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Integrals along bimonoid homomorphisms. (English) Zbl 07382618

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The notion of an integral of a bialgebra was introduced in [R. G. Larson and M. E. Sweedler, *Am. J. Math.* 91, 75–94 (1969; [Zbl 0179.05803](#))] as a generalization of the Haar measure of a group, which has been used in the study of bialgebras or Hopf algebras [M. E. Sweedler, *Ann. Math.* (2) 89, 323–335 (1969; [Zbl 0174.06903](#)); D. E. Radford, *Bull. Am. Math. Soc.* 81, 1103–1105 (1975; [Zbl 0326.16008](#)); *Am. J. Math.* 98, 333–355 (1976; [Zbl 0332.16007](#))]. The notion of a bialgebra was generalized to bimonoids in a symmetric monoidal category \mathcal{C} [M. Aguiar and S. Mahajan, *Monoidal functors, species and Hopf algebras*. Providence, RI: American Mathematical Society (AMS) (2010; [Zbl 1209.18002](#)); S. Mac Lane, *Categories for the working mathematician*. New York, NY: Springer (1998; [Zbl 0906.18001](#))], while the integral theory was generalized to the categorical setting to study bimonoids or Hopf monoids [Y. Bespalov et al., *J. Pure Appl. Algebra* 148, No. 2, 113–164 (2000; [Zbl 0961.16023](#))].

This paper introduces a notion of an integral along a bimonoid homomorphism in a symmetric monoidal category \mathcal{C} , generalizing the notions of integral and cointegral of a bimonoid. The principal objective in this paper is to establish the following theorem.

Theorem. Let A and B be bicommutative Hopf algebras in \mathcal{C} with $\xi : A \rightarrow B$ a Hopf homomorphism. Then there exists a normalized generator integral μ_ξ along ξ iff the following conditions are satisfied:

- [1] The kernel Hopf monoid $\text{Ker}(\xi)$ has a normalized integral.
- [2] The cokernel Hopf monoid $\text{Cok}(\xi)$ has a normalized cointegral.

Moreover, if a normalized integral exists, then it is unique.

Some applications are as follows.

- The author investigates the category $\text{Hopf}^{\text{bc},*}(\mathcal{C})$ of bicommutative Hopf monoids with a normalized integral and cointegral, establishing that the category $\text{Hopf}^{\text{bc},*}(\mathcal{C})$ is an abelian subcategory of $\text{Hopf}^{\text{bc}}(\mathcal{C})$ and closed under short exact sequents.
- The author introduces the notion of volume on an abelian category as a generalization of the dimension of vector spaces and the order of abelian groups, studying basic notions related with it.
- The author constructs an $\text{End}_{\mathcal{C}}(\mathbf{1})$ -valued volume vol^{-1} on the abelian category $\text{Hopf}^{\text{bc},*}(\mathcal{C})$, where $\mathbf{1}$ is the unit object of \mathcal{C} and the endomorphism set $\text{End}_{\mathcal{C}}(\mathbf{1})$ is an abelian monoid induced by the symmetric monoidal structure of \mathcal{C} .
- By using the volume vol^{-1} , the author introduces a notion of Fredholm homomorphisms between bicommutative Hopf monoids as an analogue of Fredholm operators [T. Kato, *Perturbation theory for linear operators*. Berlin: Springer-Verlag (1995; [Zbl 0836.47009](#))], investigating its index which is robust to some perturbations. A functorial assignment of integrals to Fredholm homomorphisms is constructed.
- The author expects that the result in this paper could be applied to topology. There is a topological invariant of 3-manifolds induced by a finite-dimensional Hopf algebra, called the Kuperberg invariant [G. Kuperberg, *Int. J. Math.* 2, No. 1, 41–66 (1991; [Zbl 0726.57016](#)); “Non-involutory Hopf algebras and 3-manifold invariants”, Preprint, [arXiv:q-alg/9712047](#)].

This paper gives a technical preliminary to the author’s subsequent paper [M. Kim, “A pair of homotopy-theoretic version of TQFT’s induced by a Brown functor”, Preprint, [arXiv:2006.10438](#)], which uses the results in this paper to give a generalization of the untwisted abelian Dijkgraaf-Witten theory [R. Dijkgraaf and E. Witten, *Commun. Math. Phys.* 129, No. 2, 393–429 (1990; [Zbl 0703.58011](#)); M. Wakui, *Osaka J. Math.* 29, No. 4, 675–696 (1992; [Zbl 0786.57008](#)); D. S. Freed and F. Quinn, *Commun. Math. Phys.* 156, No. 3, 435–472 (1993; [Zbl 0788.58013](#))] and the bicommutative Turaev-Viro TQFT [V. G. Turaev and O. Y. Viro, *Topology* 31, No. 4, 865–902 (1992; [Zbl 0779.57009](#)); J. W. Barrett and B. W. Westbury, *Trans. Am. Math. Soc.* 348, No. 10, 3997–4022 (1996; [Zbl 0865.57013](#))], giving a systematic way to construct a sequence of TQFT’s from (co)homology theory. The TQFT’s are constructed by using

path-integral which is formulated by some integral along bimonoid homomorphisms.

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MSC:

- [18M05](#) Monoidal categories, symmetric monoidal categories
- [16T10](#) Bialgebras
- [81T45](#) Topological field theories in quantum mechanics

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