categories. §2.7 recalls the definition of *morphisms of sites*, showing that, according to the class of sites under consideration, they induce the relatively pristine, polished or proper geometric morophisms. §2.8 investigates comorphisms of sites, providing some results about points of various classes of toposes. §2.9 presents some examples and counterexamples of reductive and coalescent sites as well as supercompactly and compactly generated toposes.

Reviewer: Hirokazu Nishimura (Tsukuba)

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18B25 Topoi

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[20] Università degli Studi dell'Insubria, Via Valleggio n. 11, 22100 Como CO Email: mrogers@uninsubria.it

[21] Gabriella Böhm, Wigner Research Centre for Physics: bohm.gabriella (at) wigner.mta.hu Maria Manuel Clementino, Universidade de Coimbra: mmc.mat.uc.pt Valeria de Paiva, Nuance Communications Inc: valeria.depaiva@gmail.com Richard Garner, Macquarie University: richard.garner@mq.edu.au Ezra Getzler, Northwestern University: getzler (at) northwestern(dot)edu

[22] Dirk Hofmann, Universidade de Aveiro: dirk@ua.pt Pieter Hofstra, Université d' Ottawa: phofstra (at) uottawa.ca Anders Kock, University of Aarhus: kock@math.au.dk Joachim Kock, Universitat Autònoma de Barcelona: kock (at) mat.uab.cat Stephen Lack, Macquarie University: steve.lack@mq.edu.au Tom Leinster, University of Edinburgh: Tom.Leinster@ed.ac.uk Matias Menni, Conicet and Universidad Nacional de La Plata, Argentina: matias.menni@gmail.com Ieke Moerdijk, Utrecht University: i.moerdijk@uu.nl Susan Niefield, Union College: niefiels@union.edu

[23] Kate Ponto, University of Kentucky: kate.ponto (at) uky.edu Robert Rosebrugh, Mount Allison University: rrosebrugh@mta.ca Jiří Rosický, Masaryk University: rosicky@math.muni.cz Giuseppe Rosolini, Università di Genova: rosolini@disi.unige.it Michael Shulman, University of San Diego: shulman@sandiego.edu Alex Simpson, University of Ljubljana: Alex.Simpson@fmf.uni-lj.si James Stasheff, University of North Carolina: jds@math.upenn.edu

[24] Tim Van der Linden, Université catholique de Louvain: tim.vanderlinden@uclouvain.be