Land, Markus
Introduction to infinity-categories. (English) Zbl 07332844
Compact Textbooks in Mathematics. Cham: Birkhäuser/Springer (ISBN 978-3-030-61523-9/pbk; 978-3-030-61524-6/ebook). ix, 296 p. (2021)

This book grew out of lecture notes for the courses "Introduction to ∞-categories" and "∞-categories" that the author taught at the University of Regensburg in the winter term 2018/2019 and the summer term 2019, respectively. The book is well-written except for lack of indices.

A synopsis of the book consisting of five chapters goes as follows.

Chapter 1 consists of 4 sections. §1.1 deals with simplicial sets, establishing the following theorem.

Theorem (Theorem 1.1.52). For a simplicial set X, the following three conditions are equivalent.

(1) X has unique extensions for $\Lambda^n_j \to \Delta^n$ if $0 < j < n$.

(2) X has unique extensions for $I^n \to \Delta^n$ for $n \geq 2$.

(3) X is isomorphic to the nerve of a category.

§1.2 gives two different definitions of ∞-categories by taking Theorem 1.1.52 as a motivation. discussing that Joyal's definition (Definition 1.2.15 taking (1) in the above theorem into account) behaves better than that of a composer (Definition 1.2.1 taking (2) in the above theorem into account). §1.3 aims to set up some combinatorial notions allowing of establishing that ∞-groupoids are in fact Kan complexes. The main result in the section is

Theorem (Theorem 1.3.37). Let $f : X \to Y$ be a (inner, left, right) fibration and let $i : A \to B$ a monomorphism. Then we have

(1) the map $< f, i >$ is a (inner, left, right) fibration.

(2) If additionally, i is (inner, left, right) anodyne, then the map $< f, i >$ is a trivial fibration.

§1.4 addresses a construction in simplicial sets mimicking the notion of joins and slices in ordinary category theory.

Chapter 2 consists of five sections. §2.1 begins with a conservative functor detecting whether a given morphism is an equivalence. Then the author demonstrates Joyal's speical horn lifting theorem (Theorem 2.1.8) claiming that conservative inner fibrations are characterized by a lifting property with respect to special horns where a particular edge is sent to an equivalence, leading to one of the central results of the book claiming that ∞-groupoids are Kan complexes (Corollary 2.1.12). §2.2 aims to establish that a natural transformation τ between functors

$$f, g : \mathcal{C} \to \mathcal{D}$$

of ∞-categories which is pointwise an equivalence is itself an equivalence viewed as a morhphism in the ∞-category of functors $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$. §2.3 aims at demonstrating that functors which are essentially surjective and fully faithful are in fact invertible (Theorem 2.3.20). §2.4 studies a further construction of ∞-categories by universally inverting a chosen set of morphisms into a given ∞-category (Dwyer-Kan localizations). The following result is presented without a proof, where a similar result holds in general for simplicial model categories [https://people.math.harvard.edu/~lurie/papers/HA.pdf, Theorem 1.3.4.20].

Theorem (Theorem 2.4.10). The canonical functors

$$\mathrm{sSet}\left[\mathrm{we}^{-1}\right] \to \mathrm{Spc}$$

and

$$\mathrm{sSet}\left[\mathrm{Joy}^{-1}\right] \to \mathrm{Cat}_\infty$$

are equivalences of ∞-categories.

§2.5 constructs an alternative join, showing that it is Joyal-equivalent to the construction in §1.4. The two main results go as follows, where the latter theorem comparing the mapping spaces of the coherent nerve of a Kan-enriched category with the mapping Kan complexes that prevail in the simplicial category is presented without a proof, while its proof can be seen in [*J. Lurie*, Higher topos theory. Princeton, NJ: Princeton University Press (2009; Zbl 1175.18001), §2.2.2] and its direct combinatorial proof was recently given in [*F. Hebestreit* and *A. Krause*, "Mapping spaces in homotopy coherent nerves", Preprint, arXiv:2011.09345].

Theorem (Theorem 2.5.14). Let $p : \mathcal{C} \to \mathcal{D}$ be an isofibration between ∞-categories and let $A \to B$ be a monomorphism which is in addition a Joyal equivalence. Then the lifting problem

$$\begin{array}{ccc} A & \to & \mathcal{C} \\ \downarrow & & \downarrow p \\ B & \to & \mathcal{D} \end{array}$$

admits a solution (i.e., an arrow from B to $\mathcal{C}$ making the whole diagram commutative).

Theorem (Theorem 2.5.35). Let $\mathcal{C}$ be a Kan-enriched category and let x and y be objects of $\mathcal{C}$. Then there is a canonical map

$$\mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{map}_{\mathrm{N}(\mathcal{C})}(x, y)$$

which is a homotopy equivalence. The homotopy class of this map is natural in x and y.

Chapter 3 consists of three sections. §3.1 defines and discusses the notion of (co)cartesian fibrations and in particular of p-(co)cartesian morphisms of X, if $p : X \to Y$ is an inner fibration between simplicial sets. The notion of a morphism of (co)cartesian fibrations over a fixed base ∞-category is introduced and it is shown that among such, equivalence can be detected fiberwise. §3.2 asks whether (co)cartesian fibrations are characterized by a lifting property, answering that it is the case in the context of marked simplicial sets, though this being not true on the nose. Marked simplicial sets are related to the previous construction of Dwyer-Kan localizations. The following theorem is presented without a proof.

Theorem (Theorem 3.2.21). There exists a simplicial model structure on marked simplicial sets whose cofibrations are monomorphisms, whose equivalences are marked equivalences, and where fibrant objects are precisely ∞-categories whose marked edges are all equivalences.

The section concludes with the following results describing certain Dwyer-Kan localizations of 1-categories to the coherent nerve of related Kan-enriched categories.

Theorem (Theorem 3.2.25). Let M be a simplicial model category. Then the canonical functor

$$\mathrm{N}(M^c)\left[W^{-1}\right] \to \mathrm{N}(M^\circ)$$

is an equivalence of ∞-categories, where M^c denotes the category of cofibrant objects, M° denotes the category of cofibrant-fibrant objects and W denotes the collection of weak equivalences.

§3.3 formulates and discusses the Grothendieck construction. The main theorem is Lurie's straightening-unstraightening equivalence (Theorem 3.3.16), comparing the ∞-category of ∞-categories with an ∞-category of cocartesian fibrations over $\mathcal{C}$.

Chapter 4 consists of four sections. §4.1 discusses terminal and initial objects as a warm-up for the later notion of limits and colimits in ∞-categories, giving several standard equivalent characterizations for an object to be terminal or initial, respectively. §4.2 establishes the ∞-categorical version of the Yoneda lemma (Proposition 4.2.10). Using the straightening-unstraightening equivalence, a bivariant mapping-space functor

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Spc}$$

is constructed by means of twisted arrow category after Lurie's construction [https://people.math.harvard.edu/~lurie/papers/HA.pdf]. A proof of the Yoneda lemma in the language of left fibrations is sketched. It is shown (Proposition 4.2.11) how one can deduce from this lemma that the functor (called the *Yoneda embedding*)

$$\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Spc})$$

associated with the bivariant mapping-space functor is fully faithful. §4.3 is concerned with limits and colimits, establishing that (co)limits, if they exist, are unique up to a contractible space of choices, It is shown that an ∞-category admits small (co)limits iff it admits small (co)products and pullbacks (pushouts). It is shown also (Theorem 4.3.37) that the ∞-category Spc of spaces and Cat_∞ of ∞-categories are of all small limits and colimits. §4.4 addresses the notion of cofinal and coinitial functors, discussing the notion of smooth and proper functors. It is demonstrated that left fibrations are smooth, while right fibrations are proper. A simple proof of the ∞-categorical version of Quillen's Theorem A claiming that cofinal functors can be detected by their behavior on slice categories is given (Theorem 4.4.20).

Chapter 5 consists of two sections. §5.1 addresses adjunctions between ∞-categories, defining them in the language of fibrations and showing that they may equivalently be described by a choice of a binatural transformation of bivariant mapping-space functors. Several sufficient criteria for a fixed functor $f : \mathcal{C} \to \mathcal{D}$ to admit an adjoint are given. It is demonstrated that (co)limits are to be functorially formed by showing that their formation assembles into an adjoint of the diagonal or constant funcor. The author discusses the notion of Bousfield localizations, which are those Dwyer-Kan localizations admitting a right adjoint, or equivalently, those functors admitting a fully faithful right adjoint. §5.2 establishes two adjoint functor theorems (Theorem 5.2.2, Theorem 5.2.14) after [*H. K. Nguyen* et al., J. Lond. Math. Soc., II. Ser. 101, No. 2, 659–681 (2020; Zbl 1448.18004)] leading to the adjoint functor theorem of Lurie [*J. Lurie*, Higher topos theory. Princeton, NJ: Princeton University Press (2009; Zbl 1175.18001)].

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MSC:

18-01 Introductory exposition (textbooks, tutorial papers, etc.) pertaining to category theory
18Nxx Higher categories and homotopical algebra

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