Caramello, Olivia; Marra, Vincenzo; Spada, Luca
General affine adjunctions, Nullstellensätze, and dualities. (English) Zbl 1469.18001
J. Pure Appl. Algebra 225, No. 1, Article ID 106470, 32 p. (2021).

Let us recall the *classical affine connection*. Let k be an algebraically closed field. As is well known, there is a contravariant Galois connection

$$\mathbb{C} : 2^{k^n} \to 2^{k[X]}$$
$$\mathbb{V} : 2^{k[X]} \to 2^{k^n}$$

where $k[X] := k[X_1, \dots, X_n]$ is the polynomial ring of n variables over k, $\mathbb{C}$ assigns the set of polynomials vanishing over S to any subset $S \subseteq k^n$, and $\mathbb{V}$ assigns the set of common solutions of R to any subset $R \subseteq k[X]$. Hilbert's *Nullstellensatz* characterizes the fixed points of the closure operator $\mathbb{C} \circ \mathbb{V}$. The Galois connection given by the pair $(\mathbb{C}, \mathbb{V})$ can be made functorial. The two functors yield a contravariant adjunction. Upon restricting each functor to the fixed points in each domain, one obtains the *classical duality* between affine algebraic varieties and their coordinate rings.

The principal objective in this paper consisting of six sections is to give an abstraction of the affine adjunction of classical algebraic geometry to algebraic and categorical settings. A synopsis of the paper goes as follows.

- The classical affine adjunction is generalized to any variety of algebras [*J. Słomiński*, Diss. Math. 18 (1959; Zbl 0178.34104); *F. E. J. Linton*, in: Proc. Conf. Categor. Algebra, La Jolla 1965, 84–94 (1966; Zbl 0201.35003)], whether finitary or infinitary, in §4, where the free algebras play the same role as the ring of polynomials in the classical affine connection, while congruences of free algebras replace ideals of rings of polynomials. The adjunction is presented in Corollary 4.8.
- The classical affine connection is lifted to the categorical context in §2. The general adjunction is presented in Theorem 2.7, while an abstract Nullstellensatz is given in Theorem 2.14.
- §3 compares the adjunction in Theorem 2.7 with the theory of dual adjunctions induced by a dualizing object [*H.-E. Porst*, Cah. Topologie Géom. Différ. Catégoriques 17, 95–107 (1976; Zbl 0342.18005); *H. E. Porst* and *W. Tholen*, in: Improving constructions in topology. . 111–136 (1991; Zbl 0761.18001)].
- §5 discusses possible applications in the context of syntactic categories and theories of presheaf type [*M. Makkai* and *G. E. Reyes*, First order categorical logic. Model-theoretical methods in the theory of topoi and related categories. Springer, Cham (1977; Zbl 0357.18002); in: Logic colloquium '81, Proc. Herbrand Symp., Marseille 1981, Stud. Logic Found. Math. 107, 217–232 (1982; Zbl 0522.03006); Contemp. Math. 30, 175–243 (1984; Zbl 0561.03034); Ann. Pure Appl. Logic 40, No. 2, 167–215 (1988; Zbl 0669.03037); Duality and definability in first order logic. Providence, RI: American Mathematical Society (AMS) (1993; Zbl 0783.03038)].
- §6 outlines the method in the prototypical cases of Boolean algebras and commutative C^*-algebras. A detailed proof of a duality theorem along the lines of the theory developed in §4 is to be seen in [*V. Marra* and *L. Spada*, Ann. Pure Appl. Logic 164, No. 3, 192–210 (2013; Zbl 1275.03099)].

Finally some comments on the literature are in order.

- A framework abstracting the classical ring-theoretic adjunction to possibly infinitary varieties of algebras was discussed in [*Y. Diers*, J. Geom. 65, No. 1–2, 54–76 (1999; Zbl 0931.18008); J. Pure Appl. Algebra 150, No. 2, 123–135 (2000; Zbl 0964.18007); *Y. Diers*, Appl. Categ. Struct. 9, No. 1, 35–40 (2001; Zbl 0973.18007)].
- Various elements of algebraic geometry were developed for finitary varieties of algebras [*È. Yu. Daniyarova* et al., Fundam. Prikl. Mat. 9, No. 3, 37–63 (2003; Zbl 1072.17004); translation in J. Math. Sci., New York 135, No. 5, 3292–3310 (2006); Fundam. Prikl. Mat. 9, No. 3, 65–87 (2003; Zbl 1072.17005); translation in J. Math. Sci., New York 135, No. 5, 3311–3326 (2006); *È. Yu. Daniyarova* and *V. N. Remeslennikov*, Algebra Logika 44, No. 3, 269–304 (2005; Zbl 1150.17009); translation in Algebra Logic 44, No. 3, 148–167 (2005); *E. Daniyarova* et al., in: Aspects of infinite groups.

A Festschrift in honor of Anthony Gaglione. Papers of the conference, Fairfield, USA, March 2007 in honour of Anthony Gaglione's 60th birthday. Hackensack, NJ: World Scientific. 80–111 (2008; Zbl 1330.08003); Algebra Logic 49, No. 6, 483–508 (2011; Zbl 1245.08005); translation from Algebra Logika 49, No. 6, 715–756 (2010); Dokl. Math. 84, No. 1, 545–547 (2011; Zbl 1264.14001); translation from Dokl. Akad. Nauk 439, No. 6, 730–732 (2011); Southeast Asian Bull. Math. 35, No. 1, 35–68 (2011; Zbl 1240.08002); Algebra Logic 51, No. 1, 28–40 (2012; Zbl 1255.08001); translation from Algebra Logika 51, No. 1, 41–60 (2012); J. Math. Sci., New York 185, No. 3, 389–416 (2012; Zbl 1256.03037); translation from Fundam. Prikl. Mat. 17, No. 1, 65–106 (2012); Dokl. Math. 90, No. 1, 450–452 (2014; Zbl 1306.08003); translation from Dokl. Akad. Nauk, Ross. Akad. Nauk 457, No. 3, 265–267 (2014); Algebra Logic 56, No. 4, 281–294 (2017; Zbl 1381.08001); translation from Algebra Logika 56, No. 4, 421–442 (2017); Int. J. Algebra Comput. 28, No. 8, 1425–1448 (2018; Zbl 1404.08006); Algebra Logic 57, No. 6, 414–428 (2019; Zbl 1429.08007); translation from Algebra Logika 57, No. 6, 639–661 (2018)].

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MSC:

18A40 Adjoint functors (universal constructions, reflective subcategories, Kan extensions, etc.)
03C05 Equational classes, universal algebra in model theory
06D35 MV-algebras

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