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Homology of categories via polygraphic resolutions. (English) Zbl 07357173

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R. Street [J. Pure Appl. Algebra 49, 283–335 (1987; Zbl 0661.18005)] defined a nerve functor

$$N_\omega : \omega\boldsymbol{Cat} \to \widehat{\Delta}$$

from the category of strict ω-categories (called simply *ω-categories*) to the category of simplicial sets, which can be used to transfer the homotopy theory of simplicial sets to ω-categories [*D. Ara* and *G. Maltsiniotis*, Adv. Math. 259, 557–654 (2014; Zbl 1308.18004); Adv. Math. 328, 446–500 (2018; Zbl 1390.18011); High. Struct. 4, No. 1, 284–388 (2020; Zbl 07173321); *A. Gagna*, Adv. Math. 331, 542–564 (2018; Zbl 1395.18008); J. Lond. Math. Soc., II. Ser. 100, No. 2, 470–497 (2019; Zbl 1430.18021); *P. Ara* et al., Banach J. Math. Anal. 14, No. 4, 1692–1710 (2020; Zbl 1454.46052); *D. Ara* and *G. Maltsiniotis*, Mém. Soc. Math. Fr., Nouv. Sér. 165, 1–203 (2020; Zbl 07362646)]. In particular, we have the following definition.

Definition. Let C be an ω-category and $k \in \mathbb{N}$. The k-th homology group $H_k(C)$ of C is the k-th homology group of its nerve $N_\omega(C)$.

On the other hand, *F. Métayer* [Theory Appl. Categ. 11, 148–184 (2003; Zbl 1020.18001)] deined *polygraphic homology groups*, observing that

(a) Every ω-category admits a polygraphic resolution that is an arrow

$$u : P \to C$$

of $\omega\boldsymbol{Cat}$, such that P is a free ω-category and u abides by some properties of formal similarities with trivial fibrations of topological spaces.

(b) Every free ω-category P is to be linearized to a chain complex $\lambda(P)$.

(c) Given two polygraphic resolutions $P \to C$ and $P' \to C$ of the same free ω-category, the homology groups of the chain complexes $\lambda(P)$ and $\lambda(P')$ coincide.

Definition. Let C be an ω-category and $k \in \mathbb{N}$. The k-th polygraphic homology group $H_k^{\mathrm{pol}}(C)$ of C is the k-th homology group of $\lambda(P)$ for any polygraphic resolution $P \to C$.

The principal objective in this paper is to establish the following theorem.

Theorem. Let C be an ω-category. For every $k \in \mathbb{N}$, we have

$$H_k(C) \simeq H_k^{\mathrm{pol}}(C).$$

The restriction of the above theorem to the case of monoids is precisely Corollary 3 of [*Y. Lafont* and *F. Métayer*, J. Pure Appl. Algebra 213, No. 6, 947–968 (2009; Zbl 1169.18002), §3.4], but the author's novelty lies in his more conceptual proof than theirs. Besides, the actural result in this paper (Theorem 8.3) is more precise than the above theorem in that

(a) The homology of an ω-category, whether polygraphic or of the nerve, is considered as a chain complex up to quasi-isomorphism, but not only a sequence of abelian groups.

(b) It is established that the polygraphic homology and homology of the nerve of a small category are naturally isomorphic with the natural isomorphism explicitly constructed.

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18N30 Strict omega-categories, computads, polygraphs
18G90 Other (co)homology theories (category-theoretic aspects)

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